# Patterns of primes in Chebotarev sets 

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#### Abstract

The work of Green and Tao shows that there are infinitely many arbitrarily long arithmetic progressions of primes. Recently, Maynard and Tao independently proved that for any $m \geq 2$, there exists $k$ (depending on $m$ ) so that for any admissible set $\mathcal{H}=\left\{h_{1}, \ldots, h_{k}\right\}$, there are infinitely many $n \in \mathbb{N}$ such that at least $m$ of $n+h_{1}, \ldots, n+h_{k}$ are prime. We obtain a common generalization of both these results for primes satisfying Chebotarev conditions. We also give an improvement of the known bound for gaps between primes in any given Chebotarev set.


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## 1. Introduction

A set $\mathcal{H}=\left\{h_{1}, \ldots, h_{k}\right\}$ of distinct non-negative integers is said to be admissible if for every (rational) prime $p$, there is an integer $a_{p}$ which is not congruent to any element in $\mathcal{H}$ modulo $p$. In other words, $|\mathcal{H}(\bmod p)|<p$ for all primes $p$. In 1904, Dickson [2] considered the following conjecture, which was also formulated later by Hardy and Littlewood in a quantitative form.

Conjecture 1.1 (Prime $\boldsymbol{k}$-tuples conjecture). Given any admissible set $\mathcal{H}$, there are infinitely many integers $n$ such that $n+h_{1}, \ldots, n+h_{k}$ are all prime.

This conjecture generalizes the famous twin prime conjecture which asserts that there are infinitely many primes $p$ such that $p+2$ is also a prime. This can be seen by choosing $\mathcal{H}=\{0,2\}$. In their groundbreaking paper [4], Goldston, Pintz, and Yıldırım developed what is now known as the GPY sieve method. Originating in
the work of Selberg, this method allowed them to show that

$$
\liminf _{n} \frac{p_{n+1}-p_{n}}{\log p_{n}}=0
$$

where $p_{n}$ denotes the $n$th prime. Considerable progress has been made since then on this problem. In 2013, Zhang [14] obtained the remarkable result

$$
\liminf _{n}\left(p_{n+1}-p_{n}\right) \leq 7 \times 10^{7}
$$

Moreover, his work established that a positive proportion of 2-element admissible sets satisfy the prime 2-tuples conjecture. Within a few months of this breakthrough, Maynard [7] and Tao (unpublished) obtained a simplified proof of bounded gaps between primes by using a "higher rank" Selberg sieve. Their method gave much better numerical results and also enabled them to show that for each $k$, the prime $k$-tuples conjecture holds for a positive proportion of admissible sets of size $k$. In particular, the current best (unconditional) bound is

$$
\liminf _{n}\left(p_{n+1}-p_{n}\right) \leq 246
$$

A crucial ingredient in these results is the Bombieri-Vinogradov theorem which shows that the primes have a level of distribution $\theta$ for any $\theta<1 / 2$. This is discussed in more detail in Sec. 3.

In a different vein, in 2008, Green and Tao [5] proved the existence of infinitely many $l$-term arithmetic progressions in the sequence of primes for every natural number $l$. Although their proof uses some ideas from the work of GPY [4], it is mostly disjoint from the methods used to examine gaps between primes. It was the foresight of Pintz [9] that led him to marry these two important results by combining the methods of GPY and Green-Tao to obtain the following result in 2010.

Theorem $1.2\left(\left[\mathbf{9}\right.\right.$, Theorem 5]). Let $\mathcal{H}=\left\{h_{1}, \ldots, h_{k}\right\}$ be an admissible set. If there exists a set $\mathcal{S}(\mathcal{H})$ and constants $c_{1}(k), c_{2}(k)>0$ such that

$$
\begin{equation*}
P^{-}\left(\prod_{j=1}^{k}\left(n+h_{j}\right)\right) \geq n^{c_{1}(k)} \text { for all } n \in \mathcal{S}(\mathcal{H}) \tag{1.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\#\{n \leq x \mid n \in \mathcal{S}(\mathcal{H})\} \geq \frac{c_{2}(k) x}{(\log x)^{k}} \tag{1.2}
\end{equation*}
$$

for all $x$ sufficiently large, then $\mathcal{S}(\mathcal{H})$ contains l-term arithmetic progressions for every natural number $l$.

This result allowed Pintz to obtain a conditional strengthening of the Green-Tao theorem under the assumption that the primes have a level of distribution $\theta>1 / 2$. More precisely, he proved the following.

Theorem 1.3. If the level of distribution $\theta$ of the primes exceeds $1 / 2$, then there exists a positive $h \leq C_{1}(\theta)$ so that there are arbitrarily long arithmetic progres-
sions of primes $p$ such that $p+h$ is the prime following $p$ for each element of the progression.

Adapting the work of Zhang, Pintz [10] was able to make this result unconditional. More recently, following the work of Maynard and Tao on gaps between primes, Pintz [11] established a common generalization of the results of Zhang, Maynard, Green-Tao, and himself as the following.

Theorem 1.4. Let $m$ be a natural number and $\mathcal{H}=\left\{h_{1}, \ldots, h_{k}\right\}$ be an admissible set of $k$ distinct non-negative integers, where $k=\left\lceil C m^{2} e^{4 m}\right\rceil$ with a sufficiently large absolute constant $C$. Then there is an $(m+1)$-element subset $\left\{h_{1}^{\prime}, \ldots, h_{m+1}^{\prime}\right\} \subseteq \mathcal{H}$ such that for every natural number l, there exist infinitely many l-term (non-trivial) arithmetic progressions of primes $p=n+h_{1}^{\prime}$, such that $n+h_{j}^{\prime}$ is also a prime for every $2 \leq j \leq m+1$. Furthermore, $n+h_{j}^{\prime}$ is always the $(j-1)$ th prime following $p$, for $2 \leq j \leq m+1$.

In this paper, we generalize this theorem for primes satisfying Chebotarev conditions. A prime is said to satisfy a Chebotarev condition if it belongs to what is known as a Chebotarev set. A subset $\mathcal{P}$ of the set of rational primes $\mathbb{P}$ is called a Chebotarev set if there is a Galois extension $K / \mathbb{Q}$ of number fields with Galois group $G$ and absolute discriminant $d_{K}$ such that

$$
\mathcal{P}=\left\{p \in \mathbb{P} \mid p \text { is unramified with }\left(\frac{K / \mathbb{Q}}{p}\right) \subseteq C\right\}
$$

Here, for $p$ unramified (or equivalently, $\left.p \nmid d_{K}\right),\left(\frac{K / \mathbb{Q}}{p}\right)$ denotes the Artin symbol at $p$, and $C$ is a union of conjugacy classes of $G$. It is clear that a Chebotarev set $\mathcal{P}$ is determined by $K$ and $C$. We will denote $\mathcal{P}(K, C)$ as $\mathcal{P}$, considering $K$ and $C$ to be fixed.

With this discussion in place, one can consider the analogous problem of bounded gaps between primes in a given Chebotarev set $\mathcal{P}$. The variant of the BombieriVinogradov theorem that plays a key role in this setting is due to Murty and Murty [8] and is discussed in Sec. 4. Adapting the method of Maynard, Thorner [13] showed that for every Chebotarev set $\mathcal{P}$, there exist infinitely many pairs of distinct primes $p, p^{\prime} \in \mathcal{P}$ such that $\left|p-p^{\prime}\right| \leq M_{\mathcal{P}}$, with $M_{\mathcal{P}}$ given by

$$
M_{\mathcal{P}}=825\left(\frac{|G|^{2} d_{K}}{|C| \phi\left(d_{K}\right)}\right)^{3} \exp \left(\frac{|G|^{2} d_{K}}{|C| \phi\left(d_{K}\right)}\right)
$$

However, Thorner did not invoke the full power of the equidistribution result of [8] and only used a level of distribution $\theta=2 /|G|-\delta$, for some small fixed $\delta>$ 0 , whenever $|G|>4$. Applying a result of Arthur-Clozel [1] along with recent numerical advances by the Polymath project [12], we improve Thorner's bound to

$$
M_{\mathcal{P}}=\kappa \frac{2|G| d_{K}}{|C| \theta \phi\left(d_{K}\right)} \exp \left(\frac{2|G| d_{K}}{|C| \theta \phi\left(d_{K}\right)}\right)
$$

where $\kappa$ is a sufficiently large absolute constant and $\theta \geq 2 /|G|-\delta$ whenever $|G|>4$. The precise level of distribution $\theta$ is given in Sec. 4.

We also generalize the method of Pintz to primes satisfying Chebotarev conditions. In this way we obtain a common extension of Theorem 1.4 of Pintz as well as Thorner's result. Our result generalizes the Green-Tao theorem, giving arbitrarily long arithmetic progressions of primes satisfying Chebotarev conditions. We state our main result as follows.

Theorem 1.5. Let $\mathcal{P}=\mathcal{P}(K, C)$ be a Chebotarev set having level of distribution $\theta$. Let $m$ be a natural number and $\mathcal{H}=\left\{h_{1}, \ldots, h_{k}\right\}$ be an admissible set of $k$ distinct non-negative integers $h_{i}$, where

$$
k=\left\lceil\kappa \exp \left(\frac{2|G| d_{K} m}{|C| \theta \phi\left(d_{K}\right)}\right)\right\rceil, \quad \kappa \text { a sufficiently large absolute constant. }
$$

Then there is an $(m+1)$-element subset $\left\{h_{1}^{\prime}, \ldots, h_{m+1}^{\prime}\right\}$ of $\mathcal{H}$ such that, for every natural number $l$, there exist infinitely many $l$-term (non-trivial) arithmetic progressions of primes $p=n+h_{1}^{\prime}$ in $\mathcal{P}$, satisfying $n+h_{j}^{\prime} \in \mathcal{P}$ for every $2 \leq j \leq m+1$. Moreover, one can impose the stricter condition that $n+h_{j}^{\prime}$ is the $(j-1)$ th prime in $\mathcal{P}$ following $p$, for $2 \leq j \leq m+1$.

This result has several interesting arithmetic applications. Indeed as demonstrated in [13], if one can associate an arithmetic object with a Chebotarev set, then the theorem applies. In this paper, however, we emphasize the sieve theoretic aspect and direct the interested reader to [13] for arithmetic applications.

