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Computational Section

Primes in the Chebotarev density theorem for all number fields (with an Appendix by Andrew Fiori) ☆

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ABSTRACT

Let L/K be any Galois extension of number fields such that $L \neq \mathbb{Q}$, and let C be a conjugacy class in $\mathrm{Gal}(L/K)$. We show that there exists a degree-one unramified prime $\mathfrak{p}$ of K such that $\sigma_{\mathfrak{p}} = C$ and $N\mathfrak{p} \leq d_L^B$ with $B = 310$. This improves upon $B = 12\,577$ obtained by Ahn and Kwon by making the argument of Lagarias, Montgomery, and Odlyzko explicit. Our improvements come from using the following key ideas: new weights, an optimized analysis on the location of the potential exceptional zeros for $\zeta_L(s)$, and a new version of Turán's power sum method which gives a stronger Deuring-Heilbronn phenomenon for $\zeta_L(s)$. We also use Fiori's numerical verification for number fields up to a certain discriminant height. Other results include a lower bound for the number of unramified primes $\mathfrak{p}$ of K such that $\sigma_{\mathfrak{p}} = C$.

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1. Introduction

Let L/K be a Galois extension of number fields with Galois group G, and let C be a conjugacy class in G. The celebrated Chebotarev density theorem asserts that

$$\#\{\mathfrak{p} \subset \mathcal{O}_K \mid N\mathfrak{p} \le x,\ \mathfrak{p} \text{ is an unramified prime with } \sigma_{\mathfrak{p}} = C\} \sim \frac{|C|}{|G|}\operatorname{Li}(x)$$

as $x \to \infty$, where $\operatorname{Li}(x)$ is the usual logarithmic integral, $\mathcal{O}_K$ is the ring of integers of K, $N = N_{K/\mathbb{Q}}$ is the absolute norm of K, and $\sigma_{\mathfrak{p}}$ denotes the Artin symbol at $\mathfrak{p}$.

As the Chebotarev density theorem generalizes Dirichlet's theorem on primes in arithmetic progressions, a natural question of finding the least (unramified) prime $\mathfrak{p}$ with $\sigma_{\mathfrak{p}} = C$ then arises from Linnik's famous theorem on the least prime in an arithmetic progression. Under the generalized Riemann hypothesis for the Dedekind zeta function $\zeta_L(s)$ of L, Lagarias and Odlyzko [LaOd77] showed that $N\mathfrak{p} \ll (\log d_L)^2$, where d_L denotes the absolute discriminant of L (cf. [BaSo96]).

In [LaMoOd79], Lagarias, Montgomery, and Odlyzko proved, unconditionally, that if $L \neq \mathbb{Q}$, then there is a constant $B > 0$ such that there is an unramified prime $\mathfrak{p}$ of K with $\sigma_{\mathfrak{p}} = C$ and $N\mathfrak{p} \le d_L^B$. Recently, Zaman [Zam17] established that $B = 40$ is valid when d_L is sufficiently large. This is improved by Ng and the authors in [KaNgWo19], who showed that $B = 16$ is admissible for sufficiently large d_L. Also, Ahn and Kwon [AhKw19-1] showed that $B = 12\,577$ is valid for *all* number fields. The main result of this article is the following theorem.

Theorem 1. *Let L/K be a Galois extension of number fields with Galois group G, and let C be a conjugacy class in G. If $L \neq \mathbb{Q}$, then there exists an unramified prime $\mathfrak{p}$ of K, of degree one, such that $\sigma_{\mathfrak{p}} = C$ and $N\mathfrak{p} \le d_L^B$ with $B = 310$.*

Throughout this article, we let $n_L \ge n_0 \ge 2$ and $d_L \ge d_0 \ge 3$. In Appendix A, numerical verifications of the bound for the least prime, with $B \le 1.7712$, are done for $n_L = n_0$ with $2 \le n_0 \le 20$ and $d_L \le d_0$, as well as for $n_L \ge n_0 \ge 21$ and $d_L \le d_0 = 10^{n_L}$ (see Table 3).

In Section 3, we prove the least prime result for the remaining (n_0, d_0) (see Table 2). The proof starts by establishing an explicit inequality (see Section 3.2) between a weighted sum over primes detecting the least one in a given class and the zeros of the Dedekind zeta function $\zeta_L(s)$. The size of the least prime then depends on how close to the vertical line $\mathfrak{Re}\, s = 1$ these zeros are. We can establish some zero-free regions: there exist absolute constants $A, A' > 0$ such that $\zeta_L(s)$ has no zeros in the region

$$\mathfrak{Re}(s) \ge 1 - \frac{1}{A \log d_L + A' n_L \log(|\mathfrak{Im}(s)| + 2)},$$

with the exception of at most one real zero β_1. It has been proven in [Kad12, Theorem 1.1] that for $|\mathfrak{Im}(s)| \le 1$, $A = 12.74$, and $A' = 0$ assuming d_L is sufficiently large.

Recently, Lee improved this to $A = 12.44$ in [Lee, Theorem 2]. Also, this was made explicit in [AhKw19-1, Proposition 6.1] with $A = A' = 29.57$. We will also need a region with no non-exceptional zeros of the form $\mathfrak{Re}(s) \geq 1 - \frac{1}{R_1 \log d_L}$ and $|\mathfrak{Im}(s)| \leq \frac{1}{R_1 \log d_L}$ with some admissible $R_1 \geq 1.24$ as discussed in Section 2.1.

We can refine the statement of Theorem 1 depending on whether $\zeta_L(s)$ possesses an exceptional real zero β_1 or not. For instance, if $\zeta_L(s)$ has no exceptional zero, then Theorem 1 is valid with $B = 10.5$. In the other case, we require a careful study depending on how close the exceptional zero β_1 is to 1, and we adapt our choice of weight (see Section 3.1) to detect the least prime accordingly. In particular, we determine that it is useful to investigate the supplementary case where β_1 is at a "very small" distance, namely $(\log d_L)^{-2} \ll (1 - \beta_1) \ll (\log d_L)^{-1.15}$.

It is known as the Deuring-Heilbronn phenomenon that the closer β_1 is to 1, the further left other zeros are "repulsed". We now describe the stronger zero-repulsion theorems we use instead of the one established in [AhKw19-1, Theorem 7.3]; in Section 2.2, we prove some versions of [KaNgWo19, Theorems 1.2 and 1.3] which are valid for all number fields.

Theorem 2. *Let $L \neq \mathbb{Q}$ be a number field of degree n_L and with absolute discriminant d_L. Assume $\zeta_L(s)$ admits an exceptional real zero β_1. Let $\beta' + it$ be another (non-trivial) zero of $\zeta_L(s)$, and let $\tau = |t| + 2$. Then, for any $\eta \in (0, 1]$, there exist c_1 and c_2 such that either $\beta' \leq 1 - \eta$ or*

$$\beta' \leq 1 - c_2 \frac{\log\left(\frac{c_1}{(1-\beta_1)\log(d_L \tau^{n_L})}\right)}{\log(d_L \tau^{n_L})}. \tag{1.1}$$

In addition, if $|t| \leq 1$, then there exist c_1' and c_2' such that either $\beta' \leq 1 - \eta$ or

$$\beta' \leq 1 - c_2' \frac{\log\left(\frac{c_1'}{(1-\beta_1)(\log d_L)}\right)}{\log d_L}. \tag{1.2}$$

Here (c_1, c_2) and (c_1', c_2') depend on η, and are respectively defined in (2.24) and (2.28).

We also investigate the case of real zeros as this case offers a stronger zero-repulsion phenomenon, as illustrated in the next theorem.

Theorem 3. *Assume $\zeta_L(s)$ admits an exceptional real zero β_1. Let β' be another real zero of $\zeta_L(s)$. Then, for any $\eta \in (0, 1]$, there exist c_1'' and c_2'' such that either $\beta' \leq 1 - \eta$ or*

$$\beta' \leq 1 - c_2'' \frac{\log\left(\frac{c_1''}{(1-\beta_1)\log d_L}\right)}{\log d_L}, \tag{1.3}$$

where (c_1'', c_2''), depend on η, are defined in (2.30). For $\eta = 1$, numerical results are listed in Table 1.

The key tool to prove Theorems 2 and 3 resides in Theorem 7 where we improve lower bounds for some modified Turán power sums. Numerical results for $\eta = 1$ are listed in Table 1. For instance, we establish that $c_2 = 0.04233 \approx \frac{1}{23.624}$, $c_2' = 0.05 = \frac{1}{20}$, and $c_2'' = 0.1008 \approx \frac{1}{9.921}$ are admissible. This enlarges the region described in [AhKw19-1, Theorem 7.3(2)], where $c_2 = \frac{1}{77}$ was obtained.[1] In addition this extends and improves [Zam17, Theorem 1.2] where Zaman proved $c_2' = \frac{1}{35.8}$ for d_L sufficiently large. In [KaNgWo19], the repulsion constant was improved to $\frac{1}{14.144}$, for d_L sufficiently large.

In Section 2.3 we deduce a bound for the possible exceptional zero β_1 of $\zeta_L(s)$.

Theorem 4. *Suppose $L \neq \mathbb{Q}$. If $\zeta_L(s)$ admits a real zero β_1, then*

$$1 - \beta_1 \geq d_L^{-c_3} \text{ with } c_3 = 11.7.$$

This improves [AhKw19-1, Corollary 7.4] where $c_3 = 114.72...$ was proven admissible. We note that for d_L sufficiently large, the bound above was established with $c_3 = 16.6$ in [Zam17, Corollary 1.4] and $c_3 = 7.072$ in [KaNgWo19, Corollary 1.3.1].

At this point, we have all the key ingredients (also summarized in Section 3.6), and we will complete the proof of Theorem 1 in Section 3. Here is a qualitative description to illustrate how they impact the final result:

- a refined case analysis (in particular, adding the "very small" case) allows us to divide Ahn and Kwon's constant of 12 577 by a factor of about 7;
- the choice of the weight and the strength of Deuring-Heilbronn phenomenon allow us to improve the result by a factor of about 3;
- finally, the numerical verifications divide the value of B by 2 (going from 620 without using Table 3, to the announced 310).

We let $\tilde{\pi}_C(x)$ denote the number of degree-one unramified primes $\mathfrak{p}$ of K such that $N\mathfrak{p} \leq x$ and $\sigma_{\mathfrak{p}} = C$. In Section 4 we apply Theorem 4 to obtain a lower bound for $\tilde{\pi}_C(x)$.

Theorem 5. *Let L/K be a Galois extension of number fields with Galois group G, and let C be a conjugacy class in G. If $L \neq \mathbb{Q}$, then for $x \geq \exp(d_L^{c_3})$, we have*

$$\tilde{\pi}_C(x) \geq m \frac{|C|}{|G|} \frac{x}{\log x},$$

where $c_3 = 11.7$ and $m = 0.4899$.

[1] The labelling for our (c_1, c_2) was (c_7, c_8) in [AhKw19-1, Theorem 7.3(2)].

This improves [AhKw19-2, Theorem 3] where Ahn and Kwon obtained instead $c_3 = 114.72...$ and $m = 0.353$ for all $L \neq \mathbb{Q}$. We also note that Thorner and Zaman [ThZa17, Theorem 3.1] showed that there are absolute constants κ_2 and κ_3 such that

$$\tilde{\pi}_C(x) \gg \frac{1}{d_L^{\kappa_2}} \frac{|C|}{|G|} \operatorname{Li}(x)$$

for $x \geq d_L^{\kappa_3}$ and d_L sufficiently large. While our range for x is more restricted, we are able to obtain a lower bound independent of L.

Notation

We recall that L is a number field of degree $n_L \geq n_0 \geq 2$ and with absolute discriminant $d_L \geq d_0 \geq 3$. We denote

$$\mathscr{L} = \log d_L \text{ and } \mathscr{L}_0 = \log d_0. \tag{1.4}$$

Together with Minkowski's bound, we consider

$$n_0 \leq n_L \leq \frac{2 \log d_L}{\log 3}, \ d_L \geq d_0, \text{ and } \log d_L \geq \mathscr{L}_0. \tag{1.5}$$

Let $\tau = |t| + 2$. We shall define

$$\delta = \delta_L(\tau) = \frac{\log(\tau^{n_L})}{\log d_L}, \tag{1.6}$$

and use various bounds for δ depending on cases. For $T_0 \geq 0$, we define

$$\Delta_0(T_0) = \mathscr{Q}_0 \log(T_0 + 2), \tag{1.7}$$

where

$$\mathscr{Q}_0 = \begin{cases} \frac{n_L}{\mathscr{L}_0} & \text{if } 2 \leq n_L \leq 20; \\ \frac{1}{\log 10} & \text{if } n_L \geq 21. \end{cases} \tag{1.8}$$

Note that for $|t| \leq T_0$, we have

$$0 \leq \delta \leq \Delta_0(T_0). \tag{1.9}$$

2. Repelling non-exceptional zeros further left

We recall [AhKw19-1, Proposition 6.1], which asserts that for any number field L, there are no zeros for $\zeta_L(s)$ in the region

$$\mathfrak{Re}(s) \geq 1 - \frac{1}{29.57\,(\mathscr{L} + n_L \log(|\mathfrak{Im}(s)| + 2))}, \tag{2.1}$$

with the exception of at most one real zero β_1. If the exceptional zero β_1 exists, then it has to be real and simple. In addition, by [Kad12] (for d_L sufficiently large) and by [AhKw14, Theorem 1] (for all $L \neq \mathbb{Q}$), we know that $\zeta_L(s)$ has at most one zero in the region

$$\beta > 1 - \frac{1}{R_0\mathscr{L}} \text{ and } |\gamma| < \frac{1}{2\mathscr{L}}, \tag{2.2}$$

for any $R_0 \geq 2$. In what follows we let β_1 denote the exceptional zero of $\zeta_L(s)$ if it exists. We shall split our considerations into several cases depending on the location (and existence) of β_1.

2.1. Enlarging the region without non-exceptional zeros

We assume that there exists $R_1 \leq R_0$ such that there is no non-trivial zeros $\rho \neq \beta_1$ of $\zeta_L(s)$ in the region

$$\beta > 1 - \frac{1}{R_1\mathscr{L}} \text{ and } |\gamma| < \frac{1}{R_1\mathscr{L}}. \tag{2.3}$$

We note that one can always take $R_1 = 2$, but show here that we can take 1.24 and 1.7 as admissible values for R_1 for certain instances:

Proposition 6. *Let L be a number field such that $L \neq \mathbb{Q}$, and let $R_0 \geq 2$. If the exceptional zero β_1 of $\zeta_L(s)$ presents in $(1 - \frac{1}{R_0\mathscr{L}}, 1)$, then there is no other zeros of $\zeta_L(s)$ in the region*

$$\beta > 1 - \frac{1}{1.7\mathscr{L}} \text{ and } |\gamma| < \frac{1}{1.7\mathscr{L}}. \tag{2.4}$$

Moreover, if $R_0 \geq 3.5$, then 1.7 in (2.4) can be improved to 1.24.