## 2. Notation

For the sake of clarity, it will be convenient to introduce some notation. We let $n, N$ denote natural numbers, $p$ a prime, and $\mathbb{P}$ the set of rational primes. The notation $n \sim N$ means that $N<n \leq 2 N$. The greatest integer less than $x$ and the least integer greater than $x$ are denoted by $\lfloor x\rfloor$ and $\lceil x\rceil$, respectively. The gcd and 1 cm of $a, b$ are written as $(a, b)$ and $[a, b]$, respectively. The functions $\phi, \mu$, and $\tau_{r}(n)$ refer to the Euler totient function, the Möbius function, and the number of representatives of $n$ as a product of $r$ natural numbers, respectively. The radical of $n$ is defined as $\operatorname{rad}(n)=\prod_{p \mid n} p$. Throughout the paper, $P^{-}(n)$ denotes the least prime divisor of $n$. Given a finite set $S$, we define the diameter of $S$ as

$$
\operatorname{diam} S=\max _{s_{i}, s_{j} \in S}\left|s_{i}-s_{j}\right|
$$

We denote the $k$-tuple of integers $\left(d_{1}, \ldots, d_{k}\right)$ by $\underline{d}$. A tuple is said to be square-free if the product of its components is square-free. For a real number $R$, the inequality $\underline{d} \leq R$ means that $\prod_{i} d_{i} \leq R$. The notion of divisibility among tuples is defined component-wise, that is,

$$
\underline{d}\left|\underline{n} \Leftrightarrow d_{i}\right| n_{i} \quad \text { for all } 1 \leq i \leq k .
$$

It follows that the notion of congruence among tuples, modulo a tuple, is also defined component-wise. We define the function $f(\underline{d})$ to mean the product of its component (multiplicative) functions acting on the corresponding components of the tuple, that is,

$$
f(\underline{d})=\prod_{i=1}^{k} f_{i}\left(d_{i}\right) .
$$

For example, if $\mu$ is the Möbius function, $\mu(\underline{d})=\prod_{i=1}^{k} \mu\left(d_{i}\right)$. Furthermore, we will use the notation $[\underline{d}, \underline{e}]$ to denote the product $\prod_{j=1}^{k}\left[d_{j}, e_{j}\right]$.

As before, $K / \mathbb{Q}$ denotes a Galois extension of number fields with Galois group $G$ and absolute discriminant $d_{K}$. For every unramified prime $p,\left(\frac{K / \mathbb{Q}}{p}\right)$ denotes the Artin symbol at $p$. We let $C$ denote a union of conjugacy classes of $G$, and let

$$
\mathcal{P}=\mathcal{P}(K, C)=\left\{p \in \mathbb{P} \mid p \text { is unramified with }\left(\frac{K / \mathbb{Q}}{p}\right) \subseteq C\right\}
$$

be a fixed Chebotarev set. The $n$th prime in our Chebotarev set $\mathcal{P}$ is denoted by $p_{n}$. We will work with a fixed admissible set

$$
\mathcal{H}=\left\{h_{1}, \ldots, h_{k}\right\} \subseteq \mathbb{N}
$$

where $k$ is a fixed natural number. Recall that admissibility of $\mathcal{H}$ means that

$$
|\mathcal{H}(\bmod p)|<p
$$

for every prime $p$. We let $H$ denote $\max _{1 \leq i \leq k}\left|h_{i}\right|$. Any constants implied by the asymptotic notation $O, \ll$ or $\gg$ may depend on $k, H$ and the field $K$. We indicate these constants whenever this is the case.

## 3. Small Gaps between Primes and Beyond

In this section, we recall the work of Maynard on bounded gaps between primes and indicate the key ideas of Pintz in this setting.

The primes are said to have level of distribution $\theta>0$ if for any fixed $A>0$, one has

$$
\sum_{q \leq x^{\theta}} \max _{y \leq x} \max _{(a, q)=1}\left|\pi(y, q, a)-\frac{\pi(y)}{\phi(q)}\right|<_{A} \frac{x}{(\log x)^{A}}
$$

where

$$
\pi(x)=\#\{p \in \mathbb{P} \mid p \leq x\}, \quad \pi(x, q, a)=\#\{p \in \mathbb{P} \mid p \leq x, p \equiv a(\bmod q)\}
$$

The celebrated Bombieri-Vinogradov theorem asserts that the above estimate holds when $0<\theta<\frac{1}{2}$, and the Elliott-Halberstam conjecture predicts that the above estimate holds for all $0<\theta<1$.

For a fixed admissible set $\mathcal{H}=\left\{h_{1}, \ldots, h_{k}\right\}$, following the GPY approach to the prime $k$-tuples conjecture, we consider the following sums:

$$
\begin{align*}
& S_{1}(N, \mathbb{P})=\sum_{\substack{n \sim N \\
n \equiv v_{0}(\bmod W)}} \omega_{n},  \tag{3.1}\\
& S_{2}(N, \mathbb{P})=\sum_{\substack{n \sim N \\
n \equiv v_{0}(\bmod W)}} \sum_{i=1}^{k} \chi_{\mathbb{P}}\left(n+h_{i}\right) \omega_{n}, \tag{3.2}
\end{align*}
$$

where $\omega_{n}$ are non-negative weights and $\chi_{\mathbb{P}}$ denotes the characteristic function of $\mathbb{P}$. Here

$$
\begin{equation*}
W=\prod_{p \leq D_{0}} p \tag{3.3}
\end{equation*}
$$

for some large enough positive number $D_{0}$ to be chosen later and $\nu_{0}$ is some residue class modulo $W$ such that $\left(\nu_{0}, W\right)=1$.

For $\rho>0$, we define the difference

$$
\begin{equation*}
S(N, \rho, \mathbb{P})=S_{2}(N, \mathbb{P})-\rho S_{1}(N, \mathbb{P}) \tag{3.4}
\end{equation*}
$$

The key idea of this approach is as follows. Suppose that one can show that there exists suitable $\rho>0$ such that $S(N, \rho, \mathbb{P})>0$. This means that the inequality

$$
\sum_{j=1}^{k} \chi_{\mathbb{P}}\left(n+h_{j}\right)-\rho>0
$$

must hold for some $n \sim N$. If one can do this for all sufficiently large $N$, one can then obtain infinitely many $n$ such that at least $\lfloor\rho+1\rfloor$ elements in $\left\{n+h_{i}\right\}_{i=1}^{k}$ are primes.

Clearly, this approach relies on good asymptotic formulas for $S_{1}(N, \mathbb{P})$ and $S_{2}(N, \mathbb{P})$. We state below the asymptotic formulas obtained by Maynard in [7].

Proposition 3.1 ([7, Proposition 4.1]). Suppose that the primes have a level of distribution $\theta>0$, and let $R=N^{\frac{\theta}{2}-\delta}$ for some small fixed $\delta>0$. Let $F$ be a smooth function on $[0,1]^{k}$ which is supported on $\mathcal{R}_{k}=\left\{\left(x_{1}, \ldots, x_{k}\right) \mid \sum_{i=1}^{k} x_{i}=1\right\}$. We define $\lambda_{\underline{d}}$ as

$$
\lambda_{\underline{d}}=\mu(\underline{d}) \underline{d} \sum_{\substack{\underline{d} \mid \underline{r} \\(\underline{r}, \bar{W}=1}} \frac{\mu(\underline{r})^{2}}{\phi(\underline{r})} F\left(\frac{\log r_{1}}{\log R}, \ldots, \frac{\log r_{k}}{\log R}\right),
$$

whenever $\left(\prod_{i=1}^{k} d_{i}, W\right)=1$, and set $\lambda_{\underline{d}}=0$ otherwise. Setting weights $\omega_{n}$ as

$$
\begin{equation*}
\omega_{n}=\left(\sum_{\underline{d} \mid \underline{n}} \lambda_{\underline{d}}\right)^{2} \tag{3.5}
\end{equation*}
$$

and choosing $D_{0}=\log \log \log N$, one has

$$
\begin{gathered}
S_{1}(N, \mathbb{P})=\left(1+O\left(\frac{1}{D_{0}}\right)\right) \frac{\phi(W)^{k} N(\log R)^{k}}{W^{k+1}} I_{k}(F), \\
S_{2}(N, \mathbb{P})=\left(1+O\left(\frac{1}{D_{0}}\right)\right) \frac{\phi(W)^{k} N(\log R)^{k+1}}{W^{k+1} \log N} \sum_{i=1}^{k} J_{k}^{(i)}(F),
\end{gathered}
$$

where

$$
\begin{gathered}
I_{k}(F)=\int_{[0,1]^{k}} F\left(t_{1}, \ldots, t_{k}\right) d t_{1} \ldots d_{k} \\
J_{k}^{(m)}(F)=\int_{[0,1]^{k-1}}\left(\int_{0}^{1} F\left(t_{1}, \ldots, t_{k}\right) d t_{m}\right)^{2} d t_{1} \ldots d_{m-1} d t_{m+1} \ldots d_{k}
\end{gathered}
$$

Using this, one can conclude the following.
Theorem 3.2 ([7, Proposition 4.2]). Let $\mathbb{P}$ have level of distribution $\theta>0$ and let $\mathcal{R}_{k}, I_{k}(F)$ and $J_{k}^{(m)}(F)$ be given as above. We let $\mathcal{S}_{k}$ denote the set of Riemann integrable functions $F:[0,1]^{k} \rightarrow \mathbb{R}$ supported on $\mathcal{R}_{k}$ with $I_{k}(F) \neq 0$ and $J_{k}^{(m)}(F) \neq 0$ for each $m$. Set

$$
M_{k}=\sup _{F \in \mathcal{S}_{k}} \frac{\sum_{m=1}^{k} J_{k}^{(m)}(F)}{I_{k}(F)}, \quad r_{k}=\left\lceil\frac{\theta M_{k}}{2}\right\rceil .
$$

Then there are infinitely many integers $n$ such that at least $r_{k}$ elements in $\left\{n+h_{i}\right\}_{i=1}^{k}$ are primes. In particular,

$$
\liminf _{n}\left(p_{n+r_{k}+1}-p_{n}\right) \leq \max _{1 \leq i, j \leq k}\left|h_{i}-h_{j}\right|
$$

Finally, Maynard established his groundbreaking work by showing good lower bounds for $M_{k}$. These lower bounds have subsequently been improved through better numerical methods in [12].

We now turn our attention to Pintz's work. Heuristically, it is expected that sieving the sequences $\left\{n+h_{i}\right\}, 1 \leq i \leq k$ produces numbers which are almost primes, in the sense that

$$
P^{-}\left(n+h_{i}\right)>n^{c}, \quad \text { for fixed } c>0
$$

In the spirit of this general principle in sieve theory, the key idea of Pintz is that one can "overlook" all the weights

$$
\omega_{n}=\left(\sum_{\underline{d} \mid \underline{n}} \lambda_{\underline{d}}\right)^{2}
$$

in the sum $S_{1}(N, \mathbb{P})$ for which there is an $i(1 \leq i \leq k)$ such that $n+h_{i}$ has a "small" prime factor $p$. This was made precise in [10] by means of the following result.

Lemma 3.3 (Pintz [10]). There is a sufficiently small constant $c_{1}(k)$ such that

$$
S_{1}^{-}(N, \mathbb{P}):=\sum_{\substack{n \sim N \\ n \equiv \equiv_{0}(\bmod W) \\ P^{-}\left(\prod_{i=1}^{k}\left(n+h_{i}\right)\right)<n^{c_{1}(k)}}} \omega_{n} \ll k, H \frac{c_{1}(k) \log N}{\log R} S_{1}(N, \mathbb{P}) .
$$

Pintz also noticed that Proposition 3.1 of Maynard holds with $D_{0}=C^{*}(k)$ for some sufficiently large constant $C^{*}(k)$ (depending only on $k$ ). Combining these ideas with his earlier work, namely Theorem 1.2, Pintz was able to derive a generalization of the results of Green-Tao and Maynard, as stated in Theorem 1.4.

## 4. Bounded Gaps Between Primes in Chebotarev Sets

In this section, we discuss the method of Thorner [13] for bounded gaps between primes in a Chebotarev set and obtain better bounds by applying the full power of the equidistribution theorem in this context. As discussed earlier, Thorner [13] adapted the work of Maynard [7] to obtain bounded gaps between primes in any given Chebotarev set. An essential ingredient in Thorner's work is a variant of the Bombieri-Vinogradov theorem due to Murty and Murty [8].

With this theorem in mind, we elaborate on the notion of the level of distribution of a Chebotarev set. Let $\mathcal{P}$ be a set of primes. We use the standard notation

$$
\pi_{\mathcal{P}}(x)=\#\{p \in \mathcal{P} \mid p \leq x\}
$$

and

$$
\pi_{\mathcal{P}}(x, q, a)=\#\{p \in \mathcal{P} \mid p \leq x, p \equiv a(\bmod q)\}
$$

A Chebotarev set $\mathcal{P}=\mathcal{P}(K, C)$ is said to have a level of distribution $\theta$ if there exists a natural number $M$ such that

$$
\begin{equation*}
\sum_{\substack{q \leq x^{\theta} \\(q, M)=1}} \max _{y \leq x} \max _{(a, q)=1}\left|\pi_{\mathcal{P}}(y, q, a)-\frac{\pi_{\mathcal{P}}(y)}{\phi(q)}\right|<_{A, K} \frac{x}{(\log x)^{A}} \tag{4.1}
\end{equation*}
$$

holds for any $A>0$. In [8], the authors prove the following.
Theorem 4.1. The average result (4.1) holds if $M=d_{K}$ and $0<\theta<\min \left(\frac{2}{|G|}, \frac{1}{2}\right)$. Moreover, assuming a special case of the Artin conjecture (AC) that all L-functions attached to all abelian twists of any non-trivial character of $G$ are entire, and setting

$$
\eta=\max _{\chi \neq \chi_{0}}|\chi(1)-2|,
$$

where the maximum runs over all non-trivial characters of $G$, one can improve the estimate (4.1) with a larger level of distribution $0<\theta<\min \left(\frac{1}{\eta}, \frac{1}{2}\right)$.

Thus, for $G$ with $\eta \leq 2$ (which can be classified by using results of the type [3, Theorem 24.6]), assuming (AC), $\mathcal{P}$ has a level of distribution $\theta$ for any $\theta<1 / 2$. On
the other hand, in [1], Arthur and Clozel obtained the following celebrated theorem which proves Langlands' reciprocity conjecture in the nilpotent case.

Theorem 4.2 ([1, p. 220]). If $K / k$ is a nilpotent Galois extension of number fields, then for each irreducible representation $\rho$ of $\operatorname{Gal}(K / k)$ of dimension n, there exists a cuspidal automorphic representation $\Pi=\Pi(\rho)$ of $G L_{n}\left(\mathbb{A}_{k}\right)$ such that the Artin L-function attached to $\rho$ coincides with the automorphic L-function attached to $\Pi$.

Applying the Rankin-Selberg theory due to Jacquet, Piatetski-Shapiro and Shalika, as well as the above theorem of Arthur and Clozel, the assumption of (AC) in Theorem 4.1 automatically holds if $K / \mathbb{Q}$ is a nilpotent Galois extension. Thus, if $G$ is nilpotent, $\mathcal{P}$ has level of distribution $\theta$ for any $0<\theta<\min \left(\frac{1}{\eta}, \frac{1}{2}\right)$ unconditionally.

In general, as mentioned in [8, Sec. 7.4], it is possible to improve the level of distribution as follows. Let

$$
d^{*}=\min _{H} \max _{\chi}[G: H] \chi(1),
$$

where the minimum is over all subgroups $H$ of $G$ satisfying

- $H \cap C \neq \varnothing$, and
- (AC) is true for $H$, i.e. all $L$-functions attached to all abelian twists of any nontrivial character of $H$ are entire;
while the maximum runs over irreducible characters of $H$. Then, $\eta$ can be replaced by $\eta^{*}$ which is defined as

$$
\eta^{*}= \begin{cases}d^{*}-2 & \text { if } d^{*} \geq 4  \tag{4.2}\\ 2 & \text { if } d^{*}<4\end{cases}
$$

In other words, one has a level of distribution $\theta$ for any $\theta \in\left(0, \frac{1}{\eta^{*}}\right)$. As discussed above, since (AC) holds when $H$ is nilpotent, we have the upper bound

$$
d^{*} \leq \min _{H} \max _{\chi}[G: H] \chi(1)
$$

where the minimum now is over all nilpotent subgroups $H$ of $G$ such that $H \cap C \neq \varnothing$, and the maximum runs over irreducible characters of $H$. However, sometimes this bound is not really practical since it requires information about all character degrees of all subgroups $H$ appearing in the minimum. To obtain a more precise bound on $d^{*}$, we recall the following.
Theorem 4.3 ([6, p. 28]). Let $G$ be a finite group and $\mathbf{Z}(G)$ its centre. Then for every irreducible character $\chi$ of $G$, one has

$$
\chi(1)^{2} \leq[G: \mathbf{Z}(G)] .
$$

Therefore, we then deduce

$$
d^{*} \leq \min _{H}[G: H][H: \mathbf{Z}(H)]^{\frac{1}{2}},
$$

where the minimum is over all nilpotent subgroups $H$ of $G$ such that $H \cap C \neq \varnothing$. It is worth noting that, as all abelian groups are nilpotent, we have

$$
d^{*} \leq\left[G: H_{C}\right]
$$

where $H_{C}$ is the largest abelian subgroup such that $H_{C} \cap C \neq \varnothing$. For the case $d^{*} \geq 4$, we will show that even this crude bound gives a better level of distribution than $\theta<2 /|G|$, which is used in [13] whenever $|G|>4$. Assuming $|G|>4$, let $H_{C}$ be the largest abelian subgroup such that $H_{C} \cap C \neq \varnothing$. We first show that $\left|H_{C}\right| \geq 2$. Clearly, if $C$ contains any non-trivial element $g$, then the cyclic group $\langle g\rangle$ has non-empty intersection with $C$. On the other hand, if $C=\{e\}$, we can simply pick $H_{C}$ to be the largest abelian subgroup of $G$.

To show that we obtain a better level of distribution, we need to prove that

$$
\frac{1}{\left[G: H_{C}\right]-2} \geq \frac{2}{|G|}
$$

This follows easily from the inequality

$$
|G|\left(\frac{2}{\left|H_{C}\right|}-1\right) \leq 4
$$

which holds since $2 /\left|H_{C}\right| \leq 1$. Thus, we have level of distribution at least

$$
\frac{1}{\left[G: H_{C}\right]-2}
$$

provided that $\left[G: H_{C}\right] \geq 4$; otherwise, we will have level of distribution $\theta$ for any $\theta<1 / 2$.

In order to apply the above discussion to the context of gaps between primes satisfying Chebotarev conditions, we now turn our attention to the results of Thorner. For the admissible set $\mathcal{H}=\left\{h_{1}, \ldots, h_{k}\right\}$, we put

$$
\operatorname{det}(\mathcal{H})=\prod_{i \neq j}\left(h_{i}-h_{j}\right)
$$

Recall that for a natural number $n$, the radical of $n$ is defined as $\operatorname{rad}(n)=\prod_{p \mid n} p$. As before, we let $W=\prod_{p \leq D_{0}} p$ for some sufficently large $D_{0}$ to be chosen later. We set $U=W / \operatorname{rad}\left(d_{K}\right)$. Note that when $D_{0}$ is sufficiently large, $\operatorname{rad}(\operatorname{det}(\mathcal{H}))$ divides $W$. By the Chinese remainder theorem and the admissibility of $\mathcal{H}$, there exists an integer $u_{0}$ such that $\left(\prod_{i=1}^{k}\left(u_{0}+h_{i}\right), U\right)=1$.

Thorner's argument is based on using the congruence condition $n \equiv u_{0}$ modulo U instead of a congruence condition modulo $W$. In analogy with the sums $S_{1}(N, \mathbb{P})$ and $S_{2}(N, \mathbb{P})$ considered earlier (see (3.1) and (3.2)), we now consider the following:

$$
\begin{align*}
& S_{1}(N, \mathcal{P})=\sum_{\substack{n \sim N \\
n \equiv u_{0}(\bmod U)}} \omega_{n}  \tag{4.3}\\
& S_{2}(N, \mathcal{P})=\sum_{\substack{n \sim N \\
n \equiv u_{0}(\bmod U)}} \sum_{i=1}^{k} \chi_{\mathcal{P}}\left(n+h_{i}\right) \omega_{n} \tag{4.4}
\end{align*}
$$

where $\omega_{n}$ are non-negative sieve parameters and $\chi_{\mathcal{P}}$ denotes the characteristic function of $\mathcal{P}$. As done in Maynard's setting, the parameters $\omega_{n}$ are chosen as in (3.5) with $\lambda_{\underline{d}}$ defined as in Proposition 3.1.

As it is convenient to examine each of the $k$ inner summands in the sum $S_{2}(N, \mathcal{P})$ separately, we define

$$
\begin{equation*}
S_{2}^{(m)}(N, \mathcal{P})=\sum_{\substack{n \sim N \\ n \equiv u_{0} \bmod U}} \chi_{\mathcal{P}}\left(n+h_{m}\right) \omega_{n} \tag{4.5}
\end{equation*}
$$

Finally, for $\rho>0$, we consider the difference

$$
\begin{equation*}
S(N, \rho, \mathcal{P})=S_{2}(N, \mathcal{P})-\rho S_{1}(N, \mathcal{P}) \tag{4.6}
\end{equation*}
$$

Recall that by definition, the support of $\lambda_{\underline{d}}$ is restricted to square-free tuples $\underline{d}$ satisfying

$$
\left(\prod_{i=1}^{k} d_{i}, W\right)=1, \quad \prod_{j=1}^{k} d_{j}<R
$$

Clearly, this implies that $\lambda_{\underline{d}} \neq 0$ only if

$$
\begin{equation*}
\left(d_{i}, d_{j}\right)=1, \quad\left(d_{i}, U\right)=1, \quad \forall i \neq j \tag{4.7}
\end{equation*}
$$

Adapting Maynard's method, Thorner proved the following result. We continue to use notation established in this section.

Proposition 4.4 ([13, Propositions 4 and 5]). Let $\mathcal{P}$ have level of distribution $\theta>0$, choose $D_{0}=\log \log \log N$ and $R=N^{\frac{\theta}{2}-\delta}$ for some small fixed $\delta>0$. Then,

$$
\begin{aligned}
& S_{1}(N, \mathcal{P})=(1+o(1)) \frac{\operatorname{rad}\left(d_{K}\right) \phi(W)^{k} N(\log R)^{k}}{W^{k+1}} I_{k}(F) \\
& S_{2}(N, \mathcal{P})=(1+o(1)) \frac{|C|}{|G|} \frac{\phi\left(\operatorname{rad}\left(d_{K}\right)\right) \phi(W)^{k} N(\log R)^{k+1}}{W^{k+1} \log N} \sum_{i=1}^{k} J_{k}^{(i)}(F),
\end{aligned}
$$

where $I_{k}(F)$ and $J_{k}^{(i)}(F)$ are as given in Proposition 3.1.
For the purpose of this paper, we need asymptotic formulas for $S_{1}$ and $S_{2}$ which give the error term explicitly in terms of $D_{0}$. These can be proved in almost the same way as done by Thorner. In essence, he shows that $S_{1}(N, \mathcal{P})$ and $S_{2}^{(m)}(N, \mathcal{P})$ are "multiples" (in terms of $d_{K}$ ) of the familiar sums $S_{1}(N, \mathbb{P})$ and $S_{2}^{(m)}(N, \mathbb{P})$, respectively. The required estimates then follow from Maynard's results. However, for the sake of completeness and clarity, we include the proof below.