Proof of Proposition 6. Suppose that the exceptional zero β_1 presents in $(1 - \frac{1}{R_0\mathscr{L}}, 1)$. Let $\beta_2 + i\gamma_2$ be another non-trivial zero of $\zeta_L(s)$ such that

$$\beta_2 > 1 - \frac{1}{R\mathscr{L}} \text{ and } |\gamma_2| < \frac{1}{R\mathscr{L}}.$$

Following [AhKw14, Sec. 2], we assume $R \geq 1.24$ and set $\sigma = 1 + \frac{1}{r\mathscr{L}}$, with $1 < \sigma \leq 6.2$, and $\frac{1}{5.2\log 3} \leq r \leq R$. As argued in [AhKw14, pp. 438–439], one has

$$\Big(\frac{1}{2}\Big(1 - \frac{1}{\sqrt{5}}\Big) + r\Big)\mathscr{L} \geq \frac{\sigma - \beta_1}{(\sigma - \beta_1)^2} + \frac{\sigma - \beta_2}{(\sigma - \beta_2)^2 + \gamma_2^2} = \frac{1}{\sigma - \beta_1} + \frac{\sigma - \beta_2}{(\sigma - \beta_2)^2 + \gamma_2^2},$$

and thus

$$\frac{1}{2}\Big(1-\frac{1}{\sqrt{5}}\Big)+r\geq\frac{rR_0}{R_0+r}+\frac{rR(R+r)}{(R+r)^2+r^2}. \tag{2.5}$$

By a numerical calculation, if $R_0 \geq 2$ and $r = 0.6$, it can be checked that (2.5) fails for $R \geq 1.7$. Also, if $R_0 \geq 3.5$ and $r = 0.6$, (2.5) fails for $R \geq 1.24$. This concludes the proof. □

2.2. Deuring-Heilbronn phenomenon for all number fields

In this section, we prove Theorems 2 and 3, that is quantitative versions of the Deuring-Heilbronn phenomenon for all number fields. To do so, we employ the Turán's power sum method as introduced by Lagarias, Montgomery, and Odlyzko in [LaMoOd79] (see also Montgomery's [Mon94, Theorem 11]). We note that Ahn and Kwon employ [LaMoOd79, Theorem 4.2] to prove their version [AhKw19-1, Theorem 7.3] of the Deuring-Heilbronn phenomenon for all number fields. The reader can also find a refinement of [LaMoOd79, Theorem 4.2] in Zaman's [Zam17, Theorem 2.3]. In [KaNgWo19, Theorem 2.2], the authors, together with Ng, appeal to Harnack's inequality to bring some improvements to these results. The following is the case $s_j = \sum_{n\geq 1} b_n z_n^j$ with $b_n = 1$ for all n.

Theorem 7. *Let $\varepsilon > 0$ and $z_1 \neq 0$. Assume that for all $n \geq 1$, z_n is a complex number satisfying $|z_n| \leq |z_1|$. For any $j \in \mathbb{N}$, set $s_j = \sum_{n\geq 1} z_n^j$ and $M = \sum_{n\geq 1} \frac{|z_n|}{|z_1|+|z_n|}$. Then there exists j_0 with $1 \leq j_0 \leq (8+\varepsilon)M$ such that*

$$\mathfrak{Re}(s_{j_0}) \geq \frac{\varepsilon}{4(8+\varepsilon)}|z_1|^{j_0}.$$

We propose here a bound for $\mathfrak{Re}\frac{\Gamma'}{\Gamma}$ which is the last tool we need to prove Theorem 2. It improves [AhKw19-1, Lemma 5.3] and will allow for sharper estimates for $\mathfrak{Re}\frac{\gamma_L'}{\gamma_L}$ than [AhKw19-1, Lemma 5.4], where γ_L is the associated gamma factor of $\zeta_L(s)$.

Lemma 8. *Let $T_0 \geq 0$. For $s = \sigma + it$ with $\sigma > 0$ and $|t| \leq T_0$, we define*

$$g(\sigma, T_0) = \max_{|t|\leq T_0}\Big(\log\Big(\frac{\sqrt{\sigma^2+t^2}}{|t|+2}\Big)-\frac{\sigma}{\sigma^2+t^2}\Big)+\frac{1}{3\sigma^2}-\log 2. \tag{2.6}$$

Then

$$\mathfrak{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big) \leq \log(|t|+2)+g(\sigma,T_0). \tag{2.7}$$

Proof. According to [AhKw19-1, p. 1429],

$$\mathfrak{Re}\frac{\Gamma'}{\Gamma}(s) \leq \log|s| - \frac{\sigma}{2(\sigma^2+t^2)}+\frac{1}{12\sigma^2}. \tag{2.8}$$

Thus,

$$\mathfrak{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{s}{2}\Big) \le \log(|t|+2) + \max_{|t|\le T_0}\Big(\log\Big(\frac{\sqrt{\sigma^2+t^2}}{|t|+2}\Big) - \frac{\sigma}{\sigma^2+t^2}\Big) + \frac{1}{3\sigma^2} - \log 2$$

from which (2.7) follows. □

Now, we are in a position to prove Theorems 2 and 3. For both proofs, we are assuming (1.5) for n_L and d_L, and we introduce $\varepsilon > 0, \sigma \ge 2$, and $\eta \in (0,1]$.

Proof of Theorem 2. We denote β_1 the exceptional real zero that we assume $\zeta_L(s)$ possesses. We let $\mathcal{S}$ be the set of all non-trivial zeros of $\zeta_L(s)$. Let $\eta \in (0,1]$ and $T_0 \ge 0$. We also denote $\beta' + it$ a non-trivial zero of $\zeta_L(s)$ such that $1-\eta \le \beta' < 1$ and $|t| \le T_0$. We shall let $\sigma \ge 2$ and let z_n run over $(\sigma-\rho)^{-2}$ and $(\sigma+it-\rho)^{-2}$ for all $\rho \in \mathcal{S}\backslash\{\beta_1\}$ in order that $|z_n|$ decreases (so $|z_1| \ge |z_2| \ge \cdots$). Thus, $|z_1| \ge \frac{1}{(\sigma-\beta')^2}$, and by the inequality $\log(1+x) \le x$, we have

$$\begin{aligned} |z_1| &\ge \frac{1}{(\sigma-1)^2}\frac{(\sigma-1)^2}{(\sigma-\beta')^2} = \frac{1}{(\sigma-1)^2}\exp\Big(-2\log\Big(\frac{\sigma-\beta'}{\sigma-1}\Big)\Big) \\ &\ge \frac{1}{(\sigma-1)^2}\exp\Big(-2\frac{1-\beta'}{\sigma-1}\Big). \end{aligned} \tag{2.9}$$

From [AhKw19-1, Eq. (7.2) and Lemma 7.1], it follows that

$$\mathfrak{Re}\sum_{n\ge1} z_n^{j_0} \le \frac{4j_0(1-\beta_1)}{(\sigma-1)^{2j_0+1}}. \tag{2.10}$$

Applying Theorem 7 for $\varepsilon > 0$, there exists j_0 with $1 \le j_0 \le (8+\varepsilon)M$ such that

$$\mathfrak{Re}\sum_{n\ge1} z_n^{j_0} \ge \frac{\varepsilon}{4(8+\varepsilon)}|z_1|^{j_0}. \tag{2.11}$$

Combining (2.9), (2.10), (2.11), and the fact $j_0 \le (8+\varepsilon)M$ gives

$$1-\beta' \ge \frac{\sigma-1}{2(8+\varepsilon)M}\log\Big(\frac{\varepsilon(\sigma-1)}{16M(8+\varepsilon)^2(1-\beta_1)}\Big). \tag{2.12}$$

We finalise by appealing to [KaNgWo19, p. 2294] to establish an upper bound for M:

$$M \le \mathcal{A}S_L(d-1,t), \tag{2.13}$$

where $\mathcal{A}$, d, and $S_L(d-1,t)$ are respectively defined in [KaNgWo19, Eq. (2.6), (2.11), and (2.12)]:

$$\mathcal{A} = \mathcal{A}(\sigma,\eta) = (\sigma - 1 + \eta)^2, \tag{2.14}$$

$$d = d(\sigma,\eta) = \sqrt{\sigma^2 + \mathcal{A}} = \sqrt{2\sigma^2 + (1-\eta)^2 - 2\sigma(1-\eta)}, \tag{2.15}$$

$$S_L(d-1,t) = \sum_{\rho} \Big(\frac{1}{|d-\rho|^2} + \frac{1}{|d+it-\rho|^2} \Big). \tag{2.16}$$

Also, [Zam17, Lemma 2.5 and Eq. (2.10)] provides a bound for S_L:

$$S_L(d-1,t) \le \frac{\log d_L}{d-1} + \frac{G_1(d-1;|t|)}{d-1} r_1 + \frac{G_2(d-1;|t|)}{d-1}(2r_2) + \frac{2}{(d-1)^2} + \frac{2}{d(d-1)}, \tag{2.17}$$

where r_1 and r_2 are the numbers of real and complex places, respectively, of L, and

$$\begin{aligned} G_1(d-1;|t|)r_1 + G_2(d-1;|t|)(2r_2) = & \frac{r_1+r_2}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{d}{2}\Big) + \frac{r_1+r_2}{2}\mathfrak{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{d+i|t|}{2}\Big) \\ & + \frac{r_2}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{d+1}{2}\Big) + \frac{r_2}{2}\mathfrak{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{d+1+i|t|}{2}\Big) \\ & - n_L \log \pi. \end{aligned} \tag{2.18}$$

Together with the bounds on Gamma from Lemma 8 and the fact that $\frac{\Gamma'}{\Gamma}(x)$ increases with real values x, we obtain

$$r_1 G_1(d-1;|t|) + 2r_2 G_2(d-1;|t|) \le \frac{\log(\tau^{n_L})}{2} + \frac{n_L}{2}\Big(\frac{\Gamma'}{\Gamma}\Big(\frac{d+1}{2}\Big) + \max_{x=d,d+1} g(x,T_0) - 2\log\pi \Big). \tag{2.19}$$

We observe that, for δ defined in (1.6), whenever $\log d_L \ge \mathscr{L}_0$,

$$\log d_L = \frac{1}{1+\delta}\log(d_L\tau^{n_L}), \quad \log(\tau^{n_L}) = \frac{\delta}{1+\delta}\log(d_L\tau^{n_L}),$$
$$n_L \le \frac{\delta}{1+\delta}\frac{1}{\log 2}\log(d_L\tau^{n_L}), \quad 1 \le \frac{1}{(1+\delta)}\frac{1}{\mathscr{L}_0}\log(d_L\tau^{n_L}).$$

We combine these with (2.19) so that (2.17) becomes

$$S_L(d-1,t) \le \frac{\log(d_L\tau^{n_L})}{d-1}\Big(\frac{b_1(d) + b_2(d,T_0)\delta}{1+\delta}\Big), \tag{2.20}$$

where we denote

$$b_1(d) = 1 + \frac{2}{\mathscr{L}_0(d-1)} + \frac{2}{\mathscr{L}_0 d}, \tag{2.21}$$

$$b_2(d,T_0) = \frac{1}{2} + \frac{1}{2\log 2}\max\Big\{ \frac{\Gamma'}{\Gamma}\Big(\frac{d+1}{2}\Big) + \max\{g(d,T_0), g(d+1,T_0)\} - 2\log\pi, 0 \Big\}. \tag{2.22}$$

Now, setting

$$\mathcal{B} = \mathcal{B}(d, T_0) = \frac{1}{d-1} \max_{\delta \in [0, \Delta_0(T_0)]} \Big(\frac{b_1(d) + b_2(d, T_0)\delta}{1+\delta} \Big), \tag{2.23}$$

we see that (2.13) becomes $M \le \mathcal{AB} \log(d_L \tau^{n_L})$. From (2.12) and (2.13), it follows that

$$1 - \beta' \ge \frac{\sigma - 1}{2(8+\varepsilon)\mathcal{AB}} \frac{\log\left(\frac{\varepsilon(\sigma-1)}{16(8+\varepsilon)^2 \mathcal{AB}} \frac{1}{(1-\beta_1)\log(d_L \tau^{n_L})} \right)}{\log(d_L \tau^{n_L})} = c_2 \frac{\log\left(\frac{c_1}{(1-\beta_1)\log(d_L \tau^{n_L})} \right)}{\log(d_L \tau^{n_L})},$$

with

$$c_1 = \frac{\varepsilon}{8(8+\varepsilon)} c_2 \quad \text{and} \quad c_2 = \frac{\sigma - 1}{2(8+\varepsilon)\mathcal{AB}}. \tag{2.24}$$

Now, taking $T_0 = \infty$, we establish the first part of Theorem 2. We note that one may bound $g(\sigma, \infty)$, trivially, by

$$g(\sigma, \infty) \le \frac{1}{2} \log\left(\frac{\sigma^2}{4} + 1 \right) + \frac{1}{3\sigma^2} - \log 2.$$

We will use this bound to control $\mathcal{B}(d, \infty)$ and calculate c_1 and c_2 for the theorem.

On the other hand, for $|t| \le 1$ (so for $T_0 = 1$), we may directly use (2.8) to deduce

$$\max_{x=d,d+1} \mathfrak{Re} \frac{\Gamma'}{\Gamma}\Big(\frac{x+it}{2} \Big) \le \max_{x=d,d+1} \Big(\log \sqrt{x^2+1} - \frac{x}{x^2+1} + \frac{1}{3x^2} \Big) - \log 2. \tag{2.25}$$

Hence, by (2.18), we have $G_1(d-1; |t|) r_1 + G_2(d-1; |t|)(2r_2) \le n_L \mathcal{H}(d)$, where we denote

$$\mathcal{H}(d) = \frac{1}{2} \frac{\Gamma'}{\Gamma}\Big(\frac{d+1}{2} \Big) + \frac{1}{2} \max_{x=d,d+1} \Big(\log \sqrt{x^2+1} - \frac{x}{x^2+1} + \frac{1}{3x^2} \Big) - \frac{\log 2 + 2 \log \pi}{2}. \tag{2.26}$$

Thus, whenever $n_L \ge n_0$ and $\frac{n_L}{\log d_L} \le \mathscr{Q}_0$, we obtain

$$\begin{aligned} S_L(d-1, t) \le & \frac{1}{d-1} \Big(\log d_L + n_L \mathcal{H}(d) + \frac{4d-2}{d(d-1)} \Big) \\ \le & \frac{\log d_L}{d-1} \Big(1 + \mathscr{Q}_0 \max \Big\{ \mathcal{H}(d) + \frac{4d-2}{d(d-1)n_0}, 0 \Big\} \Big), \end{aligned}$$

with $\mathscr{Q}_0$ as defined in (1.8). Now, setting

$$\mathcal{B}' = \mathcal{B}'(d) = \frac{1 + \mathscr{Q}_0 \max\left\{ 0, \mathcal{H}(d) + \frac{4d-2}{d(d-1)n_0} \right\}}{d-1}, \tag{2.27}$$

(2.13) gives $M \le \mathcal{AB}' \log d_L$. Applying (2.12), we arrive at $1 - \beta' \ge c_2' \frac{\log\left(\frac{c_1'}{(1-\beta_1)\log d_L} \right)}{\log d_L}$, where

$$c_1' = c_1'(\varepsilon, \sigma, \eta) = \frac{\varepsilon}{8(8+\varepsilon)} c_2' \quad \text{and} \quad c_2' = c_2'(\varepsilon, \sigma, \eta) = \frac{\sigma - 1}{2(8+\varepsilon)\mathcal{A}\mathcal{B}'}. \tag{2.28}$$

This completes the proof of Theorem 2. □

We shall now prove the version of the Deuring-Heilbronn phenomenon that *only* concerns the location of real zeros of $\zeta_L(s)$. We adapt the previous proof to the case $t = 0$.