Proposition 4.5. Let $\mathcal{P}$ have a level of distribution $\theta>0$ and $D_{0}=C^{*}(k)$ be a sufficiently large constant. As before, set $R=N^{\frac{\theta}{2}-\delta}$ for some small fixed $\delta>0$. Then we have

$$
S_{1}(N, \mathcal{P})=\left(1+O\left(\frac{1}{D_{0}}\right)\right) \frac{\operatorname{rad}\left(d_{K}\right) \phi(W)^{k} N(\log R)^{k}}{W^{k+1}} I_{k}(F)
$$

and

$$
S_{2}(N, \mathcal{P})=\left(1+O\left(\frac{1}{D_{0}}\right)\right) \frac{|C|}{|G|} \frac{\phi\left(\operatorname{rad}\left(d_{K}\right)\right) \phi(W)^{k} N(\log R)^{k+1}}{W^{k+1} \log N} \sum_{i=1}^{k} J_{k}^{(i)}(F),
$$

where $I_{k}(F)$ and $J_{k}^{(i)}(F)$ are as given in Proposition 3.1.
Proof. We first consider the non-weighted sum $S_{1}(N, \mathcal{P})$.

## The estimate for $S_{1}(N, \mathcal{P})$

Expanding the square and interchanging summation gives

$$
S_{1}(N, \mathcal{P})=\sum_{\underline{d}, \underline{e}} \lambda_{\underline{d}} \lambda_{\underline{e}} \sum_{\substack{n \sim N \\ n \equiv u_{0}(\bmod U) \\[\underline{d}, e] \mid \underline{n}}} 1 .
$$

It is clear from (4.7) that $U$ is co-prime to each $\left[d_{i}, e_{i}\right], 1 \leq i \leq k$. Note that if a prime $p$ divides $\left[d_{i}, e_{i}\right],\left[d_{j}, e_{j}\right]$ for some $i \neq j$, then $p$ must divide both $n+h_{i}$ and $n+h_{j}$. This means that $p$ is a factor of $h_{j}-h_{i}$ and hence of $W$. As this contradicts the second co-primality condition of (4.7), we must have that the $\left[d_{i}, e_{i}\right]$ 's are mutually pairwise co-prime.

As all the moduli in the inner sum above are co-prime to each other, using the Chinese remainder theorem, the congruence conditions in this sum can be written as a single congruence condition $*$ (say) modulo the product $q=U[\underline{d}, \underline{e}]$. This gives

$$
S_{1}(N, \mathcal{P})=\sum_{\underline{d}, \underline{e}} \lambda_{\underline{d}} \lambda_{\underline{e}} \sum_{\substack{n \sim N \\ n \equiv *(\bmod q)}} 1=\frac{N}{U} \sum_{\underline{d}, \underline{e}} \frac{\lambda_{\underline{d}} \lambda_{\underline{e}}}{\prod_{i=1}^{k}\left[d_{i}, e_{i}\right]}+O\left(\sum\left|\lambda_{\underline{d}}\right|\left|\lambda_{\underline{e}}\right|\right),
$$

where the final sum is over all tuples such that $[\underline{d}, \underline{e}]$ is square-free and co-prime to $U$. Since $\lambda_{\underline{d}} \lambda_{\underline{e}}=0$ if a prime $p$ divides $\left(U,\left[d_{i}, e_{i}\right]\right)$ for some $i$, one may drop the requirement that $U$ is relatively prime to each $\left[d_{i}, e_{i}\right]$. To ease notation, one may put $\lambda_{\max }=\sup _{\underline{d}}\left|\lambda_{\underline{d}}\right|$, and [7, Lemma 5.1] yields that

$$
S_{1}(N, \mathcal{P})=\frac{N}{U} \sum_{\underline{d}, \underline{e}} \frac{\lambda_{\underline{d}} \lambda_{\underline{e}}}{\prod_{i=1}^{k}\left[d_{i}, e_{i}\right]}+O\left(\lambda_{\max }^{2} R^{2}(\log R)^{2 k}\right)
$$

This expression for $S_{1}(N, \mathcal{P})$ shows that this sum is the same as $S_{1}(N, \mathbb{P})$ except that $W$ is now replaced by $U=W / \operatorname{rad}\left(d_{k}\right)$. Hence, the desired estimate for $S_{1}(N, \mathcal{P})$ follows directly from the corresponding asymptotic formula of Proposition 3.1.

## The estimate for $S_{2}(N, \mathcal{P})$

For each $m$, expanding the square and interchanging summation in the expression for $S_{2}^{(m)}(N, \mathcal{P})$ gives

$$
S_{2}^{(m)}(N, \mathcal{P})=\sum_{\underline{d}, \underline{e}} \lambda_{\underline{d}} \lambda_{\underline{e}} \sum_{\substack{n \sim N \\ n \equiv u_{0} \bmod U \\[\underline{d}, e] \mid \underline{n}}} \chi_{\mathcal{P}}\left(n+h_{m}\right) .
$$

Reasoning as in the case of $S_{1}(N, \mathcal{P})$, we can write the inner sum as a sum running over a single residue class $a_{m}$ modulo $q=U[\underline{d}, \underline{e}]$. Writing $n+h_{m}$ as $n^{\prime}$, this gives

$$
S_{2}^{(m)}(N, \mathcal{P})=\sum_{\underline{d}, \underline{e}} \lambda_{\underline{d}} \lambda_{\underline{e}} \sum_{\substack{n^{\prime} \sim N \\ n^{\prime} \equiv a_{m}+h_{m}(\bmod q)}} \chi_{\mathcal{P}}\left(n^{\prime}\right)
$$

From the construction of the residue class $a_{m}$, one has

$$
a_{m} \equiv u_{0} \quad \bmod U \quad \text { and } \quad a_{m} \equiv-h_{i} \quad\left(\bmod \left[d_{i}, e_{i}\right]\right) \quad \text { for all } 1 \leq i \leq k
$$

As $u_{0}$ is the chosen integer satisfying $\left(\prod_{i=1}^{k}\left(u_{0}+h_{i}\right), U\right)=1$, we have,

$$
\left(a_{m}+h_{m}, U\right)=1
$$

Note that $a_{m}+h_{m} \equiv h_{m}-h_{i}\left(\bmod \left[d_{i}, e_{i}\right]\right)$. Since it follows from the support of $\lambda_{\underline{d}}$ that $\left(h_{m}-h_{i},\left[d_{i}, e_{i}\right]\right)=1$ for all $i \neq m$, we obtain that $\left(a_{m}+h_{m}, q\right)=\left[d_{m}, e_{m}\right]$. We first consider the case $\left[d_{m}, e_{m}\right] \neq 1$. As $\left[d_{m}, e_{m}\right]$ divides $n^{\prime}$, we see that the summand $\chi_{\mathcal{P}}\left(n^{\prime}\right)$ survives if and only if $n^{\prime}=\left[d_{m}, e_{m}\right]$ is a prime in our Chebotarev set $\mathcal{P}$. Observing that $n^{\prime}>N$ and $\left[d_{m}, e_{m}\right]<R^{2} \leq N^{\theta-\delta}$ due to the support of the $\lambda_{\underline{d}}$ 's, we conclude that it is not possible to have $n^{\prime}=\left[d_{m}, e_{m}\right]$.

Hence, the inner sum only contributes when $d_{m}=e_{m}=1$, in which case, it is given by

$$
\frac{\pi_{\mathcal{P}}(2 N)-\pi_{\mathcal{P}}(N)}{\phi(q)}+O(E(N, q, \mathcal{P}))
$$

where

$$
E(N, q, \mathcal{P})=1+\max _{(a, q)=1}\left|\sum_{\substack{n \sim N \\ n \equiv a \bmod q}} \chi_{\mathcal{P}}(n)-\frac{1}{\phi(q)} \sum_{n \sim N} \chi_{\mathcal{P}}(n)\right|
$$

Recall that

$$
\pi_{\mathcal{P}}(x)=\sum_{n \leq x} \chi_{\mathcal{P}}(n)
$$

Thus, we have obtained

$$
\begin{aligned}
S_{2}^{(m)}(N, \mathcal{P})= & \frac{\pi_{\mathcal{P}}(2 N)-\pi_{\mathcal{P}}(N)}{\phi(U)} \sum_{\substack{\underline{d, e} \\
d_{m}=e_{m} \\
=1}} \frac{\lambda_{\underline{d}} \lambda_{\underline{e}}}{\phi([\underline{d}, \underline{e}])} \\
& +O\left(\sum_{\sum_{d_{m}}^{\underline{d}, \underline{e}}=e_{m}=1} \lambda_{\max }^{2} E(N, q, \mathcal{P})\right)
\end{aligned}
$$

where $q=U[\underline{d}, \underline{e}]$, and the sums are over tuples $\underline{d}, \underline{e}$ such that $[\underline{d}, \underline{e}]$ is square-free and co-prime to $U$.

We now proceed by first dealing with the error term. From the support of $\lambda_{\underline{d}}$, it is clear that one only needs to consider square-free $q$ with $q<R^{2} U$ and $\left(q, d_{K}\right) \stackrel{\underline{d}}{=} 1$.

Given a square-free integer $r$, it is easy to see that there are at most $\tau_{3 k}(r)$ choices of integers $d_{1}, \ldots, d_{k}, e_{1}, \ldots, e_{k}$ for which $U[\underline{d}, \underline{e}]=r$. Thus, the error term is of the order of

$$
\lambda_{\max }^{2} \sum_{\substack{r<R^{2} U \\\left(r, d_{K}\right)=1}} \mu^{2}(r) \tau_{3 k}(r) E(N, r, \mathcal{P})
$$

By the Cauchy-Schwarz inequality, the trivial bound $E(N, q, \mathcal{P}) \ll \frac{N}{q}+1$, and the assumption that $\mathcal{P}$ has a level of distribution $\theta$, this term contributes
$\ll \lambda_{\max }^{2}\left(\sum_{\substack{r<R^{2} U \\\left(r, d_{K}\right)=1}} \mu^{2}(r) \tau_{3 k}^{2}(r) \frac{N}{r}\right)^{\frac{1}{2}}\left(\sum_{\substack{r<R^{2} U \\\left(r, d_{K}\right)=1}} \mu^{2}(r) E(N, r, \mathcal{P})\right)^{\frac{1}{2}} \ll \lambda_{\max }^{2} \frac{N}{(\log N)^{A}}$,
for any $A>0$. This gives for any fixed $A>0$,

$$
S_{2}^{(m)}(N, \mathcal{P})=\frac{\pi_{\mathcal{P}}(2 N)-\pi_{\mathcal{P}}(N)}{\phi(U)} \sum_{\substack{\underline{d}, \underline{e} \\ d_{m}=e_{m}=1}} \frac{\lambda_{\underline{d}} \lambda_{\underline{e}}}{\phi([\underline{d}, \underline{e}])}+O\left(\lambda_{\max }^{2} \frac{N}{(\log N)^{A}}\right)
$$

where the sum is over tuples such that $[\underline{d}, \underline{e}]$ is square-free and relatively prime to $U$. We also note that the implicit constant above depends on the field $K$. Finally, the Chebotarev density theorem implies that

$$
\pi_{\mathcal{P}}(2 N)-\pi_{\mathcal{P}}(N)=\frac{|C|}{|G|} \frac{N}{\log N}+O\left(\frac{N}{(\log N)^{2}}\right)
$$

with the implicit constant depending on $K$. From this discussion, it is clear that $S_{2}^{(m)}(N, \mathcal{P})$ is the same as $S_{2}^{(m)}(N, \mathbb{P})$, with $W$ replaced by $U$, up to a factor of $|G| /|C|$. The required asymptotic formula for $S_{2}^{(m)}(N, \mathcal{P})$, and hence for $S_{2}(N, \mathcal{P})$ now follows from the corresponding expression for $S_{2}(N, \mathbb{P})$ in Proposition 3.1. This completes the proof.

With asymptotic formulas for $S_{1}(N, \mathcal{P})$ and $S_{2}(N, \mathcal{P})$ in place, we would now like to determine the optimum value of $\rho$ for which the inequality

$$
S_{2}(N, \mathcal{P})-\rho S_{1}(N, \mathcal{P})>0
$$

holds. This is done in the following proposition.
Proposition 4.6. Let $\mathcal{H}=\left\{h_{1}, \ldots, h_{k}\right\}$ be an admissible set, and let $\mathcal{P}$ have level of distribution $\theta>0$. Let $\mathcal{S}_{k}$ denote the set of Riemann integrable functions $F$ supported on the simplex $\mathcal{R}_{k}$, such that $I_{k}(F)$ and $J_{K}^{(i)}(F)$ do not vanish for $1 \leq$ $i \leq k$. Set

$$
M_{k}=\sup _{F \in S_{k}} \frac{\sum_{i=1}^{k} J_{k}^{(i)}(F)}{I_{k}(F)}, \quad r_{k}=\left\lceil\frac{|C| \phi\left(d_{K}\right) \theta M_{k}}{2|G| d_{K}}\right\rceil .
$$

Then there are infinitely many integers $n$ such that at least $r_{k}$ elements of $\left\{n+h_{i}\right\}_{i=1}^{k}$ are prime. In particular, if $p_{n}$ denotes the nth prime in $\mathcal{P}$, we have

$$
\liminf _{n}\left(p_{n+r_{k}-1}-p_{n}\right) \leq \max _{1 \leq i, j \leq k}\left|h_{i}-h_{j}\right| .
$$

Proof. Recall that $R=N^{\frac{\theta}{2}-\delta}$ for some small $\delta$. For this $\delta>0$, by the definition of $M_{k}$, there is a Riemann integrable function $G$ with the required support, such that $\sum_{i=1}^{k} J^{(i)}(G)>\left(M_{k}-\delta\right) I_{k}(G)$. Since $G$ is Riemann integrable, there exists a smooth function $F$ with the required support, such that

$$
\sum_{i=1}^{k} J^{(i)}(F)>\left(M_{k}-2 \delta\right) I_{k}(F)
$$

Using this smooth function $F$ for the choice of $\lambda_{\underline{d}}$, Proposition 4.5 allows us to deduce that $S(N, \rho, \mathcal{P})=S_{2}(N, \mathcal{P})-\rho S_{1}(N, \mathcal{P})$ is bounded below by

$$
\begin{equation*}
\left(1+O\left(\frac{1}{D_{0}}\right)\right) \frac{\phi(W)^{k}}{W^{k}} \frac{N(\log R)^{k}}{U} I_{k}(F)\left(\frac{|C| \phi\left(d_{K}\right)}{|G| d_{K}}\left(\frac{\theta}{2}-\delta\right)\left(M_{k}-2 \delta\right)-\rho\right) . \tag{4.8}
\end{equation*}
$$

Note that here we have used $U=W / \operatorname{rad}\left(d_{K}\right)$ as well as the identity

$$
\frac{\phi\left(\operatorname{rad}\left(d_{K}\right)\right)}{\operatorname{rad}\left(d_{K}\right)}=\frac{\phi\left(d_{K}\right)}{d_{K}}
$$

For the above expression to be positive, the term in parenthesis must be positive. Choose

$$
\rho=\frac{|C| \phi\left(d_{K}\right) \theta M_{k}}{2|G| d_{K}}-\epsilon
$$

for some small $\epsilon>0$. Then by choosing $\delta$ sufficiently small (depending on $\epsilon$ ), one can ensure that $S(N, \rho, \mathcal{P})>0$ for all large $N$. This means that there are infinitely many integers $n$ for which at least $\lfloor\rho+1\rfloor$ elements of $\left\{n+h_{i}\right\}_{i=1}^{k}$ are prime. Since we have

$$
\lfloor\rho+1\rfloor=\left\lceil\frac{|C| \phi\left(d_{K}\right) \theta M_{k}}{2|G| d_{K}}\right\rceil
$$

for $\epsilon$ sufficiently small, the result follows.
For the above result to be effective, we need a good lower bound for $M_{k}$. For this purpose, we utilize the bound

$$
\begin{equation*}
M_{k}>\log k-c \tag{4.9}
\end{equation*}
$$

for all $k \geq c$, for some absolute constant $c$. This is given in [12, Theorem 23].
With all this in place, we obtain the following result. As mentioned in the remark following the proof of this theorem, this gives better bounds than those previously obtained, for $m$-gaps between primes satisfying given Chebotarev conditions.

Theorem 4.7. Let $\mathcal{P}=\mathcal{P}(K, C)$ be a Chebotarev set having level of distribution $\theta$. Let $m$ be a natural number and $\mathcal{H}=\left\{h_{1}, \ldots, h_{k}\right\}$ be an admissible set of $k$ distinct non-negative integers, where

$$
k \geq\left\lceil\kappa \exp \left(\frac{2|G| d_{K} m}{|C| \phi\left(d_{K}\right) \theta}\right)\right\rceil,
$$

for a sufficiently large absolute constant $\kappa$. Then, there are infinitely many $n$ such that at least $m+1$ of the $n+h_{i}$ 's are in $\mathcal{P}$. In particular, we obtain

$$
\begin{equation*}
\liminf _{n \rightarrow \infty}\left(p_{n+m}-p_{n}\right) \leq c_{0}\left(\frac{2|G| d_{K} m}{|C| \phi\left(d_{K}\right) \theta}\right) \exp \left(\frac{2|G| d_{K} m}{|C| \phi\left(d_{K}\right) \theta}\right), \tag{4.10}
\end{equation*}
$$

for a sufficiently large absolute constant $c_{0}$.
Proof. To obtain at least $m+1$ primes among $n+h_{i}$ 's, we need $r_{k}$ of Proposition 4.6 to satisfy $r_{k} \geq m+1$. Using the expression for $r_{k}$ in the above mentioned proposition, we need to find $k$ such that

$$
\frac{|C| \phi\left(d_{K}\right) \theta M_{k}}{2|G| d_{K}}>m
$$

holds. Using the lower bound (4.9) for $M_{k}$, it suffices to find $k$ satisfying the inequality

$$
\log k-c>\frac{2|G| d_{K} m}{|C| \phi\left(d_{K}\right) \theta}
$$

From this, it is clear that the required lower bound for $k$ holds, with the (absolute) constant $\kappa$ only depending on the absolute constant $c$. As given in [12, Theorem 17], the minimal diameter of an admissible set of size $k$ is of the order of $k \log k$. Choosing

$$
k=\left\lceil\kappa \exp \left(\frac{2|G| d_{K} m}{|C| \phi\left(d_{K}\right) \theta}\right)\right\rceil,
$$

yields the desired bound for $\liminf _{n \rightarrow \infty}\left(p_{n+m}-p_{n}\right)$.
Remark 4.8. In our result, $\theta$ is the level of distribution of $\mathcal{P}$, satisfying the bound

$$
\theta<\min \left(\frac{1}{\eta^{*}}, \frac{1}{2}\right),
$$

with $\eta^{*}$ as defined in (4.2). The above theorem improves upon the work of Thorner, who uses $\theta=\frac{2}{|G|}-\delta$ and Maynard's lower bound for $M_{k}$ to obtain (c.f. [13, Remark 1]),

$$
\liminf _{n \rightarrow \infty}\left(p_{n+m}-p_{n}\right) \ll m^{3} \exp \left(\frac{|G|^{2} d_{K} m}{|C| \phi\left(d_{K}\right)}\right)
$$

It is clear that when $|G|$ is large, the improvement in the level of distribution $\theta$ does matter.

The case $m=1$ in the above theorem gives us the following improved bound for gaps between primes satisfying certain Chebotarev conditions.

Corollary 4.9. Let $\mathcal{P}=\mathcal{P}(K, C)$ be a Chebotarev set having level of distribution $\theta$. There exist infinitely many pairs of distinct primes $p_{1}, p_{2} \in \mathcal{P}$ such that

$$
\left|p_{1}-p_{2}\right| \leq c_{0}\left(\frac{2|G| d_{K}}{|C| \phi\left(d_{K}\right) \theta}\right) \exp \left(\frac{2|G| d_{K}}{|C| \phi\left(d_{K}\right) \theta}\right)
$$

for a sufficiently large absolute constant $c_{0}$.
Remark 4.10. This improves upon the result of Thorner, who obtains the bound

$$
825\left(\frac{|G|^{2} d_{K}}{|C| \phi\left(d_{K}\right)}\right)^{3} \exp \left(\frac{|G|^{2} d_{K}}{|C| \phi\left(d_{K}\right)}\right)
$$

for gaps between primes in a given Chebotarev set $\mathcal{P}(K, C)$. It is possible to compute explicitly the value of the absolute constant $c_{0}$ appearing in our bound, but for the sake of conceptual clarity, we do not do so here.

## 5. Arithmetic Progressions of Chebotarev Primes

In this section, we generalize the method of Pintz to primes satisfying Chebotarev conditions. Our exposition is self-contained and simplified. In the spirit of Pintz's work, we would like to neglect all the weights

$$
\omega_{n}=\left(\sum_{\underline{d} \mid \underline{n}} \lambda_{\underline{d}}\right)^{2}
$$

in the sums $S_{1}(N, \mathcal{P}), S_{2}(N, \mathcal{P})$, for which there exists $1 \leq i \leq k$, such that $n+h_{i}$ has a "small" prime factor $p$. In order to make this more precise, we first prove the following lemma. As this lemma lies at the crux of the method and is of interest in its own right, we give a detailed and lucid proof.

Lemma 5.1. For any $1 \leq j \leq k$ and any prime $p>D_{0}$ with $\frac{\log p}{\log R}<\epsilon$, for some sufficiently small $\epsilon>0$, we have as $R \rightarrow \infty$,

$$
S_{1, p}^{(j)}:=\sum_{\substack{n \sim N \\ n \equiv u_{0}(\bmod U) \\ p \mid n+h_{j}}} \omega_{n} \ll \frac{(\log p)^{2}}{p(\log R)^{2}} \frac{N(\log R)^{k}}{U} .
$$

The implicit constant above depends only on $k$.