Proof of Theorem 3. The proof repeats most of the argument from the previous proof, so we shall keep using the notation with $t = 0$ and the sequence z_n running over $(\sigma - \rho)^{-2}$ for all the non-trivial zeros $\rho \in \mathcal{S} \backslash \{\beta_1\}$, with z_n satisfying $|z_1| \geq |z_2| \geq \cdots$. It follows from [KaNgWo19, Proof of Lemma 2.7] that $2M \leq \mathcal{A} S_L(d-1, 0)$. We observe that

$$\frac{S_L(d-1,0)}{2} \leq \frac{1}{d-1}\Big(\frac{\log d_L}{2} + \frac{n_L}{2}\Big(\frac{\Gamma'}{\Gamma}\Big(\frac{d+1}{2}\Big) - \log \pi\Big) + \frac{2d-1}{d(d-1)}\Big).$$

Thus, setting

$$\mathcal{B}'' = \mathcal{B}''(d) = \frac{1}{d-1}\Big(\frac{1}{2} + \mathscr{Q}_0 \max\Big\{0, \frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{d+1}{2}\Big) - \frac{\log \pi}{2} + \frac{2d-1}{d(d-1)n_0}\Big\}\Big), \tag{2.29}$$

we have $M \leq \mathcal{A}\frac{S_L(d-1,0)}{2} \leq \mathcal{A}\mathcal{B}''(\log d_L)$ whenever $n_L \geq n_0$, and $\frac{n_L}{\log d_L} \leq \mathscr{Q}_0$. Hence, arguing as before, we have $1 - \beta' \geq c_2'' \frac{\log\left(\frac{c_1''}{(1-\beta_1)\log d_L}\right)}{\log d_L}$, where c_1'', c_2'' depend on $(8+\varepsilon), \sigma, \eta$ and are given by

$$c_1'' = \frac{\varepsilon}{8(8+\varepsilon)} c_2'' \quad \text{and} \quad c_2'' = \frac{\sigma - 1}{2(8+\varepsilon)\mathcal{A}\mathcal{B}''}. \tag{2.30}$$

This completes the proof of Theorem 3. □

2.3. Distance of the exceptional zero to 1-line

With Theorem 3 in hand, we shall prove Theorem 4 in this section.

Proof of Theorem 4. We let $\eta \in (0, 1]$ be a parameter to be chosen later. We consider β a non-exceptional real zero of $\zeta_L(s)$. We assume $\beta \geq 1 - \eta$, and we consider c_3 such that

$$1 - \beta_1 = d_L^{-c_3}.$$

By Theorem 3, whenever $\mathscr{L} \geq \mathscr{L}_0$, we have

$$\eta \geq 1 - \beta \geq c_2'' \frac{\log c_1'' - \log(1-\beta_1) - \log\log d_L}{\log d_L} \geq c_2'' \frac{\log c_1'' + c_3\mathscr{L} - \log \mathscr{L}}{\mathscr{L}},$$

where c_1'' and c_2'' are defined in (2.30). Thus,

$$c_3 \leq \frac{\eta}{c_2''} + \frac{\log \mathscr{L} - \log c_1''}{\mathscr{L}} \leq \frac{\eta}{c_2''} + \frac{\log \mathscr{L}_0 - \log c_1''}{\mathscr{L}_0}, \tag{2.31}$$

which completes the proof of Theorem 4. □

Remark. We note that if there are only two real zeros (i.e. β_1 and $1-\beta_1$), then we can only take the trivial bound $\beta = 1-\beta_1 > 0$ (i.e. $\eta = 1$). However, if there is a third real zero β', by the symmetry of zeros of $\zeta_L(s)$, we may take $\beta = \beta' \geq 1/2$ (so $\eta = 1/2$ is admissible), which reduces the admissible value for c_3 and, consequently, would push β_1 away from the 1-line a bit further. In other words, while Deuring-Heilbronn describes how an exceptional real zero would push further left other zeros in its vicinity, the other real zeros also do the same to the exceptional zero. We note that this phenomenon would hold for Dirichlet L-functions as well.

2.4. Numerical results

As Theorems 2, 3, and 4 are "trivial" if $\zeta_L(s)$ does not admit the exceptional zero β_1, it is sufficient to calculate $c_1, c_2, c_1', c_2', c_1'', c_2'', c_3$ for L such that $\zeta_L(s)$ may admit the exceptional zero β_1. To do so, we first summarize some cases that the non-existence of β_1 is known.

For $n_L = 2$, we recall that there is a (quadratic) Dirichlet character χ_L modulo d_L such that $\zeta_L(s) = \zeta(s)L(s,\chi_L)$, where $\zeta(s)$ is the Riemann zeta function, and $L(s,\chi_L)$ is the Dirichlet L-function attached to χ_L. It is known that $\zeta(\sigma)$ is non-vanishing for $\sigma > 0$. Moreover, Platt [Pla16] showed that for any Dirichlet character χ modulo q, with $q \leq 400\,000$, $L(s,\chi)$, the Dirichlet L-function attached to χ, has no positive real zeros. Hence, for $n_L = 2$, $\zeta_L(s)$ has no positive real zeros if $d_L \leq 400\,000$.

For $n_L = 3$ and $d_L \leq 239$, Tollis [Tol97] verified the generalized Riemann hypothesis for $\zeta_L(s)$ to $|\mathfrak{Im}(s)| \leq 92$. Also, for $n_L = 4$ and $d_L \leq 320$, Tollis [Tol97] verified the generalized Riemann hypothesis for $\zeta_L(s)$ to $|\mathfrak{Im}(s)| \leq 40$. Consequently, $\zeta_L(s)$ has no positive real zeros whenever $n_L = 3$ and $d_L \leq 239$ or $n_L = 4$ and $d_L \leq 320$. Hence, for $2 \leq n_L \leq 4$, we only need to compute $c_1, c_2, c_1', c_2', c_1'', c_2'', c_3$ for L with $d_L \geq d_0$, where d_0 given in Table 1.

For fields L of degree $n_L \geq 5$, instead of trying to confirm the non-existence of the exceptional zero of $\zeta_L(s)$, we use the following lower bounds d_0 of d_L to calculate $c_1, c_2, c_1', c_2', c_1'', c_2'', c_3$. For $5 \leq n_L \leq 8$, by [LMFDB], it can be checked that all the fields L satisfy that $d_L \geq d_0$, where d_0 is given in Table 1. For $9 \leq n_L \leq 20$, as remarked by Fiori, by the lower bounds for the minimal root discriminant in [DyD80], we know that $d_L \geq d_0$, where d_0 is given in Table 1. In addition, for $n_L \geq 21$, by [DyD80], one has $d_L > 10^{n_L}$, and thus d_0 can be taken as $d_0 = 10^{n_L}$ for these fields.

Table 1
Values for $(c_1, c_2, c_1', c_2', c_1'', c_2'', c_3)$ as defined in Theorems 2, 3, and 4.

n_0	d_0	c_1	c_2	c_1'	c_2'	c_1''	c_2''	c_3
2	400 000	$5.174 \cdot 10^{-5}$	0.04233	$7.904 \cdot 10^{-5}$	0.06466	$1.581 \cdot 10^{-4}$	0.1293	8.608
3	239	$1.276 \cdot 10^{-4}$	0.05697	$1.133 \cdot 10^{-4}$	0.05059	$2.275 \cdot 10^{-4}$	0.1015	11.69
4	320	$1.047 \cdot 10^{-4}$	0.04974	$1.052 \cdot 10^{-4}$	0.05000	$2.121 \cdot 10^{-4}$	0.1008	11.69
5	1 609	$8.485 \cdot 10^{-5}$	0.04960	$8.790 \cdot 10^{-5}$	0.05139	$1.774 \cdot 10^{-4}$	0.1037	11.08
6	9 747	$7.056 \cdot 10^{-5}$	0.05019	$7.371 \cdot 10^{-5}$	0.05243	$1.488 \cdot 10^{-4}$	0.1058	10.65
7	184 607	$5.852 \cdot 10^{-5}$	0.05397	$5.802 \cdot 10^{-5}$	0.05351	$1.168 \cdot 10^{-4}$	0.1077	10.23
8	1 257 728	$5.118 \cdot 10^{-5}$	0.05411	$5.105 \cdot 10^{-5}$	0.05397	$1.028 \cdot 10^{-4}$	0.1087	10.04
9	$2.290 \cdot 10^{7}$	$4.450 \cdot 10^{-5}$	0.05620	$4.321 \cdot 10^{-5}$	0.05457	$8.690 \cdot 10^{-5}$	0.1097	9.831
10	$1.560 \cdot 10^{8}$	$4.008 \cdot 10^{-5}$	0.05609	$3.916 \cdot 10^{-5}$	0.05480	$7.878 \cdot 10^{-5}$	0.1102	9.728
11	$3.910 \cdot 10^{9}$	$3.602 \cdot 10^{-5}$	0.05792	$3.436 \cdot 10^{-5}$	0.05526	$6.906 \cdot 10^{-5}$	0.1110	9.579
12	$2.740 \cdot 10^{10}$	$3.323 \cdot 10^{-5}$	0.05774	$3.187 \cdot 10^{-5}$	0.05538	$6.406 \cdot 10^{-5}$	0.1113	9.517
13	$7.560 \cdot 10^{11}$	$3.037 \cdot 10^{-5}$	0.05914	$2.862 \cdot 10^{-5}$	0.05574	$5.749 \cdot 10^{-5}$	0.1120	9.410
14	$5.430 \cdot 10^{12}$	$2.755 \cdot 10^{-5}$	0.05899	$2.607 \cdot 10^{-5}$	0.05582	$5.236 \cdot 10^{-5}$	0.1121	9.370
15	$1.610 \cdot 10^{14}$	$2.527 \cdot 10^{-5}$	0.06010	$2.359 \cdot 10^{-5}$	0.05610	$4.736 \cdot 10^{-5}$	0.1126	9.288
16	$1.170 \cdot 10^{15}$	$2.424 \cdot 10^{-5}$	0.05987	$2.273 \cdot 10^{-5}$	0.05614	$4.565 \cdot 10^{-5}$	0.1127	9.261
17	$3.700 \cdot 10^{16}$	$2.273 \cdot 10^{-5}$	0.06080	$2.108 \cdot 10^{-5}$	0.05638	$4.231 \cdot 10^{-5}$	0.1132	9.196
18	$2.730 \cdot 10^{17}$	$2.172 \cdot 10^{-5}$	0.06062	$2.021 \cdot 10^{-5}$	0.05639	$4.057 \cdot 10^{-5}$	0.1132	9.177
19	$9.030 \cdot 10^{18}$	$2.010 \cdot 10^{-5}$	0.06140	$1.852 \cdot 10^{-5}$	0.05660	$3.718 \cdot 10^{-5}$	0.1136	9.123
20	$6.740 \cdot 10^{19}$	$1.908 \cdot 10^{-5}$	0.06122	$1.765 \cdot 10^{-5}$	0.05661	$3.542 \cdot 10^{-5}$	0.1136	9.109
21+	10^{n_L}	$1.819 \cdot 10^{-5}$	0.06141	$1.679 \cdot 10^{-5}$	0.05669	$3.370 \cdot 10^{-5}$	0.1138	9.082

3. The least prime in the Chebotarev density theorem - proof of Theorem 1

3.1. Choosing a weight to detect the least prime

Let $x \geq 1$. Let $\theta > 1$ be a parameter to be chosen later. We consider the kernel

$$k(s) = k_\theta(s) = \Big(\frac{x^{\theta(s-1)} - x^{s-1}}{s-1}\Big)^2 \tag{3.1}$$

and recall its inverse Mellin transform is

$$\hat{k}(u) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} k(s)u^{-s}ds = \begin{cases} u^{-1}\log(x^{2\theta}/u) & \text{if } x^{\theta+1} \leq u \leq x^{2\theta}; \\ u^{-1}\log(u/x^2) & \text{if } x^2 \leq u \leq x^{\theta+1}; \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

In [AhKw19-1,LaMoOd79], θ was chosen to be 2. Note that

$$\hat{k}(u) \leq \frac{(\theta-1)(\log x)}{u}, \tag{3.3}$$

$$k(1) = k_\theta(1) = ((\theta-1)\log x)^2, \tag{3.4}$$

$$k(\sigma) = \Big(\frac{x^{\theta(\sigma-1)} - x^{\sigma-1}}{\sigma-1}\Big)^2 = \Big(\frac{1 - x^{(\theta-1)(\sigma-1)}}{\sigma-1}\Big)^2 x^{2(\sigma-1)} \leq \frac{1}{(\sigma-1)^2}x^{2(\sigma-1)} \text{ if } \sigma < 1, \tag{3.5}$$

$$|k(\sigma+it)| \leq \frac{\Big(x^{\theta(\sigma-1)} + x^{\sigma-1}\Big)^2}{|s-1|^2} \leq k(\sigma)\Big(\frac{1 + x^{(\theta-1)(\sigma-1)}}{1 - x^{(\theta-1)(\sigma-1)}}\Big)^2 \frac{(\sigma-1)^2}{(\sigma-1)^2 + t^2}, \tag{3.6}$$

$$k(1) - k(\beta_1) = (\log x)^2 \phi_\theta((1-\beta_1)\log x), \text{ where } \phi_\theta(v) = (\theta-1)^2 - \Big(\frac{e^{-v} - e^{-\theta v}}{v}\Big)^2. \tag{3.7}$$

We shall require the following lemma regarding the properties of ϕ_θ.

Lemma 9. *Let $\theta > 1$. Then*

(i) *$\phi_\theta(v)$ is increasing for $v > 0$;*
(ii) *for any $v > 0$, we have $\phi_\theta(v) \geq (\theta-1)^2(1-e^{-2v})$.*
In particular, if $b > 0$, then for any $0 \leq v \leq b$, we have $\phi_\theta(v) \geq 2(\theta-1)^2 e^{-2b} v$.