Proof. It is enough to show this for $j=1$. By choice of the weights $\omega_{n}$, we have

$$
S_{1, p}^{(1)}=\sum_{\substack{n \sim N \\ n \equiv u_{0}(\bmod U) \\ p \mid n+h_{1}}}\left(\sum_{\substack{\underline{d}<R \\ \underline{d} \mid \underline{n}}} \lambda^{\underline{d}}\right)^{2}
$$

Interchanging summation gives

$$
S_{1, p}^{(1)}=\sum_{\underline{d}, \underline{e}<R} \lambda_{\underline{d}} \lambda_{\underline{e}} \sum_{\substack{n \sim N \\ n \equiv u_{0}(\bmod U) \\ n \equiv-h_{j}\left(\bmod \left[d_{j}, e_{j}\right]\right) \forall j \\ n \equiv-h_{1}(\bmod p)}} 1 .
$$

A moment's reflection allows us to see that $p$ must be co-prime to $U$. Indeed, if $p$ divides $U$, then we get $u_{0} \equiv-h_{1}(\bmod p)$. This is a contradiction because the construction of $u_{0}$ implies in particular that $u_{0}+h_{1}$ is co-prime to all primes dividing $U$. Similarly one can show that $p$ must be co-prime to all $\left[d_{j}, e_{j}\right]$ for $2 \leq$ $j \leq k$. Moreover, the $\left[d_{j}, e_{j}\right]$ 's are themselves mutually co-prime and also co-prime to $U$. This means that one can apply the Chinese remainder theorem to the inner sum above to obtain a sum running over some residue class $a$ modulo the product $q=U\left[d_{1}, e_{1}, p\right]\left[d_{j}, e_{j}\right]$, where $\left[d_{1}, e_{1}, p\right]$ denotes the lcm of $\left[d_{1}, e_{1}\right]$ and $p$. Thus,

$$
S_{1, p}^{(1)}=\sum_{\underline{d}, \underline{e}<R} \lambda_{\underline{d}} \lambda_{\underline{e}} \sum_{\substack{n \sim N \\ n \equiv a(\bmod q)}} 1=\frac{N}{p U} \sum_{\underline{d}, \underline{e}<R} \frac{\lambda_{\underline{d}} \lambda_{\underline{e}}}{\frac{\left[d_{1}, e_{1}, p\right]}{p} \prod_{j=2}^{k}\left[d_{j}, e_{j}\right]}+O\left(\sum_{\underline{d}, \underline{e}<R} \lambda_{\max }^{2}\right) .
$$

As done while proving the asymptotic formula for $S_{1}(N, \mathcal{P})$ in Proposition 4.5, it can be shown that the error term above is of the order of $N /(\log N)^{A}$ for any $A>0$. We proceed to analyze the main term.

For any multiplicative function $f(n)$, it can be checked that $f([n, p]) / p$ is also a multiplicative function of $n$. For our fixed prime $p$, we define the multiplicative function $g(n)=[n, p] / p$. We also define the "Möbius inverse" of $g$, denoted $g^{\prime}$, by the equation $g^{\prime}(n)=\sum_{d \mid n} \mu(n / d) g(d)$. Then, observe that for any prime $\ell$, we have

$$
g(\ell)=\left\{\begin{array}{ll}
\ell & \text { if } \ell \neq p, \\
1 & \text { if } \ell=p,
\end{array} \quad g^{\prime}(\ell)= \begin{cases}\ell-1 & \text { if } \ell \neq p \\
0 & \text { if } \ell=p\end{cases}\right.
$$

Following the notation set by Pintz in [10], let us denote the sum in the main term for $S_{1, p}^{(1)}$ as $T_{p, 1}^{(1)}$, that is,

$$
T_{p, 1}^{(1)}=\sum_{\underline{d}, \underline{e}<R} \frac{\lambda_{\underline{d}} \lambda_{\underline{e}}}{\frac{\left[d_{1}, e_{1}, p\right]}{p} \prod_{j=2}^{k}\left[d_{j}, e_{j}\right]} .
$$

We diagonalize the quadratic form $T_{p, 1}^{(1)}$ following the procedure of the classical Selberg sieve to get

$$
\begin{aligned}
& T_{p, 1}^{(1)}=\sum_{\underline{d}, \underline{e}<R} \frac{\lambda_{\underline{d}} \lambda_{\underline{e}}}{} \\
&=\sum_{\underline{d}, \underline{e}<R} \frac{\lambda_{\underline{d}} \lambda_{\underline{e}}}{g\left(\left[d_{1}, e_{1}\right]\right) \prod_{j=2}^{k}\left[d_{j}, e_{j}\right]} \\
& g\left(d_{1}\right) g\left(e_{1}\right) \prod_{j=2}^{k} d_{j} e_{j}
\end{aligned}\left(\left(d_{1}, e_{1}\right)\right) \prod_{j=2}^{k}\left(d_{j}, e_{j}\right) \quad l
$$

$$
\begin{aligned}
& =\sum_{\underline{d}, \underline{e}<R} \frac{\lambda_{\underline{d}} \lambda_{\underline{e}}}{g\left(d_{1}\right) g\left(e_{1}\right) \prod_{j=2}^{k} d_{j} e_{j}} \sum_{\underline{r} \mid \underline{d}, \underline{e}} g^{\prime}\left(r_{1}\right) \prod_{j=2}^{k} \phi\left(r_{j}\right) \\
& =\sum_{\underline{r}} g^{\prime}\left(r_{1}\right) \prod_{j=2}^{k} \phi\left(r_{j}\right)\left(\sum_{\underline{d}, \underline{r} \mid \underline{d}} \frac{\lambda_{\underline{d}}}{g\left(d_{1}\right) \prod_{j=2}^{k} d_{j}}\right)^{2}
\end{aligned}
$$

Observe that in the sum above we may assume $p \nmid r_{1}$ because whenever $p$ divides $r_{1}$, the summand vanishes as $g^{\prime}(p)=0$. For $\underline{r}$ such that $p \nmid r_{1}$, we make the change of variable

$$
\omega_{\underline{r}}=\mu(\underline{r}) g^{\prime}\left(r_{1}\right) \prod_{r=2}^{k} \phi\left(r_{j}\right) \sum_{\underline{r} \mid \underline{d}} \frac{\lambda_{\underline{d}}}{g\left(d_{1}\right) \prod_{j=1}^{k} d_{j}},
$$

so that we have the convenient expression

$$
\begin{equation*}
T_{p, 1}^{(1)}=\sum_{\underline{r}, p \nmid r_{1}} \frac{\mu(\underline{r})^{2}}{g^{\prime}\left(r_{1}\right) \prod_{j=2}^{k} \phi\left(r_{j}\right)}\left(\omega_{\underline{r}}\right)^{2} . \tag{5.1}
\end{equation*}
$$

Recall that Maynard's choice of parameters $\lambda_{\underline{d}}$ in terms of the test function $F$ corresponds to the choice

$$
\begin{equation*}
y_{\underline{r}}=F\left(\frac{\log r_{1}}{\log R}, \ldots, \frac{\log r_{k}}{\log R}\right) \tag{5.2}
\end{equation*}
$$

where

$$
y_{\underline{r}}=\mu(\underline{r}) \phi(\underline{r}) \sum_{\underline{r} \mid \underline{d}} \frac{\lambda_{\underline{d}}}{\underline{d}} .
$$

As $\omega_{\underline{r}}$ and $y_{\underline{r}}$ differ only in terms of the functions being evaluated on the first component of the tuple, we try to relate them so as to obtain some information about $T_{p, 1}^{(1)}$. Using $g^{\prime}\left(r_{1}\right)=\phi\left(r_{1}\right)$, for $p \nmid r_{1}$ along with the definition of $g\left(d_{1}\right)$, depending on whether $d_{1}$ is a multiple of $p$ or not, we have

$$
\begin{aligned}
\omega_{\underline{r}} & =\mu(\underline{r}) \phi(\underline{r})\left(\sum_{\underline{r}|\underline{d}, p| d_{1}} \frac{\lambda_{\underline{d}}}{g\left(d_{1}\right) \prod_{j=2}^{k} d_{j}}+\sum_{\underline{r} \mid \underline{d}, p \nmid d_{1}} \frac{\lambda_{\underline{d}}}{g\left(d_{1}\right) \prod_{j=2}^{k} d_{j}}\right) \\
& =\mu(\underline{r}) \phi(\underline{r})\left(\sum_{\underline{r}|\underline{d}, p| d_{1}} \frac{p \lambda_{\underline{d}}}{\underline{d}}+\sum_{\underline{r} \mid \underline{d}, p \nmid d_{1}} \frac{\lambda_{\underline{d}}}{\underline{d}}\right)=\mu(\underline{r}) \phi(\underline{r})\left(\phi(p) \sum_{\underline{r}|\underline{d}, p| d_{1}} \frac{\lambda_{\underline{d}}}{\underline{d}}+\sum_{\underline{r} \mid \underline{d}} \frac{\lambda_{\underline{d}}}{\underline{d}}\right) \\
& =\mu(p) y_{p r_{1}, r_{2}, \ldots, r_{k}}+y_{\underline{r}} .
\end{aligned}
$$

Plugging this back into (5.1) and choosing $y_{\underline{r}}$ as in (5.2), as done by Maynard, we obtain

$$
T_{p, 1}^{(1)}=\sum_{\underline{r}<R, p \nmid r_{1}} \frac{\mu(\underline{r})^{2}}{\phi(\underline{r})}\left(y_{\underline{r}}-y_{p r_{1}, r_{2}, \ldots, r_{k}}\right)^{2}
$$

$$
=\sum_{\underline{r}, p \nmid r_{1}} \frac{\mu(\underline{r})^{2}}{\phi(\underline{r})}\left(F\left(\frac{\log r_{1}}{\log R}, \ldots, \frac{\log r_{k}}{\log R}\right)-F\left(\frac{\log r_{1}+\log p}{\log R}, \ldots, \frac{\log r_{k}}{\log R}\right)\right)^{2}
$$

keeping in mind that the function $F(\underline{t})$ is zero outside the simplex

$$
\mathcal{R}_{k}=\left\{\underline{t} \in[0,1]^{k} \mid \sum t_{i} \leq 1\right\} .
$$

We recall that Maynard's choice of the test function $F$ is given by

$$
F(\underline{t})=\left\{\begin{array}{lc}
\prod_{i=1}^{k} g\left(k t_{i}\right) & \text { if } \sum_{i=1}^{k} t_{i} \leq 1,  \tag{5.3}\\
0 & \text { otherwise }
\end{array}\right.
$$

for some smooth, compactly supported function $g$, depending only on $k$. As the function $g_{k}(t):=g(k t)$ is smooth, we have as $h \rightarrow 0$,

$$
\begin{equation*}
\left|g_{k}(t+h)-g_{k}(t)\right|=(1+o(1))\left|g_{k}^{\prime}(t)\right||h|=(1+o(1))\left|g^{\prime}(t)\right||k h| . \tag{5.4}
\end{equation*}
$$

Note the implicit constant above depends on $g$ and hence only on $k$. As $g$ and $g^{\prime}$ are smooth, compactly supported functions, they can both be bounded absolutely in terms of $k$.

Going back to the final expression for $T_{p, 1}^{(1)}$ obtained above, from the above discussion and the condition that $\log p / \log R$ can be made as small as necessary, we have

$$
\left(F\left(\frac{\log r_{1}}{\log R}, \ldots, \frac{\log r_{k}}{\log R}\right)-F\left(\frac{\log r_{1}+\log p}{\log R}, \ldots, \frac{\log r_{k}}{\log R}\right)\right)^{2} \leq C(k)\left(\frac{\log p}{\log R}\right)^{2},
$$

for all $R$ sufficiently large. Here $C(k)$ is a constant depending on the suprema of $|g(t)|$ and $\left|g^{\prime}(t)\right|$ in their support.

This gives

$$
T_{p, 1}^{(1)} \ll k\left(\frac{\log p}{\log R}\right)^{2} \sum_{\underline{r}<R, p \nmid r_{1}} \frac{\mu(\underline{r})^{2}}{\phi(\underline{(x)}} \ll k_{k}\left(\frac{\log p}{\log R}\right)^{2}(\log R)^{k},
$$

using the elementary estimate $\sum_{t \leq x} 1 / t=\log x+O(1)$.
Recalling that $S_{1, p}^{(1)}=\frac{N}{p U} T_{p, 1}^{(1)}$, we derive the bound

$$
S_{1, p}^{(1)} \ll_{k} \frac{(\log p)^{2}}{p(\log R)^{2}} \frac{N(\log R)^{k}}{U} .
$$

With this lemma in place, we can now estimate the contribution of $n$ 's having small prime factors to our sum $S_{1}(N, \mathcal{P})$.

Lemma 5.2. Given any $\epsilon(k)>0$, there exists a sufficiently small constant $c_{1}(k)$, such that

$$
S_{1}^{-}(N, \mathcal{P}):=\sum_{\substack { n \sim N \\
\begin{subarray}{c}{n=u_{0}(\bmod U) \\
P^{-}\left(\prod_{i=1}^{k}\left(n+h_{i}\right)\right)<n^{c_{1}(k)}{ n \sim N \\
\begin{subarray} { c } { n = u _ { 0 } ( \operatorname { m o d } U ) \\
P ^ { - } ( \prod _ { i = 1 } ^ { k } ( n + h _ { i } ) ) < n ^ { c _ { 1 } ( k ) } } }\end{subarray}} \omega_{n} \leq \epsilon(k) \frac{N(\log R)^{k}}{U},
$$

as $N \rightarrow \infty$.

Proof. Observe that the above sum of $S_{1}^{-}(N, \mathcal{P})$ runs over only those $n$ 's for which each of the $n+h_{i}$ 's have small prime factors. Clearly, this means

$$
S_{1}^{-}(N, \mathcal{P}) \leq \sum_{j=1}^{k} \sum_{p \leq(2 N)^{c_{1}(k)}} S_{1, p}^{(j)} \leq \sum_{p \leq(2 N)^{c_{1}(k)}} k S_{1, p}^{(1)}
$$

Letting $c_{1}(k)$ be sufficiently small, all primes $p<(2 N)^{c_{1}(k)}$ satisfy the conditions of the previous lemma. Applying the bound obtained in Lemma 5.1 to each such $p$, we see that

$$
S^{-}(N, \mathcal{P}) \leq c(k) \frac{N(\log R)^{k}}{U} \sum_{p \leq(2 N)^{c_{1}(k)}} \frac{(\log p)^{2}}{p(\log R)^{2}}
$$

as $R \rightarrow \infty$, where $c(k)$ is a constant depending only on $k$. Using the asymptotic formula

$$
\sum_{p \leq x} \frac{(\log p)^{2}}{p}=\frac{(\log x)^{2}}{2}+O(\log x)
$$

which can be obtained by partial summation, we get

$$
S^{-}(N, \mathcal{P}) \leq c(k) \frac{N(\log R)^{k}}{U} \frac{\left(c_{1}(k) \log N\right)^{2}}{(\log R)^{2}}
$$

By choosing $c_{1}(k)$ to be sufficiently small and noting that $R=N^{\frac{\theta}{2}-\delta}$, we obtain

$$
S_{1}^{-}(N, \mathcal{P}) \leq \epsilon(k)\left(\frac{N(\log R)^{k}}{U}\right)
$$

The expression for $S_{1}(N, \mathcal{P})$ from Proposition 4.5 suggests that one can neglect the contribution of $S_{1}^{-}(N, \mathcal{P})$ in the sum $S_{1}(N, \mathcal{P})$ if $\epsilon(k)$ is chosen appropriately. We will elucidate two consequences of this simple fact that play a crucial role in Pintz's strategy.

Note that for all $n$ satisfying

$$
\begin{equation*}
P^{-}\left(\prod_{i=1}^{k}\left(n+h_{i}\right)\right) \geq n^{c_{1}(k)} \tag{5.5}
\end{equation*}
$$

each $n+h_{i}$ has a bounded number of prime factors, with the explicit bound depending on $h_{i}$ as well as $c_{1}(k)$. From the definition of $\omega_{n}$, one then has

$$
\omega_{n}=\left(\sum_{d_{i} \mid n+h_{i} \forall i} \lambda_{\underline{d}}\right)^{2} \ll_{c_{1}(k), H} \lambda_{\max }^{2}
$$

where $\lambda_{\max }=\sup _{\underline{d}} \lambda_{\underline{d}}$ and $H$ is the maximum of the $\left|h_{i}\right|$ 's. From [7, Eqs. (5.9) and (6.3)], we have

$$
\lambda_{\max }<_{k} y_{\max }(\log R)^{k}<_{k}(\log R)^{k}
$$

where $y_{\max }=\sup _{\underline{\underline{r}}} y_{\underline{r}}$. Here we have used the fact that the choice of the test function $F=F_{k}$ is only dependent on $k$. Putting everything together, it follows that when
$n$ is an "almost prime" in the sense of (5.5), we have

$$
\begin{equation*}
\left(\sum_{d_{i} \mid n+h_{i} \forall i} \lambda_{\underline{d}}\right)^{2}<_{k, H}(\log R)^{2 k} \tag{5.6}
\end{equation*}
$$

This is an important point which will be useful later.
Additionally, Lemma 5.2 allows one to overlook the contribution to the sum $S_{2}(N, \mathcal{P})$ from those $n$ which are not of the form (5.5).

Lemma 5.3. Given any $\epsilon(k)>0$, there exists a sufficiently small constant $c_{1}(k)$, such that

$$
S_{2}^{-}(N, \mathcal{P}):=\sum_{\substack{n \sim N \\ n \equiv u_{0}(\bmod U) \\ P^{-}\left(\prod_{i=1}^{k}\left(n+h_{i}\right)\right)<n^{c_{1}(k)}}} \sum_{i=1}^{k} \chi_{\mathcal{P}}\left(n+h_{i}\right) \omega_{n} \leq \epsilon(k) \frac{N(\log R)^{k}}{U},
$$

as $N \rightarrow \infty$.

Proof. By the triangle inequality, we have

$$
\begin{aligned}
&\left|S_{2}^{-}(N, \mathcal{P})\right| \leq \sum_{\substack{n \sim N \\
n \equiv u_{0}(\bmod U) \\
P^{-}\left(\prod_{i=1}^{k}\left(n+h_{i}\right)\right)<n^{c_{1}(k)}}} \sum_{i=1}^{k}\left|\chi_{\mathcal{P}}\left(n+h_{i}\right)\right| \omega_{n} \\
& \leq \sum_{\substack{n \sim N \\
n \equiv u_{0}(\bmod U) \\
P^{-}\left(\prod_{i=1}^{k}\left(n+h_{i}\right)\right)<n^{c_{1}(k)}}} k \omega_{n},
\end{aligned}
$$

since $\chi_{\mathcal{P}}(n)$ is absolutely bounded by 1 for all $n$. The right-hand side above is simply $k S_{1}^{-}(N, \mathcal{P})$, which together with Lemma 5.2 , completes the proof.

We set

$$
S_{1}^{+}(N, \mathcal{P})=S_{1}(N, \mathcal{P})-S_{1}^{-}(N, \mathcal{P}), \quad S_{2}^{+}(N, \mathcal{P})=S_{2}(N, \mathcal{P})-S_{2}^{-}(N, \mathcal{P})
$$

From Lemmas 5.2 and 5.3 , we see that by choosing $c_{1}(k)$ sufficiently small, one has that $S_{1}^{-}(N, \mathcal{P})$ and $S_{2}^{-}(N, \mathcal{P})$ are both $<_{k}\left(N(\log R)^{k} / U\right)$. We now show that Theorem 4.7 goes through verbatim for natural numbers $n$ satisfying the additional hypothesis of being almost primes in the sense of (5.5).

Theorem 5.4. Let $\mathcal{P}=\mathcal{P}(K, C)$ be a Chebotarev set having a level of distribution $\theta$. Let $m$ be a natural number and $\mathcal{H}=\left\{h_{1}, \ldots, h_{k}\right\}$ be an admissible set of $k$ distinct non-negative integers, where

$$
k \geq\left\lceil\kappa \exp \left(\frac{2|G| d_{K} m}{|C| \phi\left(d_{K}\right) \theta}\right)\right\rceil
$$

for a sufficiently large absolute constant $\kappa$. Then, there are infinitely many $n$ such that at least $m+1$ of the $n+h_{i}$ 's are in $\mathcal{P}$ and moreover

$$
P^{-}\left(\prod_{i=1}^{k}\left(n+h_{i}\right)\right) \geq n^{c_{1}(k)}
$$

Proof. Recall from (4.8) that the difference $S_{2}(N, \mathcal{P})-\rho S_{1}(N, \mathcal{P})$ is bounded below by

$$
\left(1+O\left(\frac{1}{D_{0}}\right)\right) \frac{\phi(W)^{k}}{W^{k}} \frac{N(\log R)^{k}}{U} I_{k}(F)\left(\frac{|C| \phi\left(d_{K}\right)}{|G| d_{K}}\left(\frac{\theta}{2}-\delta\right)\left(M_{k}-2 \delta\right)-\rho\right) .
$$

Theorem 4.7 implies that for some fixed $m \in \mathbb{N}$, if $\rho$ is chosen to be $\rho_{m}$, where $\left\lfloor\rho_{m}+1\right\rfloor=m+1$, and $k$ satisfies

$$
\begin{equation*}
k \geq\left\lceil\kappa \exp \left(\frac{2|G| d_{K} m}{|C| \phi\left(d_{K}\right) \theta}\right)\right\rceil, \tag{5.7}
\end{equation*}
$$

then the term

$$
\frac{|C| \phi\left(d_{K}\right)}{|G| d_{K}}\left(\frac{\theta}{2}-\delta\right)\left(M_{k}-2 \delta\right)-\rho
$$

in parenthesis above is a positive constant, depending on the choice of $k$ and the field $K$. Indeed, this is exactly how the proof of Theorem 4.7 proceeded. Let us denote the above positive constant by $C(K, k)$. In other words, we have

$$
S_{2}(N, \mathcal{P})-\rho_{m} S_{1}(N, \mathcal{P}) \geq\left(1+O\left(\frac{1}{D_{0}}\right)\right) \frac{\phi(W)^{k}}{W^{k}} \frac{N(\log R)^{k}}{U} I_{k}(F) C(K, k)
$$

where the right-hand side is positive. By Lemmas 5.2 and 5.3 , choosing $c_{1}(k)$ to be sufficiently small, we can write

$$
S_{2}(N, \mathcal{P})=S_{2}^{+}(N, \mathcal{P})+S_{2}^{-}(N, \mathcal{P}) \leq S_{2}^{+}(N, \mathcal{P})+O_{k}\left(\frac{N(\log R)^{k}}{U}\right)
$$

and similarly for $S_{1}(N, \mathcal{P})$. This gives for $k$ chosen as in (5.7),

$$
\begin{aligned}
S_{2}^{+}(N, \mathcal{P})-\rho_{m} S_{1}^{+}(N, \mathcal{P}) \geq & \left(1+O\left(\frac{1}{D_{0}}\right)+O_{k}(1)\right) \\
& \times \frac{\phi(W)^{k}}{W^{k}} \frac{N(\log R)^{k}}{U} I_{k}(F) C(K, k)
\end{aligned}
$$