Proof of Lemma 9. We first note that

$$\begin{aligned}\phi_\theta'(v) &= -2\Big(\frac{e^{-v} - e^{-\theta v}}{v}\Big)((-v^{-2})(e^{-v} - e^{-\theta v}) + (v^{-1})(-e^{-v} + \theta e^{-\theta v})) \\ &= 2\Big(\frac{e^{-v} - e^{-\theta v}}{v^2}\Big)\Big(e^{-v}\Big(1 + \frac{1}{v}\Big) - e^{-\theta v}\Big(\theta + \frac{1}{v}\Big)\Big).\end{aligned}$$

We claim $\phi_\theta'(v) \geq 0$ for $v \geq 0$ and so ϕ_θ is increasing. More precisely, letting $f(\theta) = e^{-\theta v}(\theta + \frac{1}{v})$, we claim $f(1) \geq f(\theta)$ whenever $\theta > 1$. This follows from $f'(\theta) = -ve^{-\theta v}(\theta + \frac{1}{v}) + e^{-\theta v} = -\theta v e^{-\theta v} \leq 0$.

Secondly, by the mean value theorem, for $\theta > 1$, we have

$$\Big|\frac{e^{-v} - e^{-\theta v}}{1-\theta}\Big| = \max_{t \in [1,\theta]} |ve^{-tv}| \leq ve^{-v},$$

and thus $\big(\frac{e^{-v} - e^{-\theta v}}{v}\big)^2 \leq (\theta-1)^2 e^{-2v}$. Hence, we obtain

$$\phi_\theta(v) = (\theta-1)^2 - \Big(\frac{e^{-v} - e^{-\theta v}}{v}\Big)^2 \geq (\theta-1)^2(1-e^{-2v}).$$

Lastly, for $c > 0$, letting $g_c(v) = (1-e^{-cv}) - cve^{-cb}$, we note that $g(v) \geq 0$ for all $v \in [0,b]$. Indeed, as $g_c(0) = 0$ and $g_c'(v) = ce^{-cv} - ce^{-cb} \geq 0$ for all $v \leq b$, we know that $g_c(v)$ is non-negative and increasing. Now, the last part of the lemma follows from using $g_c(v)$ with $c = 2$. $\square$

3.2. Explicit inequalities

Throughout our discussion (from Section 3.2 to Section 3.4), we let L/K be a Galois extension of number fields with Galois group G, and we let C be a conjugacy class in G. We let $\mathcal{P}(C)$ denote the set of the unramified primes $\mathfrak{p} \subset \mathcal{O}_K$ of degree one such that $\sigma_\mathfrak{p} = C$. In addition, we choose d_0 according to Table 2 and assume $\log d_L \geq \mathscr{L}_0 = \log d_0$. We let $x \geq 101$ and let k be a weight to be chosen later. We recall that β_1 denotes the possible exceptional zero, appearing in $(1 - \frac{1}{R_0\mathscr{L}}, 1)$, of $\zeta_L(s)$. By (2.3), we know that

there is at most one zero of $\zeta_L(s)$ in $(1-\frac{1}{R_1\mathscr{L}}, 1-\frac{1}{R_0\mathscr{L}}]$; we shall denote such a zero by β' if it exists.

Following [AhKw19-1] and [LaMoOd79], our goal is to show that there exists $c_4 > 0$ such that for $x = d_L^{c_4} = e^{c_4\mathscr{L}}$, we have

$$\mathcal{S}_C = \sum_{\mathfrak{p}\in\mathcal{P}(C)} (\log N\mathfrak{p})\hat{k}(N\mathfrak{p}) > 0, \tag{3.8}$$

which implies that there is a prime $\mathfrak{p} \in \mathcal{P}(C)$ such that

$$N\mathfrak{p} \le x^{2\theta} = d_L^{2\theta c_4}.$$

Thus we want to estimate

$$B = 2\theta c_4. \tag{3.9}$$

Proposition 10. *Let $k(s)$ be the kernel given as in (3.1), with $x = d_L^{c_4} \ge 101$, and $\mathcal{S}_C$ be the sum given as in (3.8). Then we have*

$$\begin{aligned}\frac{|G|}{|C|}\mathcal{S}_C \ge &(1-\delta(\beta_1))(\theta-1)^2(\log x)^2 + \delta(\beta_1)\phi_\theta((1-\beta_1)\log x)(\log x)^2 \\ &- (1-\delta(\beta_1))\delta(\beta')|k(\beta')| \\ &- \sum_{\rho\in\mathcal{S}\setminus\{\beta_1,\beta',1-\beta_1\}} |k(\rho)| - \alpha_3(\theta-1)\frac{|G|}{|C|}\mathscr{L}\frac{\log x}{x} \\ &- \delta(\beta_1)\frac{1}{(1-\frac{1}{2\mathscr{L}})^2}x^{-2(1-\frac{1}{2\mathscr{L}})} - c_5\mathscr{L}x^{-2},\end{aligned}$$

where ϕ_θ, α_3, and c_5 are defined in (3.7), (3.15), and (3.20), respectively. Here $\delta(\beta_1) = 1$ if the exceptional zero β_1 exists for $\zeta_L(s)$, and $\delta(\beta_1) = 0$ otherwise. $\delta(\beta')$ is defined the same way for β_1.

Proof. Let $g_C \in G$ be a representative of C, and let

$$\Psi_C(s) = -\frac{|C|}{|G|}\sum_\chi \overline{\chi}(g_C)\frac{L'}{L}(s,\chi,L/K), \tag{3.10}$$

where the sum is over the irreducible characters χ of $G = \mathrm{Gal}(L/K)$, and $L(s,\chi,L/K)$ is the Artin L-function attached to χ. It follows from the orthogonality property of irreducible characters of G that

$$I_C = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Psi_C(s)k(s)ds = \sum_{\mathfrak{p}}\sum_{m=1}^{\infty} J(\mathfrak{p}^m)(\log N\mathfrak{p})\hat{k}(N\mathfrak{p}^m), \tag{3.11}$$

where for ramified primes $\mathfrak{p}$, $|J(\mathfrak{p}^m)| \leq 1$, for unramified primes $\mathfrak{p}$, $J(\mathfrak{p}^m) = 1$ if $\sigma_{\mathfrak{p}}^m = C$, and $J(\mathfrak{p}^m) = 0$ otherwise. By (3.3), as argued in [AhKw19-1, Lemmata 3.1(1) and 3.3(1)] (see also [LaMoOd79, Lemmata 3.1-3.3]), for $x \geq 2$, one has

$$\sum_{\mathfrak{p} \text{ ramified}} \sum_{m \geq 1} J(\mathfrak{p}^m)(\log N\mathfrak{p})\hat{k}(N\mathfrak{p}^m) \leq 2(\theta - 1)\frac{\log x}{x^2} \log d_L. \tag{3.12}$$

Also, for $x \geq 101$, one has

$$\sum_{\mathfrak{p},m} J(\mathfrak{p}^m)(\log N\mathfrak{p})\hat{k}(N\mathfrak{p}^m) \leq 16.08\alpha_0(\theta - 1)\frac{\log x}{x} n_K, \text{ with } \alpha_0 = 1.25506, \tag{3.13}$$

where the sum is over $\mathfrak{p}, m$ such that $N\mathfrak{p}^m \neq p$ for any (rational) prime p. Hence, by Minkowski's bound, for $x \geq 101$, we have as in [AhKw19-1, Proposition 3.5]

$$|I_C - \mathcal{S}_C| \leq \frac{2(\theta - 1)\log x}{x^2} \log d_L + 16.08\alpha_0(\theta - 1)n_K \frac{\log x}{x} \leq \alpha_3(\theta - 1)\frac{\log x}{x} \log d_L, \tag{3.14}$$

where

$$\alpha_3 = \frac{2}{101} + \frac{32.16\alpha_0}{\log 3} = 36.7595\ldots. \tag{3.15}$$

The second step is to relate $\Psi_C(s)$ to the zeros of the Dedekind zeta function $\zeta_L(s)$. To do so, we recall that by Deuring's reduction [Deu35], denoting the fixed field of g_C by E, one has

$$\Psi_C(s) = -\frac{|C|}{|G|} \sum_{\psi} \overline{\psi}(g_C) \frac{L'}{L}(s, \psi, L/E),$$

where the sum is over the irreducible characters ψ of $\mathrm{Gal}(L/E)$. As L/E is abelian, it follows from Artin reciprocity that each Artin L-function $L(s, \psi, L/E)$ corresponds to a Hecke L-function. This allows us to use the classical method of contour integration to find a lower bound of I_C (in terms of $k(s)$ and the zeros of $\zeta_L(s)$). Indeed, one has

$$I_C = \frac{|C|}{|G|} \sum_{\psi} \overline{\psi}(g_C) \Big(\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} -\frac{L'}{L}(s, \psi, L/E) k(s) ds \Big).$$

Proceeding exactly as in [AhKw19-1, Sec. 4] and [LaMoOd79, Sec. 3] (see also [LaOd77]), an application of Cauchy's integral formula gives

$$\frac{|G|}{|C|} I_C \geq k(1) - \sum_{\rho \in \mathcal{S}} |k(\rho)| - n_L k(0) - \sum_{\psi} |V(\psi)|, \tag{3.16}$$

where

$$V(\psi) = \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} -\frac{L'}{L}(s, \psi, L/E)k(s)ds.$$

Denoting $A(\psi) = d_E N_{E/\mathbb{Q}}(\mathfrak{f}_\psi)$, where $\mathfrak{f}_\psi$ is the conductor of ψ, we appeal to a bound for the L-term of the form as given in [LaOd77, Lemma 6.2] and as given explicitly in [Das21, Lemma 2.19]:

$$\Big|\frac{L'}{L}(-\frac{1}{2}+it, \psi, L/E)\Big| \leq \log A(\psi)+n_E v(t), \text{ with } v(t) = \log(\sqrt{0.25+t^2}+1)+4.452+\frac{83}{5}, \tag{3.17}$$

with $4.452 + \frac{83}{5} = 21.052$. (Note this is an improvement on [Win13, Lemma 5.1] where instead $v(t) = \log(\sqrt{0.25+t^2}+2) + \frac{19683}{812}$.)

Using the bound (3.6) for k, we have for any $x \geq 101$,

$$\Big|k\Big(-\frac{1}{2}+it\Big)\Big| \leq k\Big(-\frac{1}{2}\Big)\Big(\frac{1+101^{-\frac{3}{2}(\theta-1)}}{1-101^{-\frac{3}{2}(\theta-1)}}\Big)^2 \frac{9}{9+4t^2}.$$

Hence, we deduce

$$|V(\psi)| \leq \frac{1}{2\pi}k\Big(-\frac{1}{2}\Big)\Big(\frac{1+101^{-\frac{3}{2}(\theta-1)}}{1-101^{-\frac{3}{2}(\theta-1)}}\Big)^2 \int_{-\infty}^{\infty} (\log A(\psi) + n_E v(t))\frac{9}{9+4t^2}dt.$$

Now, from (3.16) and the conductor-discriminant formula $\sum_\psi \log A(\psi) = \log d_L$,

$$\frac{|G|}{|C|}I_C \geq k(1) - \sum_{\rho\in\mathcal{S}} |k(\rho)| - W_0 k\Big(-\frac{1}{2}\Big)\log d_L - n_L\Big(k(0) + W_1 k\Big(-\frac{1}{2}\Big)\Big), \tag{3.18}$$

where, as before, $\mathcal{S}$ denotes the set of all non-trivial zeros of $\zeta_L(s)$,

$$W_0 = \frac{1}{\pi}\int_0^\infty W(t)dt, \;\; W_1 = \frac{1}{\pi}\int_0^\infty v(t)W(t)dt, \;\; W(t) = \Big(\frac{1+101^{-\frac{3}{2}(\theta-1)}}{1-101^{-\frac{3}{2}(\theta-1)}}\Big)^2 \frac{9}{9+4t^2}, \tag{3.19}$$

and $v(t)$ is defined in (3.17). Together with Minkowski's bound and (3.5), we deduce, for $x \geq 101$,

$$W_0 k\Big(-\frac{1}{2}\Big)\log d_L + n_L\Big(k(0) + W_1 k\Big(-\frac{1}{2}\Big)\Big) \leq c_5(\log d_L)x^{-2},$$

where

$$c_5 = \frac{2}{\log 3} + \frac{4}{909}\Big(W_0 + \frac{2}{\log 3}W_1\Big). \tag{3.20}$$

Hence, putting (3.14) and (3.18) together, if $\zeta_L(s)$ has no exceptional zero, we have

$$\frac{|G|}{|C|}\mathcal{S}_C \geq k(1) - \sum_{\rho\in\mathcal{S}} |k(\rho)| - \frac{c_5\mathscr{L}}{x^2} - \alpha_3(\theta-1)\frac{|G|}{|C|}\frac{\log x}{x}\mathscr{L}, \tag{3.21}$$

where c_5 and α_3 are defined in (3.20) and (3.15). We then use (3.4) to bound $k(1)$. Otherwise, if $\zeta_L(s)$ admits an exceptional zero β_1, the term $k(1) - k(\beta_1)$ appears. By (3.5) and (3.7), we know

$$k(1) - k(\beta_1) = (\log x)^2\phi_\theta((1-\beta_1)\log x) \text{ and } |k(1-\beta_1)| \leq \frac{x^{-2\beta_1}}{\beta_1^2} \leq \frac{x^{-2(1-\frac{1}{2\mathscr{L}})}}{(1-\frac{1}{2\mathscr{L}})^2}. \tag{3.22}$$

Thus, in the case the exceptional zero β_1 appears, we have

$$\begin{aligned}\frac{|G|}{|C|}\mathcal{S}_C \geq &(\log x)^2\phi_\theta((1-\beta_1)\log x) + \frac{1}{(1-\frac{1}{2\mathscr{L}})^2}x^{-2(1-\frac{1}{2\mathscr{L}})} \\ &- \sum_{\rho\in\mathcal{S}\setminus\{\beta_1,1-\beta_1\}} |k(\rho)| - \frac{c_5\mathscr{L}}{x^2} - \alpha_3(\theta-1)\frac{|G|}{|C|}\frac{\log x}{x}\mathscr{L}.\end{aligned} \tag{3.23}$$

Finally, writing the sum in (3.23) as

$$\sum_{\rho\in\mathcal{S}\setminus\{\beta_1,1-\beta_1\}} |k(\rho)| = (1-\delta(\beta_1))\delta(\beta')|k(\beta')| + \sum_{\rho\in\mathcal{S}\setminus\{\beta_1,\beta',1-\beta_1\}} |k(\rho)|,$$

we conclude the proof. □

We now focus on controlling the size of the above sums over the zeros. This is done by means of zero-free regions, Deuring-Heilbronn repulsion, and zero-density estimates for zeros of $\zeta_L(s)$. First, we have some explicit estimate for $N_L(T)$ which counts the number of non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta_L(s)$ such that $|\gamma| \leq T$. We refer to the recent improvement of [HaShWo]:

Lemma 11. *Let $T \geq 1$. Then*

$$\left|N_L(T) - \frac{T}{\pi}\log\left(d_L\left(\frac{T}{2\pi e}\right)^{n_L}\right)\right| \leq 0.296(\log d_L + n_L\log T) + 3.971n_L + 3.969. \tag{3.24}$$

In addition, we require a sharper estimate for the number of non-trivial zeros close to 1. Denoting $Z(r;s) = \{\rho\in\mathcal{S} \mid |\rho - s| \leq r\}$, we consider $n(r;s) = |Z(r;s)|$, the number of non-trivial zeros ρ of $\zeta_L(s)$ such that $|\rho - s| \leq r$.