In particular, since the function $F$ (chosen as in (5.3)) depends only on $k$, we can write

$$
\begin{equation*}
S_{2}^{+}(N, \mathcal{P})-\rho_{m} S_{1}^{+}(N, \mathcal{P}) \gg_{k, K} \frac{\phi(W)^{k}}{W^{k}} \frac{N(\log R)^{k}}{U} \tag{5.8}
\end{equation*}
$$

It is here that the choice of $D_{0}$ as in Proposition 4.5, different from that chosen by Maynard is crucial. Recall that $W=\prod_{p<D_{0}} p$, and $D_{0}$ is chosen to be a sufficiently
large constant depending only on $k$. This means that the ratio $\phi(W)^{k} / W^{k}$ depends only on $k$ and can be absorbed into the implicit constant in (5.8). This gives

$$
\begin{equation*}
S_{2}^{+}(N, \mathcal{P})-\rho_{m} S_{1}^{+}(N, \mathcal{P}) \gg_{k, K} \frac{N(\log R)^{k}}{U} \tag{5.9}
\end{equation*}
$$

with the implicit constant depending on $k$ and the field $K$. In particular, if we have $S_{2}^{+}(N, \mathcal{P})>\rho_{m} S_{1}^{+}(N, \mathcal{P})$, then as in the key idea of the GPY approach, the inequalities

$$
\sum_{i=1}^{k} \chi_{\mathcal{P}}\left(n+h_{i}\right) \geq(m+1), \quad P^{-}\left(\prod_{i=1}^{k}\left(n+h_{i}\right)\right)<n^{c_{1}(k)}
$$

hold for some $n \sim N$. From (5.9), it is clear that this can be done for all $N$ sufficiently large, giving infinitely many such $n$ as needed.

The above proof can be used to derive the following result, which is in fact a stronger version of Theorem 5.4.

Theorem 5.5. Fix a Chebotarev set $\mathcal{P}=\mathcal{P}(K, C)$ having a level of distribution $\theta$. Let $m$ be a natural number and $\mathcal{H}=\left\{h_{1}, \ldots, h_{k}\right\}$ be an admissible set of $k$ distinct non-negative integers, where

$$
k \geq\left\lceil\kappa \exp \left(\frac{2|G| d_{K} m}{|C| \phi\left(d_{K}\right) \theta}\right)\right\rceil
$$

for a sufficiently large absolute constant $\kappa$. For $c_{1}(k)$ chosen to be sufficently small, let $S_{\mathcal{P}}(\mathcal{H})$ denote the set
$\left\{n \in \mathbb{N}\right.$ : at least $m+1$ of the $n+h_{i}$ 's are in $\left.\mathcal{P}, \quad P^{-}\left(\prod_{i=1}^{k}\left(n+h_{i}\right)\right) \geq n^{c_{1}(k)}\right\}$.
Then $\#\left\{n \leq x: n \in S_{\mathcal{P}}(\mathcal{H})\right\} \geq c_{\mathcal{P}}(k) x(\log x)^{-k}$, for some constant $c_{\mathcal{P}}(k)>0$.
Proof. Consider the number of elements in $S_{\mathcal{P}}(\mathcal{H})$ between $N$ and $2 N$, given by the sum

$$
\sum_{\substack{n \sim N \\ n \in S_{\mathcal{P}}(\mathcal{H})}} 1
$$

In this sum, we could attach to each $n \in S_{\mathcal{P}}(\mathcal{H})$ the weight

$$
\sum_{i=1}^{k} \chi_{\mathcal{P}}\left(n+h_{i}\right)-\rho_{m}
$$

where $\rho_{m}$ satisfies $\left\lfloor\rho_{m}+1\right\rfloor=m+1$. From the construction of $S_{\mathcal{P}}(\mathcal{H})$, it is evident that this weight is positive. It is also clear that it is bounded above by $k-m$. Hence, we can write

$$
\sum_{\substack{n \sim N \\ n \in S_{\mathcal{P}}(\mathcal{H})}} 1>_{k} \sum_{\substack{n \sim N \\ n \in S_{\mathcal{P}}(\mathcal{H})}}\left(\sum_{i=1}^{k} \chi_{\mathcal{P}}\left(n+h_{i}\right)-\rho_{m}\right)
$$

We would like to introduce sieve parameters into the sum on the right-hand side to make it more familiar. In order to do this, note that since (5.5) holds for any $n \in S_{\mathcal{P}}(\mathcal{H})$, we have the bound (5.6) for all such $n$. This can be restated as

$$
1>_{k, H} \frac{1}{(\log R)^{2 k}}\left(\sum_{\underline{d} \mid \underline{n}} \lambda_{\underline{d}}\right)^{2}
$$

for all $n \in S_{\mathcal{P}}(\mathcal{H})$. The dependence of the implicit constant upon the admissible set $\mathcal{H}$ can be thought of as a dependence on $k$. Combining this with the previous bound gives

$$
\sum_{\substack{n \sim N \\ n \in S_{\mathcal{P}}(\mathcal{H})}} 1>_{k} \frac{1}{(\log R)^{2 k}} \sum_{\substack{n \sim N \\ n \in S_{\mathcal{P}}(\mathcal{H})}}\left(\sum_{i=1}^{k} \chi_{\mathcal{P}}\left(n+h_{i}\right)-\rho_{m}\right)\left(\sum_{\underline{d} \underline{\underline{n}}} \lambda_{\underline{d}}\right)^{2}
$$

Continuing with this train of thought, we see that the sum appearing on the righthand side is greater than the order of $S_{2}^{+}(N, \mathcal{P})-\rho_{m} S_{1}^{+}(N, \mathcal{P})$, which was studied by us in the proof of Theorem 5.4. Using the estimate (5.9) for this difference, we obtain

$$
\sum_{\substack{n \sim N \\ n \in S_{\mathcal{P}}(\mathcal{H})}} 1>_{k, K} \frac{N}{(\log R)^{k}}
$$

thus completing the proof.

We are now ready to prove Theorem 1.5, which is a special application of the general result of Pintz stated in Theorem 1.2.

Proof of Theorem 1.5. We can construct the set $S_{\mathcal{P}}(\mathcal{H})$ as in Theorem 5.5. As this set is infinite and the number of $(m+1)$-element subsets of $\mathcal{H}$ are finite, there is a subset $\mathcal{H}^{\prime} \subseteq \mathcal{H}$ given by $\left\{h_{1}^{\prime}, \ldots, h_{m+1}^{\prime}\right\}$, such that the set
$S_{\mathcal{P}}\left(\mathcal{H}^{\prime}\right)=\left\{n \in \mathbb{N}: n+h_{j}^{\prime} \in \mathcal{P}\right.$ for all $\left.1 \leq j \leq m+1, P^{-}\left(\prod_{i=1}^{k}\left(n+h_{i}\right)\right) \geq n^{c_{1}(k)}\right\}$ satisfies the condition

$$
\begin{equation*}
\#\left\{n \leq x: n \in S_{\mathcal{P}}\left(\mathcal{H}^{\prime}\right)\right\} \geq c_{\mathcal{P}}^{\prime}(k) \frac{x}{(\log x)^{k}} \tag{5.10}
\end{equation*}
$$

for some constant $c_{\mathcal{P}}^{\prime}(k)>0$.
Thus, this set satisfies the hypotheses of Theorem 1.2. Applying Theorem 1.2 then yields $l$-term arithmetic progressions in $S_{\mathcal{P}}\left(\mathcal{H}^{\prime}\right)$ for every $l \in \mathbb{N}$. This means that there are infinitely many arbitrarily long progressions of primes $n+h_{1}^{\prime} \in \mathcal{P}$, such that $n+h_{j}^{\prime} \in \mathcal{P}$ for every $2 \leq j \leq m+1$.

We can obtain the extra condition that the primes $n+h_{i}^{\prime}$ occur "consecutively" in $\mathcal{P}$ as follows. Pick $\mathcal{H}^{\prime}$ to be an $(m+1)$-element subset of $\mathcal{H}$ with minimal diameter,
such that (5.10) holds. Consider all elements $h_{i} \in \mathcal{H} \backslash \mathcal{H}^{\prime}$ which can be added to the set $\mathcal{H}^{\prime}$ without increasing its diameter. Of these elements, we examine first those elements $h_{j_{0}}$ such that

$$
\#\left\{n \leq x: n \in S_{\mathcal{P}}(\mathcal{H}), n+h_{j_{0}} \in \mathcal{P}\right\}=o\left(x(\log x)^{-k}\right)
$$

as $x \rightarrow \infty$. In this case, we simply delete such $n$ 's from our set $S_{\mathcal{P}}(\mathcal{H})$ and apply Theorem 1.2 to the remaining set. On the other hand, there may be elements $h_{j_{0}}$ such that

$$
\#\left\{n \leq x: n \in S_{\mathcal{P}}(\mathcal{H}), n+h_{j_{0}} \in \mathcal{P}\right\} \gg\left(x(\log x)^{-k}\right)
$$

as $x \rightarrow \infty$. In this case, we consider the new admissible set $\left\{h_{1}^{\prime}, \ldots, h_{k}^{\prime}, h_{j_{0}}\right\}$. Since the diameter of this new set is at most the diameter of $\mathcal{H}^{\prime}$ by hypothesis, it is possible to pick a new $(m+1)$-element subset $\mathcal{H}^{\prime \prime}$ of this set, satisfying condition (5.10), with diam $\mathcal{H}^{\prime \prime}<\operatorname{diam} \mathcal{H}^{\prime}$. This contradicts the minimality of diam $\mathcal{H}^{\prime}$.

However, we may still have intermediate primes in $\mathcal{P}$ of the form $n+h$ for some $1 \leq h \leq H, h \notin \mathcal{H}$, such that when $h$ is added into the set $\mathcal{H}^{\prime}$, the diameter does not increase. Once again, for a given $h$, if there are only finitely many such $n$ 's, we can simply delete them from the set we are considering. If there are infinitely many such $n$ 's, then the set $\left\{h_{1}, \ldots, h_{k}, h\right\}$ must be admissible. We note that all those $n \in S_{\mathcal{P}}(\mathcal{H})$, for which $n+h$ is a prime, satisfy the inequality

$$
\begin{equation*}
P^{-}\left((n+h) \prod_{i=1}^{k}\left(n+h_{i}\right)\right)>n^{c_{1}(k)} \tag{5.11}
\end{equation*}
$$

As stated in [11, Eq. (2.20)], from standard Selberg sieve estimates, the number of $n \leq x$ such that $(5.11)$ holds is $O\left(x /(\log x)^{k+1}\right)$. As there are at most $H$ possibilities for $h$, we see that

$$
\#\left\{n \in S_{\mathcal{P}}(\mathcal{H}): n+h \text { is prime for some } h \notin \mathcal{H}\right\} \ll \frac{x H}{(\log x)^{k+1}}
$$

Hence, we can delete such $n$ from our set $S_{\mathcal{P}}(\mathcal{H})$ and still apply Theorem 1.2. This completes the proof.

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