Lemma 12. *Let $n_0 \geq 2, \mathscr{Q}_0 > 0, 0 < r \leq 1$, and $\alpha > 0$. If $n_0 \leq n_L \leq \mathscr{Q}_0\mathscr{L}$, then*

$$n(r;1) \leq \left(\frac{1+\alpha}{\alpha}\right)^2(1+\alpha r\omega(\alpha)\mathscr{L}), \tag{3.25}$$

where

$$\omega(\alpha) = \frac{1}{2} + \frac{\mathscr{Q}_0}{2} \max\Big\{\frac{\Gamma'}{\Gamma}\Big(\frac{2+\alpha}{2}\Big) - \log\pi + \frac{2}{n_0}, 0\Big\}. \tag{3.26}$$

Proof. We let $0 < r \le 1$ and $\alpha > 0$ and set $s_0 = 1 + \alpha r$. Putting together the classical explicit formula (see, e.g., [AhKw19-1, p. 1442])

$$\sum_{\rho\in\mathcal{S}} \mathfrak{Re}\frac{1}{s_0-\rho} = \frac{1}{2}\log d_L + \mathfrak{Re}\Big(\frac{1}{s_0} + \frac{1}{s_0-1}\Big) + \mathfrak{Re}\frac{\gamma_L'}{\gamma_L}(s_0) + \mathfrak{Re}\frac{\zeta_L'}{\zeta_L}(s_0), \tag{3.27}$$

with the facts that $\mathfrak{Re}\frac{\zeta_L'}{\zeta_L}(s_0) = \frac{\zeta_L'}{\zeta_L}(1+\alpha r) \le 0$, and

$$\frac{\gamma_L'}{\gamma_L}(s_0) = \frac{(r_1+r_2)}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{s_0}{2}\Big) + \frac{r_2}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{s_0+1}{2}\Big) - \frac{n_L}{2}\log\pi \le \frac{n_L}{2}\Big(\frac{\Gamma'}{\Gamma}\Big(\frac{2+\alpha}{2}\Big) - \log\pi\Big),$$

yields

$$\sum_{\rho\in\mathcal{S}} \mathfrak{Re}\frac{1}{s_0-\rho} \le \frac{1}{\alpha r} + \frac{1}{2}\log d_L + \frac{n_L}{2}\Big(\frac{\Gamma'}{\Gamma}\Big(\frac{2+\alpha}{2}\Big) - \log\pi\Big) + 1.$$

Note that

$$\begin{aligned}\sum_{\rho\in\mathcal{S}} \mathfrak{Re}\frac{1}{s_0-\rho} &\ge \sum_{\rho\in Z((1+\alpha)r;s_0)} \mathfrak{Re}\frac{1}{s_0-\rho} \ge \frac{\alpha r}{((1+\alpha)r)^2} n((1+\alpha)r; s_0)\\ &= \frac{\alpha}{(1+\alpha)^2 r} n((1+\alpha)r; s_0).\end{aligned}$$

Observing that $Z(r;1) \subseteq Z((1+\alpha)r; 1+\alpha r) = Z((1+\alpha)r; s_0)$, (3.25) follows from

$$\begin{aligned}n(r;1) &\le n((1+\alpha)r; s_0)\\ &\le \frac{(1+\alpha)^2 r}{\alpha}\Big(\frac{1}{\alpha r} + \frac{1}{2}\log d_L + \frac{n_L}{2}\Big(\frac{\Gamma'}{\Gamma}\Big(\frac{2+\alpha}{2}\Big) - \log\pi\Big) + 1\Big). \quad \square\end{aligned}$$

In addition, we require the following result on sums over the zeros close to 1.

Corollary 12.1. *Assume* (1.5), $\alpha > 0, r > 0$ *and that* $\frac{1}{r\mathscr{L}} \le 1$. *If* $n_0 \le n_L \le \mathscr{Q}_0\mathscr{L}$, *then*

$$\sum_{\frac{1}{r\mathscr{L}} \le |\rho-1| \le 1} \frac{1}{|\rho-1|^2} \le \Big(\Big(\frac{1+\alpha}{\alpha}\Big)^2 (r^2 + 2r\alpha\omega(\alpha)) - n\Big(\frac{1}{r\mathscr{L}}; 1\Big) r^2\Big)\mathscr{L}^2, \tag{3.28}$$

where $\omega(\alpha)$, *depending on* n_0 *and* $\mathscr{Q}_0$, *is defined as in* (3.26).

Proof. We start with

$$\sum_{\frac{1}{r\mathscr{L}}\le|\rho-1|\le 1}\frac{1}{|\rho-1|^2}=\int_{\frac{1}{r\mathscr{L}}}^{1}\frac{1}{u^2}dn(u;1)=n(1;1)-n\Big(\frac{1}{r\mathscr{L}};1\Big)(r\mathscr{L})^2+2\int_{\frac{1}{r\mathscr{L}}}^{1}\frac{n(u;1)}{u^3}du. \tag{3.29}$$

It follows from Lemma 12 that

$$n(1;1)\le\Big(\frac{1+\alpha}{\alpha}\Big)^2(\alpha\omega(\alpha)\mathscr{L}+1),\text{and}$$
$$\int_{\frac{1}{r\mathscr{L}}}^{1}\frac{1+\alpha u\omega(\alpha)\mathscr{L}}{u^3}du=\Big(\frac{r^2}{2}+\alpha\omega(\alpha)r\Big)\mathscr{L}^2-\alpha\omega(\alpha)\mathscr{L}-\frac{1}{2}. \tag{3.30}$$

We conclude by putting together (3.29) and (3.30). □

We now control non-trivial zeros $\rho=\beta+i\gamma$ of $\zeta_L(s)$ that are not equal to β_1, β', or $1-\beta_1$.

Proposition 13. *Let $k(s)$ be the kernel given as in (3.1), with $x=d_L^{c_4}$, and $\mathcal{S}_C$ be the sum given as in (3.8). Let R_1 be defined as in (2.3). If $n_0\le n_L\le\mathscr{Q}_0\mathscr{L}$, then*

$$\begin{aligned}\frac{|G|}{|C|}\mathcal{S}_C\ge&(1-\delta(\beta_1))(\theta-1)^2c_4^2\mathscr{L}^2+\delta(\beta_1)\phi_\theta\Big(c_4(1-\beta_1)\mathscr{L}\Big)c_4^2\mathscr{L}^2\\&-(1-\delta(\beta_1))\delta(\beta')|k(\beta')|-c_6\mathscr{L}-\sideset{}{'}\sum_{\frac{1}{R_1\mathscr{L}}\le|\rho-1|\le 1}|k(\rho)|\\&-\mathscr{Q}_0\alpha_3(\theta-1)c_4\mathscr{L}^3e^{-c_4\mathscr{L}}-\delta(\beta_1)\frac{1}{(1-\frac{1}{2\mathscr{L}})^2}e^{-c_4(2\mathscr{L}-1)}-c_5\mathscr{L}e^{-2c_4\mathscr{L}},\end{aligned}$$

where $\delta(\beta_1)=1$ (resp., $\delta(\beta')=1$) if the exceptional zero β_1 (resp., real zero β') exists for $\zeta_L(s)$, $\delta(\beta_1)=0$ (resp., $\delta(\beta')=0$) otherwise, ϕ_θ, α_3, c_5, and c_6 are defined in (3.7), (3.15), (3.20), and (3.33), respectively, and, as later, the primed sum is over non-trivial zeros $\rho\ne 1-\beta_1$ of $\zeta_L(s)$.

Proof. We shall note that if $\rho=\beta+i\gamma$ is such that $|\rho-1|<\frac{1}{R_1\mathscr{L}}$, then $1-\beta<\frac{1}{R_1\mathscr{L}}$ and $|\gamma|<\frac{1}{R_1\mathscr{L}}$. Hence, (2.3) forces that ρ is either β_1 or β' if $|\rho-1|<\frac{1}{R_1\mathscr{L}}$. Thus, we can consider the splitting

$$\sum_{\rho\in\mathcal{S}\setminus\{\beta_1,\beta',1-\beta_1\}}|k(\rho)|=\sideset{}{'}\sum_{\frac{1}{R_1\mathscr{L}}\le|\rho-1|\le 1}|k(\rho)|+\sum_{|\rho-1|>1}|k(\rho)|. \tag{3.31}$$

To bound $\sum_{|\rho-1|>1}|k(\rho)|$, we appeal to (3.6) to bound k:

$$|k(\rho)| \leq \frac{(x^{\theta(\beta-1)} + x^{\beta-1})^2}{|\rho-1|^2} \leq \frac{4}{|\rho-1|^2}. \tag{3.32}$$

Recalling that $n(r;1)$ denotes the number of non-trivial zeros ρ of $\zeta_L(s)$ such that $|\rho-1| \leq r$, we notice that $n(r;1) \leq N_L(r)$. Using (3.24), we derive

$$\sum_{|\rho-1|>1} |k(\rho)| \leq 4 \int_1^\infty \frac{1}{r^2} dn(r;1) \leq 8 \int_1^\infty \frac{n(r;1)}{r^3} dr \leq c_6 \mathscr{L},$$

where

$$c_6 = 8\Big(\frac{1}{\pi} + 0.148 + \mathscr{Q}_0 \max\Big\{0, \int_1^\infty \frac{\frac{r}{\pi}\log(\frac{r}{2\pi e}) + 0.296\log r + 3.971 + \frac{3.969}{n_0}}{r^3} dr\Big\}\Big). \tag{3.33}$$

We conclude by putting together Proposition 10 with the above and the facts that $\frac{|G|}{|C|} \leq n_L$ and $x = e^{c_4 \mathscr{L}}$. $\square$

Now, it remains to bound $\sum'_{\frac{1}{R_1\mathscr{L}} \leq |\rho-1| \leq 1} |k(\rho)|$. We shall proceed with the argument by considering cases depending on the existence of the exceptional zero β_1 and the distance of the exceptional zero β_1, if it exists, to the 1-line.

3.3. Non-exceptional case

Proposition 14. *Let $\mathcal{S}_C$ be the sum given as in (3.8). Assume there is no exceptional zero β_1 in $(1 - \frac{1}{R_0\mathscr{L}}, 1)$. If $n_0 \leq n_L \leq \mathscr{Q}_0\mathscr{L}$ and $\mathscr{L} \geq \mathscr{L}_0$, then*

$$\mathscr{L}^{-2}\frac{|G|}{|C|}\mathcal{S}_C \geq (\theta-1)^2 c_4^2 - R_0^2 e^{-\frac{2c_4}{R_0}} - c_7(c_4) e^{\frac{-2c_4}{29.57(1+\Delta_0(1))}} - \mathcal{E}_0(c_4), \tag{3.34}$$

where

$$\mathcal{E}_0(t) = \frac{c_6}{\mathscr{L}_0} + \mathscr{Q}_0\alpha_3(\theta-1)t\mathscr{L}_0 e^{-t\mathscr{L}_0} + \frac{c_5}{\mathscr{L}_0} e^{-2t\mathscr{L}_0}, \tag{3.35}$$

and c_5, c_6, and c_7 are defined as in (3.20), (3.33), and (3.37), respectively.

Proof. We begin by recalling that $\beta' \leq 1 - \frac{1}{R_0\mathscr{L}}$ (if exists) and that for any non-trivial zero $\rho = \beta + i\gamma \neq \beta'$, with $|\gamma| \leq 1$, the zero-free region [AhKw19-1, Proposition 6.1] gives

$$1 - \beta > (29.57\log(d_L\tau^{n_L}))^{-1} \geq (29.57\mathscr{L}(1+\delta_L(3)))^{-1} \geq (29.57\mathscr{L}(1+\Delta_0(1)))^{-1},$$

where δ_L and Δ_0 are defined in (1.6) and (1.7), respectively, and the last inequality is due to (1.9). Therefore, by (3.5), we have

$$|k(\beta')| \leq \frac{1}{(\beta'-1)^2} x^{2(\beta'-1)} \leq (R_0\mathscr{L})^2 x^{-\frac{2}{R_0\mathscr{L}}},$$

and by using the first bound of (3.6), we have for the other zeros that

$$\begin{aligned}|k(\rho)| &\leq \frac{x^{2(\beta-1)}(1+x^{(\theta-1)(\beta-1)})^2}{|\rho-1|^2} \\ &\leq \frac{x^{-2(29.57\mathscr{L}(1+\Delta_0(1)))^{-1}}(1+x^{(1-\theta)(29.57\mathscr{L}(1+\Delta_0(1)))^{-1}})^2}{|\rho-1|^2}.\end{aligned}$$

By Corollary 12.1, putting $x = e^{c_4\mathscr{L}}$, we then derive

$$(1-\delta(\beta_1))\delta(\beta')|k(\beta')| + \sideset{}{'}\sum_{\frac{1}{2\mathscr{L}} \leq |\rho-1| \leq 1} |k(\rho)| \leq (R_0\mathscr{L})^2 e^{-\frac{2c_4}{R_0}} + c_7(c_4)\mathscr{L}^2 e^{\frac{-2c_4}{29.57(1+\Delta_0(1))}}, \tag{3.36}$$

where

$$c_7(t) = 4(1+\alpha\omega(\alpha))\Big(\frac{1+\alpha}{\alpha}\Big)^2\Big(1+e^{\frac{(1-\theta)t}{29.57(1+\Delta_0(1))}}\Big)^2, \tag{3.37}$$

and $\omega(\alpha)$ is defined as in (3.26). For $n_L \leq \mathscr{Q}_0\mathscr{L}$, we apply Proposition 13 with $R_1 = 2$ and (3.36) to obtain

$$\begin{aligned}\frac{|G|}{|C|}\mathcal{S}_C \geq &(\theta-1)^2 c_4^2\mathscr{L}^2 - c_6\mathscr{L} - (R_0\mathscr{L})^2 e^{-\frac{2c_4}{R_0}} - c_7(c_4)\mathscr{L}^2 e^{\frac{-2c_4}{29.57(1+\Delta_0(1))}} \\ &- \mathscr{Q}_0\alpha_3(\theta-1)c_4\mathscr{L}^3 e^{-c_4\mathscr{L}} - c_5\mathscr{L}e^{-2c_4\mathscr{L}}.\end{aligned}$$

We conclude with the fact that $\mathscr{L} \geq \mathscr{L}_0$. □

3.4. Exceptional case

Assume that the exceptional zero β_1 of $\zeta_L(s)$ presents such that $\beta_1 \geq 1 - \frac{1}{R_0\mathscr{L}}$. Following Proposition 6, we take $R_0 = 20$ and $R_1 = 1.24$. We recall $R = 29.57$. We let $\varepsilon_1 > 0$, $\sigma_1 \geq 1$, and $\eta \in (0,1]$ be parameters (to be chosen later) to compute $c_1'(\varepsilon_1,\sigma_1,\eta)$ and $c_2'(\varepsilon_1,\sigma_1,\eta)$ defined in (2.28). These parameters will be chosen to make c_4 and thus B as small as possible.

Proposition 15. *Let $\mathcal{S}_C$ be the sum given as in (3.8). Assume that the exceptional zero β_1 of $\zeta_L(s)$ exists, that is $\beta_1 \geq 1 - \frac{1}{R_0\mathscr{L}}$. Let $\eta \in (0,1]$ satisfying*

$$\eta \geq c_2'(\varepsilon_1,\sigma_1,\eta)\frac{\log\left(\frac{c_1'(\varepsilon_1,\sigma_1,\eta)}{(1-\beta_1)\mathscr{L}}\right)}{\mathscr{L}} \tag{3.38}$$

(with $c_1'(\varepsilon_1,\sigma_1,\eta)$ and $c_2'(\varepsilon_1,\sigma_1,\eta)$ defined in (2.28)). If $n_0 \leq n_L \leq \mathscr{Q}_0\mathscr{L}$ and $\mathscr{L} \geq \mathscr{L}_0$, then

$$\begin{aligned}\mathscr{L}^{-2}\frac{|G|}{|C|}\mathcal{S}_C \geq & \phi_\theta((1-\beta_1)c_4\mathscr{L})c_4^2 - c_{11}((1-\beta_1)\mathscr{L})^{2c_4c_8} - \frac{c_6}{\mathscr{L}} \\ & - \mathscr{Q}_0\alpha_3(\theta-1)c_4\mathscr{L}e^{-c_4\mathscr{L}} - \frac{1}{(\mathscr{L}-\frac{1}{2})^2}e^{-c_4(2\mathscr{L}-1)} - \frac{c_5}{\mathscr{L}}e^{-2c_4\mathscr{L}},\end{aligned} \tag{3.39}$$

where α_3, c_5, c_6, c_8, and c_{11} are defined in (3.15), (3.20), (3.33), (3.42), *and* (3.45), *respectively.*

Proof. We start by estimating the sum over low-lying zeros $\sum'_{\frac{1}{R_1\mathscr{L}}\leq|\rho-1|\leq 1}|k(\rho)|$. Let $\rho=\beta+it$ denote a non-exceptional zero of $\zeta_L(s)$ such that $\frac{1}{R_1\mathscr{L}}\leq|\rho-1|\leq 1$. We shall consider the following two situations:

(a) We first suppose that $(1-\beta_1)\mathscr{L}\leq c_1'(\varepsilon_1,\sigma_1,\eta)^a$. Applying Theorem 2, we have either $\beta\leq 1-\eta$ or

$$1-\beta\geq c_2'(\varepsilon_1,\sigma_1,\eta)\frac{\log\left(\frac{c_1'(\varepsilon_1,\sigma_1,\eta)}{(1-\beta_1)\mathscr{L}}\right)}{\mathscr{L}}\geq c_9\frac{\log\left(\frac{1}{(1-\beta_1)\mathscr{L}}\right)}{\mathscr{L}},$$

where

$$c_9=c_2'(\varepsilon_1,\sigma_1,\eta)\Big(1-\frac{1}{a}\Big).$$

(b) Secondly, we consider the situation that $(1-\beta_1)\mathscr{L}>c_1'(\varepsilon_1,\sigma_1,\eta)^a$. By [AhKw19-1, Proposition 6.1] and (1.9), we know that

$$1-\beta>(29.57\log(d_L\tau^{n_L}))^{-1}\geq(29.57\mathscr{L}(1+\Delta_0(1)))^{-1}$$

as $\tau=|t|+2$ and $|t|\leq 1$. Thus, by choosing

$$c_{10}=\frac{1}{a\cdot 29.57(1+\Delta_0(1))\log(1/c_1'(\varepsilon_1,\sigma_1,\eta))},$$

we have

$$1-\beta\geq c_{10}\frac{1}{\mathscr{L}}\log\left(\frac{1}{c_1'(\varepsilon_1,\sigma_1,\eta)^a}\right)\geq c_{10}\frac{\log\left(\frac{1}{(1-\beta_1)\mathscr{L}}\right)}{\mathscr{L}}.$$

To summarise, for any non-exceptional zero $\rho=\beta+it$ such that $\frac{1}{R_1\mathscr{L}}\leq|\rho-1|\leq 1$, assuming (3.38), we have

$$1-\beta\geq D:=c_8\frac{\log\left(\frac{1}{(1-\beta_1)\mathscr{L}}\right)}{\mathscr{L}},\ \text{with } c_8=\min\{c_9,c_{10}\}. \tag{3.40}$$

We chose a such that $c_9=c_{10}$, i.e.

$$a = 1 + \frac{1}{29.57(1+\Delta_0(1))c_2'(\varepsilon_1,\sigma_1,\eta)\log(1/c_1'(\varepsilon_1,\sigma_1,\eta))}. \tag{3.41}$$

Thus, c_8 is given by

$$c_8 = \frac{1}{1/c_2'(\varepsilon_1,\sigma_1,\eta) + 29.57(1+\Delta_0(1))\log(1/c_1'(\varepsilon_1,\sigma_1,\eta))}. \tag{3.42}$$

Applying the bounds (3.6) on k with $1-\beta \geq D$ gives

$$|k(\rho)| \leq \frac{x^{2(\beta-1)}(1+x^{(\theta-1)(\beta-1)})^2}{|\rho-1|^2} \leq \frac{x^{-2D}(1+x^{-(\theta-1)D})^2}{|\rho-1|^2}. \tag{3.43}$$

Recalling that we set $x = d_L^{c_4}$, and by the definition of the exceptional zero with respect to the zero-free region (2.2), that $D \geq c_8 \frac{\log R_0}{\mathscr{L}}$, we have

$$x^{-2D} = ((1-\beta_1)\mathscr{L})^{2c_4c_8} \text{ and } (1+x^{-(\theta-1)D})^2 \leq \left(1+e^{-(\theta-1)c_4c_8\log R_0}\right)^2.$$

We apply Corollary 12.1 with $n(\frac{1}{R_1\mathscr{L}};1) = 1$:

$$\sum_{\frac{1}{R_1\mathscr{L}} \leq |\rho-1| \leq 1}{}' \frac{1}{|\rho-1|^2} \leq \left(\left(\frac{1+\alpha}{\alpha}\right)^2 (R_1^2 + 2R_1\alpha\omega(\alpha)) - R_1^2\right)\mathscr{L}^2,$$

which together with (3.43), gives

$$\sum_{\frac{1}{R_1\mathscr{L}} \leq |\rho-1| \leq 1}{}' |k(\rho)| \leq c_{11}\mathscr{L}^2((1-\beta_1)\mathscr{L})^{2c_4c_8}, \tag{3.44}$$

with

$$c_{11} = \left(1+e^{-(\theta-1)c_4c_8\log R_0}\right)^2 R_1\left(\left(\frac{1+\alpha}{\alpha}\right)^2 (R_1 + 2\alpha\omega(\alpha)) - R_1\right), \tag{3.45}$$

where $\omega(\alpha)$, depending on $\mathscr{Q}_0$ and n_0, is defined in (3.26). Note that c_9, c_{10}, and c_{11} all depend on ε_1, σ_1, and η. We conclude by using Proposition 13 and (3.44). □

For the rest of the article, $n_0 \leq n_L \leq \mathscr{Q}_0\mathscr{L}$ and $\mathscr{L} \geq \mathscr{L}_0$, with $\mathscr{L}_0$ and $\mathscr{Q}_0$ as defined in (1.4) and (1.8), respectively. For each (n_0, d_0) listed in Table 2, we choose $\varepsilon_1 > 0, \sigma_1 \geq 1$, and $\eta \in (0,1]$ which give $c_1'(\varepsilon_1,\sigma_1,\eta)$ and $c_2'(\varepsilon_1,\sigma_1,\eta)$ as defined in (2.28). We then obtain c_8 as defined in (3.42), and define

$$c_4 = \frac{1}{2c_8} + 0.001. \tag{3.46}$$

To prove Theorem 1, we shall split our consideration depending on the distance of β_1 to 1-line. We shall introduce further parameters:

$$\varepsilon_2 > 0,\ \sigma_2 \geq 1,\ \kappa \geq 1,\ 0 < \lambda \leq 1,\ 0 < \mu \leq 1,\ \text{and } 1 < \nu \leq 2.$$

For the rest of the article, we denote

$$C_1 = c_1'(\varepsilon_2, \sigma_2, 0.5) \text{ and } C_2 = c_2'(\varepsilon_2, \sigma_2, 0.5), \tag{3.47}$$

where c_1' and c_2' are defined in (2.28). We assume

$$\frac{(\kappa C_1)^2}{\mathscr{L}} < \frac{\mu}{c_4 \mathscr{L}^{\nu-1}} < \frac{\lambda}{c_4} < \frac{1}{R_0}.$$

We explore the cases:

- "medium" when $\frac{\lambda}{c_4} \leq (1-\beta_1)\mathscr{L} < \frac{1}{R_0}$,
- "small" when $\frac{\mu}{c_4 \mathscr{L}^{\nu-1}} \leq (1-\beta_1)\mathscr{L} \leq \frac{\lambda}{c_4}$,
- "very small" when $\frac{(\kappa C_1)^2}{\mathscr{L}} \leq (1-\beta_1)\mathscr{L} \leq \frac{\mu}{c_4 \mathscr{L}^{\nu-1}}$, and
- "extremely small" when $(1-\beta_1)\mathscr{L} < \frac{(\kappa C_1)^2}{\mathscr{L}}$.

For the "extremely small" case, we will use a different weight k and thus leave this for the end.

In each of the first three cases, we use the weight k_θ as defined in (3.1), and obtain $B = 2\theta c_4$ as given by (3.9). The choice of parameters will ensure B to be as small as possible while keeping the expression in the right of (3.39) positive. We note that if $(1-\beta_1)\mathscr{L} \geq \frac{(\kappa C_1)^2}{\mathscr{L}}$, then the condition (3.38), which is needed to apply Proposition 15, is satisfied when

$$\frac{c_2'(\varepsilon_1, \sigma_1, \eta)}{\mathscr{L}_0}\Big(\max\Big\{\log\Big(\frac{c_1'(\varepsilon_1, \sigma_1, \eta)}{(\kappa C_1)^2}\Big), 0\Big\} + \log \mathscr{L}_0\Big) \leq \eta. \tag{3.48}$$

3.4.1. "Medium" case

Assume $\frac{\lambda}{c_4} \leq (1-\beta_1)\mathscr{L} < \frac{1}{R_0}$. Lemma 9(i) yields

$$\phi_\theta((1-\beta_1)c_4\mathscr{L})c_4^2 - c_{11}((1-\beta_1)\mathscr{L})^{2c_4c_8} \geq \phi_\theta(\lambda)c_4^2 - c_{11}\big((1-\beta_1)\mathscr{L}\big)^{2c_4c_8}. \tag{3.49}$$

Hence, Proposition 15 gives

$$\mathscr{L}^{-2}\frac{|G|}{|C|}\mathcal{S}_C \geq \phi_\theta(\lambda)c_4^2 - c_{11}R_0^{-2c_4c_8} - \mathcal{E}_1(c_4), \tag{3.50}$$

where c_{11} is defined in (3.45), and $\mathcal{E}_1(t)$ is defined by

$$\mathcal{E}_1(t) = \frac{c_6}{\mathscr{L}_0} + \mathscr{Q}_0\alpha_3(\theta-1)\mathscr{L}_0 t e^{-t\mathscr{L}_0} + \frac{1}{(\mathscr{L}_0 - \frac{1}{2})^2}e^{-t(2\mathscr{L}_0-1)} + \frac{c_5}{\mathscr{L}_0}e^{-2t\mathscr{L}_0}. \tag{3.51}$$

3.4.2. "Small" case

Assume $\frac{\mu}{c_4\mathscr{L}^{\nu-1}} \le (1-\beta_1)\mathscr{L} \le \frac{\lambda}{c_4}$. As $(1-\beta_1)\mathscr{L} \le \frac{\lambda}{c_4}$ and $2c_4c_8 > 1$, Lemma 9(ii) yields

$$\phi_\theta((1-\beta_1)c_4\mathscr{L})c_4^2 - c_{11}((1-\beta_1)\mathscr{L})^{2c_4c_8} \ge 2(\theta-1)^2e^{-2\lambda}c_4^3(1-\beta_1)\mathscr{L} - c_{11}\Big(\frac{\lambda}{c_4}\Big)^{2c_4c_8-1}((1-\beta_1)\mathscr{L}).$$

Hence, by Proposition 15, we have

$$((1-\beta_1)\mathscr{L})^{-1}\mathscr{L}^{-2}\frac{|G|}{|C|}\mathcal{S}_C \ge 2(\theta-1)^2e^{-2\lambda}c_4^3 - c_{11}\Big(\frac{\lambda}{c_4}\Big)^{2c_4c_8-1} - c_6\frac{c_4\mathscr{L}_0^{\nu-2}}{\mu} - \mathcal{E}_2(c_4), \tag{3.52}$$

with

$$\mathcal{E}_2(t) = \Big(\mathscr{Q}_0\alpha_3(\theta-1)\mathscr{L}_0^2c_4e^{-t\mathscr{L}_0} + \frac{1}{\mathscr{L}_0(1-\frac{1}{2\mathscr{L}_0})^2}e^{-t(2\mathscr{L}_0-1)} + c_5e^{-2t\mathscr{L}_0}\Big)\frac{t\mathscr{L}_0^{\nu-2}}{\mu}. \tag{3.53}$$

3.4.3. "Very small" case

Assume $\frac{(\kappa C_1)^2}{\mathscr{L}} \le (1-\beta_1)\mathscr{L} \le \frac{\mu}{c_4\mathscr{L}^{\nu-1}}$. Then Lemma 9(ii) yields

$$\phi_\theta((1-\beta_1)c_4\mathscr{L})c_4^2 - c_{11}((1-\beta_1)\mathscr{L})^{2c_4c_8} \ge 2(\theta-1)^2e^{-\frac{2\mu}{\mathscr{L}_0^{\nu-1}}}c_4^3(1-\beta_1)\mathscr{L} - c_{11}\Big(\frac{\mu}{c_4\mathscr{L}_0^{\nu-1}}\Big)^{2c_4c_8-1}(1-\beta_1)\mathscr{L},$$

which, combined with Proposition 15, yields

$$((1-\beta_1)\mathscr{L})^{-1}\mathscr{L}^{-2}\frac{|G|}{|C|}\mathcal{S}_C \ge 2(\theta-1)^2e^{-\frac{2\mu}{\mathscr{L}_0^{\nu-1}}}c_4^3 - c_{11}\Big(\frac{\mu}{c_4\mathscr{L}_0^{\nu-1}}\Big)^{2c_4c_8-1} - \frac{c_6}{(\kappa C_1)^2} - \mathcal{E}_3(c_4), \tag{3.54}$$

with

$$\mathcal{E}_3(t) = \Big(\mathscr{Q}_0\alpha_3(\theta-1)\mathscr{L}_0^2te^{-t\mathscr{L}_0} + \frac{1}{\mathscr{L}_0(1-\frac{1}{2\mathscr{L}_0})^2}e^{-t(2\mathscr{L}_0-1)} + c_5e^{-2t\mathscr{L}_0}\Big)\frac{1}{(\kappa C_1)^2}. \tag{3.55}$$

3.4.4. "Extremely small" case

We assume $1-\beta_1 < (\kappa C_1)^2\mathscr{L}^{-2}$. Here we use the weight as introduced by [LaMoOd79], and used by [AhKw19-1]:

$$k(s) = x^{s^2+s},$$

with $x = d_L^{c_{12}} \geq 10^{10}$. Thus, here B is given by[2]

$$B = 5c_{12}.$$

First, we have the following explicit inequality from combining [AhKw19-1, (3.2), Proposition 4.7, line 9 of p. 1453, and line -1 of p. 1452]:

$$\frac{|G|}{|C|} \sum_{\mathfrak{p}\in\mathcal{P}(C), N\mathfrak{p}\leq x^5} (\log N\mathfrak{p})\hat{k}(N\mathfrak{p}) \geq k(1) - k(\beta_1) - \sum_{|\gamma|\leq 1} |k(\rho)| \\ - 19.17x\mathscr{L} - 1.8292\mathscr{L} - 5.4568x(\log x)^{\frac{1}{2}} n_L \mathscr{L}, \tag{3.56}$$

where, as later, the sum on the right is over non-trivial zeros $\rho = \beta + i\gamma \neq \beta_1$ of $\zeta_L(s)$.[3]

We now emphasize how to improve the estimate for $k(1) - k(\beta_1)$ as well as the contribution of the low-lying zeros of $\zeta_L(s)$. As β_1 is the closest to 1, we use the mean value theorem to deduce

$$\begin{aligned} k(1) - k(\beta_1) &\geq (1-\beta_1)(2\beta_1+1)(\log x)x^{\beta_1^2+\beta_1} \\ &= c_{12}(1-\beta_1)\mathscr{L}(2\beta_1+1)e^{2c_{12}\mathscr{L}} e^{c_{12}(\beta_1^2+\beta_1-2)\mathscr{L}}. \end{aligned}$$

Since

$$2\beta_1 + 1 \geq 3 - \frac{2(\kappa C_1)^2}{\mathscr{L}_0^2} \text{ and } \beta_1^2 + \beta_1 - 2 = (\beta_1+2)(\beta_1-1) \geq -\frac{3(\kappa C_1)^2}{\mathscr{L}^2},$$

we have

$$k(1) - k(\beta_1) \geq \phi_0(c_{12})\Big(\mathscr{L}(1-\beta_1)e^{2c_{12}\mathscr{L}}\Big)c_{12}, \tag{3.57}$$

where

$$\phi_0(t) = e^{-\frac{3(\kappa C_1)^2}{\mathscr{L}_0}t}\Big(3 - \frac{2(\kappa C_1)^2}{\mathscr{L}_0^2}\Big). \tag{3.58}$$

We split the sum over the zeros $\rho = \beta + i\gamma \neq \beta_1$, with $|\gamma| \leq 1$, into $\Sigma = \Sigma_1 + \Sigma_2$, where

$$\sum\nolimits_1 = \sum_{\beta<1/2} + \frac{1}{2}\sum_{\beta=1/2} \text{ and } \sum\nolimits_2 = \frac{1}{2}\sum_{\beta=1/2} + \sum_{\beta>1/2}.$$

[2] We observe that the power 5 seems to be the smallest real number in order to control the sum over the higher power norm primes. Namely in [AhKw19-2, Lemma 3.4] we can see that the bound for the sum over the primes satisfying $N\mathfrak{p} > x^a$ is given by the integral $\int_{(a-1)(\log x)}^{\infty} \left(\frac{(t+\log x)t}{2\log x} - 1\right)\frac{\exp\left(t - \frac{t^2}{4\log x}\right)}{\sqrt{4\pi\log x}}dt$. So for the very least, one needs a non-positive power in the exponent term, which means $1 - \frac{(a-1)}{4} \leq 0$, giving $a \geq 5$.

[3] The values appearing in (3.56) are respectively called $c_{20}, c'_{15}, \alpha_4$ in [AhKw19-1].

We use the bound $|k(\rho)| \le x^{\beta^2+\beta} \le x^{3/4}$ in the first part, together with the bound $N_L(1) \le c_{13}\mathscr{L}$, as given in (3.24), where

$$c_{13} = \frac{1}{\pi} + 0.296 + \mathscr{Q}_0\Big(3.971 - \frac{\log(2\pi e)}{\pi}\Big) + \frac{3.969}{\mathscr{L}_0}. \tag{3.59}$$

Thus, $\Sigma_1 \le \frac{N_L(1)}{2} x^{\frac{3}{4}} \le \frac{c_{13}}{2}\mathscr{L} x^{\frac{3}{4}}$. Now, we assume $c_{12} > \frac{4}{5}c_3$ and deduce

$$\sum\nolimits_1 \le \frac{c_{13}}{2} e^{(c_3 - \frac{5}{4}c_{12})\mathscr{L}}\Big((1-\beta_1)\mathscr{L}e^{2c_{12}\mathscr{L}}\Big) \le \frac{c_{13}}{2} e^{-(\frac{5}{4}c_{12} - c_3)\mathscr{L}_0}\Big((1-\beta_1)\mathscr{L}e^{2c_{12}\mathscr{L}}\Big) \tag{3.60}$$

since $\mathscr{L}x^{\frac{3}{4}} = \big(x^2\mathscr{L}(1-\beta_1)\big)\frac{x^{-5/4}}{(1-\beta_1)}$, where $x = e^{c_{12}\mathscr{L}}$, and $1-\beta_1 \ge d_L^{-c_3} = e^{-c_3\mathscr{L}}$ according to Theorem 4.

For Σ_2, we apply $\beta^2 \le 2\beta = 2 - 2(1-\beta)$ so that

$$|k(\rho)| \le \frac{x^{-2(1-\beta)}}{(1-\beta_1)\mathscr{L}}\Big((1-\beta_1)\mathscr{L}x^2\Big) = \frac{e^{-2c_{12}(1-\beta)\mathscr{L}}}{(1-\beta_1)\mathscr{L}}\Big((1-\beta_1)\mathscr{L}e^{2c_{12}\mathscr{L}}\Big). \tag{3.61}$$

Together with Deuring-Heilbronn phenomenon, as described in Theorem 2:

$$(1-\beta)\mathscr{L} \ge C_2 \log C_1 - C_2 \log\big((1-\beta_1)\mathscr{L}\big),$$

we obtain

$$e^{-2c_{12}(1-\beta)\mathscr{L}} \le C_1^{-2c_{12}C_2}((1-\beta_1)\mathscr{L})^{2c_{12}C_2}. \tag{3.62}$$

Combining (3.61) and (3.62) with $(1-\beta_1)\mathscr{L} < (\kappa C_1)^2\mathscr{L}^{-1}$, we have

$$|k(\rho)| \le \frac{C_1^{2c_{12}C_2-2}\kappa^{4c_{12}C_2-2}}{\mathscr{L}^{2c_{12}C_2-1}}\Big((1-\beta_1)\mathscr{L}e^{2c_{12}\mathscr{L}}\Big) \tag{3.63}$$

provided that $c_{12} > \frac{1}{2C_2}$. Thus, we derive

$$\sum\nolimits_2 \le \frac{c_{13}}{2}\kappa^2\Big(\frac{\kappa^2 C_1}{\mathscr{L}_0}\Big)^{2c_{12}C_2-2}\Big((1-\beta_1)\mathscr{L}e^{2c_{12}\mathscr{L}}\Big), \tag{3.64}$$

as long as $c_{12} > \frac{1}{C_2}$. Combining (3.56), (3.57), (3.60), and (3.64) finally gives that for

$$c_{12} > \max\Big(\frac{1}{C_2}, c_3\Big),$$

$$\Big((1-\beta_1)\mathscr{L}e^{2c_{12}\mathscr{L}}\Big)^{-1}\frac{|G|}{|C|}\sum_{\mathfrak{p}\in\mathcal{P}(C), N\mathfrak{p}\le d_L^{5c_{12}}}(\log N\mathfrak{p})\hat{k}(N\mathfrak{p})$$
$$\ge \phi_0(c_{12})c_{12} - \frac{c_{13}\kappa^2}{2}\Big(\frac{\kappa^2 C_1}{\mathscr{L}_0}\Big)^{2c_{12}C_2-2} - \mathcal{E}_4(c_{12}), \tag{3.65}$$

where

$$\mathcal{E}_4(t) = \frac{c_{13}}{2} e^{-(\frac{5}{4}t - c_3)\mathscr{L}_0} + \left(19.17 + 5.4568\mathscr{Q}_0\mathscr{L}_0^{\frac{3}{2}} t^{1/2}\right) e^{-(t-c_3)\mathscr{L}_0} + 1.8292 e^{-(2t-c_3)\mathscr{L}_0}. \tag{3.66}$$

3.5. Numerical results

We choose ε_1, σ_1, and η to make c_4 as small as possible while (3.50), (3.52), (3.54), and (3.65) are satisfied.

Note that the choice for ν and η are balanced in the inequalities (3.52) and (3.54) as $\frac{c_4 \mathscr{L}_0^{\nu-2}}{\mu}$ and $\frac{\mu}{c_4 \mathscr{L}_0^{\nu-1}}$ both have to be small enough. Similarly, the choice of κ is balanced by the inequalities (3.54) and (3.65) as both $\frac{c_6}{(\kappa C_1)^2}$ and $\frac{\kappa^2 C_1}{\mathscr{L}_0}$ need to be small. The choice for ε_2 and σ_2 is to make $\frac{c_6}{(\kappa C_1)^2}$ as small as possible in (3.54). That choice was satisfactory to keep (3.65) valid. Finally, we choose θ in each case to make B as small as possible. We recall that

$$R_0 = 20 \quad \text{and} \quad R_1 = 1.24.$$

We detail the case when $n_0 = 9$. From Table 3, for $\mathscr{L} \le \log(2.29 \cdot 10^7)$, we have

$$B \le 1.7712.$$

In the case where $\mathscr{L} > \log(2.29 \cdot 10^7)$, we first investigate the case where there are no exceptional zero. Choosing $\theta = 33.27$ and $\alpha = 2.56$ and solving in c_4 so that (3.34) is satisfied then lead to

$$B = 42.3849.$$

In the case the exceptional zero β_1 exists, we set

$$\varepsilon_1 = 5.57, \ \sigma_1 = 4.45, \ \eta = 0.025$$

so that

$$c_1'(\varepsilon_1, \sigma_1, \eta) = 0.002509182, \text{ and } c_2'(\varepsilon_1, \sigma_1, \eta) = 0.04890427.$$

This gives

$$c_8 = 0.003324331, \text{ and } c_4 = 150.4072.$$

We also take

$$\varepsilon_2 = 5.97, \ \sigma_2 = 4.5, \ \kappa = 23, \ \lambda = 0.2, \ \mu = 0.1, \text{ and } \nu = 1.15,$$

i.e., we work over the ranges:

$$\frac{0.003330581}{\mathscr{L}} < \frac{0.0006648618}{\mathscr{L}^{0.15}} < 0.001329724 < 0.05.$$

We find in the "extremely small" case that

$$c_3 = 9.85\ldots,\ c_{12} = 34.97\ldots, \text{and } B = 5c_{12} = 174.8780.$$

Finally, we choose α and θ in each other case and obtain

Case	Medium	Small	Very small
θ	1.02	1.02	1.029
α	5.85	0.17	0.67
B	306.8307	306.8307	309.5380

For all the remaining degrees $n_L \neq 9$, we then do the calculations using the same parameter values except for ε_2 which we choose optimally for each degree.

We find all admissible values for B are no larger than the above 309.5380 (see Table 2 for the values of B for each case). This proves that for any Galois extension L/K of number fields with Galois group G, a conjugacy class C in G, $n_L = n_0$ and $d_L \geq d_0$, there exists an unramified prime $\mathfrak{p}$ of K, of degree one, such that $\sigma_{\mathfrak{p}} = C$ and $N\mathfrak{p} \leq d_L^B$ with admissible (n_0, d_0, B) recorded in Table 2. Together with Fiori's numerical verifications recorded in Table 3, this concludes the proof of Theorem 1.

3.6. A summary of key ideas

We list here what we have done differently from [AhKw19-1].

(i) For d_L relatively small, the least prime is bounded numerically (see Appendix).
(ii) If the exceptional zeros exist, we have an enlarged region that contains no non-exceptional (Proposition 6). (Cf. [AhKw19-1, Eq. (6.1)].)
(iii) We obtain a better version of the Deuring-Heilbronn phenomenon (cf. [AhKw19-1, Theorem 7.3]) which is due to

- the refined Turán's power sum method established in [KaNgWo19] (cf. [LaMoOd79, Theorem 4.2]), and
- better estimates for $\frac{\Gamma'}{\Gamma}$, i.e., Lemma 8 and (2.25), (cf. [AhKw19-1, Lemmata 5.3 and 5.4]).

In addition, we do not follow [AhKw19-1, Theorem 7.3]) to split our consideration into imaginary and non-imaginary cases but only consider the non-trivial zeros of $\zeta_L(s)$ (as the Deuring-Heilbronn phenomenon is "trivial" for the trivial zeros of $\zeta_L(s)$).
(iv) Regarding the weight k:

Table 2
Table of admissible (n_0, d_0, B).

n_0	2	3	4	5	6
d_0	10^{10}	10^{10}	10^8	10^8	10^8
B	223.2	231.7	249.1	259.8	271.1
n_0	7	8	9	10	11
d_0	10^8	10^7	$2.29 \cdot 10^7$	$1.56 \cdot 10^8$	$3.91 \cdot 10^9$
B	280.5	303.3	309.6	309.4	303.0
n_0	12	13	14	15	16
d_0	$2.74 \cdot 10^{10}$	$7.56 \cdot 10^{11}$	$5.43 \cdot 10^{12}$	$1.61 \cdot 10^{14}$	$1.17 \cdot 10^{15}$
B	303.2	298.4	298.8	295.1	295.6
n_0	17	18	19	20	21+
d_0	$3.70 \cdot 10^{16}$	$2.73 \cdot 10^{17}$	$9.03 \cdot 10^{18}$	$6.74 \cdot 10^{19}$	10^{n_L}
B	292.5	293.0	290.4	291.0	290.2

- we change the weight k as in (3.1) for the "non-extremely-small" cases.
- we use better estimates for $k(1) - k(\beta_1)$, namely, (3.5) (together with Lemma 9) and (3.57).

(iv) Regarding the position of the exceptional zero β_1 of the Dedekind zeta function:

- we consider a more refined splitting for the position of β_1 (Section 3.4; cf. [AhKw19-1, Sec. 8]); our "small" case (when $(1-\beta_1)\mathscr{L}$ is smaller than constant size) is giving the worst B. Splitting this further with a "very small" case allowed further improvement.

(v) Regarding the zeros of the Dedekind zeta function:

- we use the improved zero-density estimates (3.24), established in [HaShWo], and (3.25) (cf. [AhKw19-1, Propositions 5.5 and 5.6]).
- we obtain a better estimate for zeros ρ with $|\mathfrak{Im}(\rho)| \leq 1$ (see Propositions 13 and 15; cf. [AhKw19-1, Lemma 8.2]) by
  - noticing that there are no non-exceptional zeros close to 1 (see (3.31)), and
  - balancing the use of zero-free region and zero repulsion in the comparison of (a) and (b) done in the proof of Proposition 15 (cf. [AhKw19-1, (i) and (ii) of the proof of Lemma 8.2]).
- a consequence of not over-counting low-lying zeros is to improve the value of B in the "medium" case.
- we realize that for the "extremely small" case the contribution of the zeros ρ, with $\mathfrak{Re}(\rho) \leq \frac{1}{2}$, to $\sum_\rho |k(\rho)|$ is small (see (3.60); cf. [AhKw19-1, Lemma 8.5]). By this observation, we essentially half the size of the sum.
- we use the stronger repulsion of zeros ρ, with $|\mathfrak{Im}(\rho)| \leq 1$, from Theorem 2 instead of the general one as used in [AhKw19-1, Theorem 7.3(2)].

Lastly, as we used a different labelling for the constants from [AhKw19-1], we provide here the following table for comparison:

Ahn-Kwon's notation	c_7	c_8	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}	c_{15}	c_{16}	c_{21}	c_{23}	$\check{c}$
our new notation	c_1	c_2	c_3	c_9	$c_8 = c_{10}$	c_6	c_{11}	c_5	c_4	c_{13}	c_{12}	$8+\varepsilon$

Note that in [AhKw19-1], there is no constant corresponding to our c_7 as the non-exceptional case is not treated separately there.

4. Lower bound for the Chebotarev density theorem

Proof of Theorem 5. Throughout this section, we shall adapt the notation used in [AhKw19-2]. We let $c_6 = 11.7$ and $1-\beta_1 \geq d_L^{-c_6}$.[4] We shall assume $\log x \geq d_L^{c_6}$. Let $a \in (1,2]$. We consider the kernel $k_a(s)$ defined by

$$k_a(s) = \frac{(x^s-1)(a^s-1)}{s^2 \log a}.$$

Recall that the inverse Mellin transform of $k_a(s)$ is

$$\widehat{k_a}(u) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} k_a(s)u^{-s}ds = \begin{cases} \frac{\log u}{\log a} & \text{if } 1<u<a, \\ 1 & \text{if } a \leq u \leq x, \\ \frac{1}{\log a}\log(\frac{ax}{u}) & \text{if } x<u\leq ax, \\ 0 & \text{if } u > ax. \end{cases}$$

By [AhKw19-2, Lemma 2.3], we know that

$$\mathcal{D}_C = \sum_{\mathfrak{p}\in\mathcal{P}(C)} (\log N\mathfrak{p})\widehat{k_a}(N\mathfrak{p}) \leq \tilde{\pi}_C(ax)\log x. \tag{4.1}$$

Let $x_0 = 3^{c_6}$. Then for $x \geq \exp(x_0)$, one has

$$\frac{|G|}{|C|}\mathcal{D}_C \geq c_{43}(a)x, \tag{4.2}$$

where

$$c_{43}(a) = 0.49\frac{a-1}{\log a} - c_{41}e^{-c_{39}\sqrt{\frac{x_0}{2}}} - \Big(c_{35}x_0\log x_0 + c_{40}x_0\Big)e^{-\frac{x_0}{2}}$$

and c_{35}, c_{39}, c_{40}, and c_{41} are constants, depending on x_0 and a, which are defined in [AhKw19-2, pp. 304–306]. By (4.1) and (4.2), we have

[4] c_6 in this section is our c_3 from Theorem 4.

$$\frac{|G|}{|C|}\tilde{\pi}_C(ax)\log x \geq c_{43}(a)x.$$

Hence, replacing x by x/a, together with a simplification, we deduce

$$\tilde{\pi}_C(x) \geq \frac{c_{43}(a)}{a}\frac{|C|}{|G|}\frac{x}{\log x - \log a} \geq \frac{c_{43}(a)}{a}\frac{|C|}{|G|}\frac{x}{\log x}.$$

Finally, we conclude the proof by choosing a as close to 1 as possible. Namely, for $a = 1.0001$, we find $m = \frac{c_{43}(a)}{a} = 0.489975\ldots$. $\square$

Remark. Note that, for $a \approx 1$, $c_{25} = 2(50.313...)\frac{a+1}{\log a}$, $c_{35} \approx \frac{2}{c_6 \log 3}c_8$, $c_{39} \approx \frac{1}{\sqrt{29.57}}$, $c_{40} \approx \sqrt{a} \approx 1$, and $c_{41} \approx \frac{5.7868c_{39}}{c_6}\frac{a+1}{\log a}$. Note that, as $\frac{a-1}{\log a} \approx 1$ when $a \approx 1$, then $m = \frac{c_{43}(a)}{a} \approx 0.49$. This value 0.49 comes from the choice of the kernel $k_a(s)$, so this may be improved by using a different kernel.

Note that for $a = 2$ we find $m = \frac{c_{43}(a)}{a} = 0.353460\ldots$.

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Appendix A. Numerical verification of the least prime in the Chebotarev density theorem - by Andrew Fiori[5]

A.1. Introduction

The purpose of this appendix is to document partial numerical verification of different bounds for the least prime in the Chebotarev density theorem. There are several known asymptotic bounds for the least prime [BaSo96,LaMoOd79,Zam17,AhKw19-1, KaNgWo19], the shape of these bounds depends on whether one assumes the GRH. For each of these bound shapes we document the worst case behaviour for fields up to some bound on the discriminant and the Galois type of the field. Here Galois type refers to the Galois closure of the field.

The numerical verification documented here involved two essential steps:

1. Obtaining proven complete tabulations of number fields up to some discriminant bound. We describe our methods in Section A.2.

[5] The majority of computations for this work were provided on systems supported by Compute Canada. Additional hardware at the University of Lethbridge was purchased through an NSERC RTI grant and supported by funding from the University of Lethbridge.

2. For each field, and each automorphism of that field, finding the least prime in the Chebotarev density theorem. We describe our methods in Section A.3.

A summary of our results by degree is in Section A.4. More complete tables of results are available from [FioWeb1]. Each step of this process made extensive use of the PARI-GP library [PARI].

A.2. Tabulation of number fields

Many researchers have made contributions towards the development of tabulations of number fields with small discriminant. In particular, we refer the reader to the following (incomplete) list of references: [JoRo14,KlMa01,Sim98,Hun57,Poh82,Mar85,Bel97, CoDyDOl03,ScPoDyD94,Oli90,BeMaOl90,Oli92,DyD84,DyD87,PoMaDyD90,Sel99,DJ10, Sel01,DrJo09,Voi08]. Moreover, several efforts have been made to construct reference databases particularly by the PARI group, Kluners and Malle, and Jones and Roberts.

The above work and databases form the basis for our own tabulations which in some cases extend these. The tabulations we have made are available in [FioWeb2]. Note that extending these tabulations is an ongoing project.

A.2.1. Methods for tabulations of number fields

For degree two fields tabulations are essentially trivial, and for degree three fields the work of Belabas [Bel97] provides an efficient method to obtain tabulations. For higher degree fields there are several strategies which may be applicable depending on the type of the field.

- For large degrees the Minkowski bounds provide strong lower bounds for the minimal root discriminant of a field, see for example [DyD80] We used this as the basis for completion results for degrees 9 and higher. Note that for a field of degree n_L which is 21 or larger we have $d_L > 10^{n_L}$.
- If the type of the field being considered is primitive then a search based on the geometry of numbers using ideas from [Hun57,Poh82] provides a method to obtain a complete tabulation up to a chosen bound on the discriminant. We used this method to extend the tabulations for relevant field types and in particular to improve upon existing tabulations in degrees $4, 5, 6$, and 8.
- If the type of the field being tabulated admits an automorphism one may use Kummer theory to build all possible relative extensions based on complete tabulations of the relevant lower degree field. PARI-GP includes complete implementations of the aspects of class field theory and Kummer theory necessary to implement these methods. We used this method to extend the tabulations for relevant field types in particular to improve the completion in degree $4, 6, 8$.
- If the type of field being tabulated admits a subfield, over which the relative Galois group (of the relative Galois closure) is solvable, one can use Kummer theory in a

more elaborate way building off the ideas used in [CoDyDOl03]. Such strategies also work if the Galois group over $\mathbb{Q}$ is solvable. PARI-GP includes complete implementations of the aspects of class field theory and Kummer theory necessary to implement these methods. We used this method to extend the tabulations for relevant field types in particular to improve the completion in degree $6, 9$.

- If the field being tabulated admits a subfield then the geometry of numbers methods of [Mar85] may be used to build all relative extensions. This method may be used even when Kummer theory can be used. Kummer theory will typically be simpler. We have not made use these methods in our search as none of the extension degrees for which we can obtain results would benefit.

A.3. Verification of least prime

We now document our method of verification.

Suppose $L/\mathbb{Q}$ is a fixed field with absolute discriminant d_L. Let $G = \mathrm{Aut}_{\mathbb{Q}}(L)$ and $\sigma \in G$.

What we verify is that there exists a *small* prime p of $\mathbb{Q}$ which is unramified in L, for which there exists a prime $\mathcal{P}$ of L over p such that

1. $\sigma(\mathcal{P}) = \mathcal{P}$.
2. $\forall x \in \mathcal{O}_L, \sigma(x) \equiv x^p \pmod{\mathcal{P}}$,

where small means relative to one of the types of bound under consideration relative to the discriminant d_L of L.

We note that this formulation differs from how one typically states the conditions on the least prime in the Chebotarev density theorem. We state the following claims without proof:

Claim. If $K = L^\sigma$ is the fixed field of σ then $\mathfrak{p} = \mathcal{P} \cap K$ satisfies the usual conditions of the least prime in the Chebotarev density theorem if and only if $\mathcal{P}$ satisfies these slightly different conditions. Hence, we may find the smallest value of $N_{K/\mathbb{Q}}(\mathfrak{p}) = p$ by finding the smallest value of p.

Claim. If $K \subset L^\sigma$ is the fixed field of σ then if $\mathcal{P}$ satisfies these slightly different conditions then $\mathfrak{p} = \mathcal{P} \cap K$ satisfies the conditions of the least prime in the Chebotarev density theorem with respect to the image of σ in $\mathrm{Aut}_K(L)$. Hence, we may find an upper bound for the smallest value of $N_{K/\mathbb{Q}}(\mathfrak{p}) = p$ by finding the smallest value of p.

The above two claims are useful because they reduce the total number of cases that needed to be checked, but also because they make clear that an explicit check for any given field and any given automorphism is actually computationally practical. Indeed,

Table 3
Table of worst case bounds.

$n_0 = [L : \mathbb{Q}]$	d_{min}	Verification Height d_0	A	B	C
2	3	16	1.7712	5.7997	136.0600
2		$16 < d_L \leq 10^{10}$	0.8071	1.5802	1.0389
3	23	10^{10}	0.6591	0.8803	0.5661
4	117	10^8	0.6098	0.9979	0.6823
5	1609	10^8	0.3559	0.6668	0.5297
6	9747	10^8	0.4335	0.6668	0.5297
7	184607	10^8	0.2490	0.2050	0.1890
8	1257728	10^7	0.2925	0.5047	0.5327
9	$2.29 \cdot 10^7$	$2.29 \cdot 10^7$	0.2183	0.1692	0.1658
10	$1.56 \cdot 10^8$	$1.56 \cdot 10^8$	0.2295	0.4319	0.5069
11	$3.91 \cdot 10^9$	$3.91 \cdot 10^9$	0.1228	0.1047	0.1272
12	$2.74 \cdot 10^{10}$	$2.74 \cdot 10^{10}$	0.2107	0.3269	0.3393
13	$7.56 \cdot 10^{11}$	$7.56 \cdot 10^{11}$	0.1365	0.0627	0.0690
14	$5.43 \cdot 10^{12}$	$5.43 \cdot 10^{12}$	0.1582	0.4006	0.5268
15	$1.61 \cdot 10^{14}$	$1.61 \cdot 10^{14}$	0.1206	0.0866	0.1201
16	$1.17 \cdot 10^{15}$	$1.17 \cdot 10^{15}$	0.1463	0.2769	0.3410
17	$3.70 \cdot 10^{16}$	$3.70 \cdot 10^{16}$	0.1069	0.0478	0.0689
18	$2.73 \cdot 10^{17}$	$2.73 \cdot 10^{17}$	0.1279	0.3371	0.4778
19	$9.03 \cdot 10^{18}$	$9.03 \cdot 10^{18}$	0.0909	0.0327	0.0484
20+	$6.74 \cdot 10^{19}$	$6.74 \cdot 10^{19}$	0.1230	0.3370	0.4741

PARI-GP includes all the tools necessary to make such a check, and hence find the smallest p.

Remark 16. Because p is expected to be small, the runtime of this search is typically very small.

We remark also that PARI-GP has all the tools necessary to compute $\mathrm{Aut}_{\mathbb{Q}}(L)$.

A.4. Summary of results

Table 3 summarizes the results we have obtained.

More complete tables, as well as searchable lists of all results for all fields referenced can be found at [FioWeb1].

There are essentially three types of bounds considered

A bound d_L^A which is based on the GRH-unconditional results.
B bound $B(\log d_L)^2$ which is based on the GRH-conditional results.
C bound $C(3e^\gamma/\pi)^2 \frac{(\log d_L)^2(\log(2\log\log d_L))^2}{(\log\log d_L)^2}$ which is based on an upper bound for the worst known family of fields [Fio18].

For each row of the table and each type of bound, there are no fields L with $n_L = n_0$ and $d_L \leq d_0$, for which the least prime in the Chebotarev density theorem exceeds the bounds. Additionally, for each row, and each type of bound, there exists a field L, with $n_L = n_0$, which (up to the precision given) realizes the given bound. In many cases these worst case fields will have absolute discriminants larger than the verification height. One

can thus interpret each row as giving a *lower bound* on the upper bound for the least prime in the Chebotarev theorem. The column d_{min} documents a lower bound for the absolute discriminant of fields of that degree, so that for all L with $n_L = n_0$ we have $d_L \geq d_{min}$. We recall that for degrees 9+ there are in fact no fields whatsoever with $n_L = n_0$ and $d_L \leq d_0$. For degree 21 and higher this verification height, and d_{min} can be taken as at least $10^{[L:Q]}$.

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