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On generalizations of the Titchmarsh divisor problem

by

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1. Introduction and statement of results. The Titchmarsh divisor problem is a well-known problem in analytic number theory, concerned with the asymptotic behaviour of the summatory function of the number of divisors of shifted primes. To formulate this precisely, let a be a fixed integer and let $\tau(n)$ denote the number of positive divisors of n. In 1930, Titchmarsh [16] showed that

$$\sum_{p\le x}\tau(p-a)=O(x).$$

He also gave the following explicit asymptotic formula for this sum under the generalized Riemann hypothesis for Dirichlet L-functions:

$$\sum_{p\le x}\tau(p-a)=x\prod_{p\nmid a}\left(1+\frac{1}{p(p-1)}\right)\prod_{p\mid a}\left(1-\frac{1}{p}\right)+O\left(\frac{x\log\log x}{\log x}\right). \tag{1.1}$$

The above formula was first proved unconditionally by Linnik [13] via the dispersion method. Moreover, by applying the celebrated Bombieri–Vinogradov theorem, Halberstam [11] and Rodriquez [15] independently gave another proof.

Subsequently, Fouvry [10, Corollaire 2] as well as Bombieri, Friedlander, and Iwaniec [2, Corollary 1] showed that for any $A>1$,

$$\sum_{p\le x}\tau(p-a)=cx+c_1\operatorname{Li}(x)+O\left(\frac{x}{(\log x)^A}\right), \tag{1.2}$$

where c is the constant expressed by the double product in (1.1), c_1 is an effectively computable constant depending on a, the implied constant depends only on a and A, and $\operatorname{Li}(x)$ is the usual logarithmic integral.

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Over the years, various variants of the classical Titchmarsh divisor problem have been studied in the literature. For instance, in [1], Akbary and Ghioca formulated a generalization of this problem in the setting of abelian varieties. In order to describe this in more detail, let us set up some notation. Let $K/\mathbb{Q}$ be a Galois extension of number fields with Galois group G and absolute discriminant d_K. For every unramified prime p, let σ_p denote the Artin symbol at p. If C is a union of conjugacy classes of G, we let

$$\mathcal{P}(K,C) = \{p \in \mathbb{P} : p \text{ is unramified with } \sigma_p \subseteq C\}$$

denote the corresponding Chebotarev set of primes; here and later, $\mathbb{P}$ denotes the set of rational primes. Letting $a = 1$ for simplicity in the classical Titchmarsh divisor problem, we have

$$\sum_{p\le x} \tau(p-1) = \sum_{p\le x} \sum_{m|p-1} 1.$$

For a given prime $p \le x$, the inner sum above can be viewed as computing the number of $m \in \mathbb{N}$ such that p splits completely in the cyclotomic extension $\mathbb{Q}(\zeta_m)$, where ζ_m denotes a primitive mth root of unity. More concisely, letting $\mathcal{P}_m = \mathcal{P}(\mathbb{Q}(\zeta_m), \mathrm{id})$, we have

$$\sum_{p\le x} \tau(p-1) = \sum_{p\le x} \sum_{\substack{m \\ p\in\mathcal{P}_m}} 1.$$

This interpretation led Akbary and Ghioca [1] to formulate the following interesting generalization of the Titchmarsh divisor problem.

Problem (Generalized Titchmarsh divisor problem, version 1). *Let* $\mathcal{F} = \{\mathcal{F}_m : m \in \mathbb{N}\}$ *be a family of finite Galois extensions of* $\mathbb{Q}$. *For each* m, *let* $\mathcal{D}_m$ *be a union of conjugacy classes of* $\mathrm{Gal}(\mathcal{F}_m/\mathbb{Q})$ *and set* $\mathcal{D} = \{\mathcal{D}_m : m \in \mathbb{N}\}$. *Define*

$$\tau_{\mathcal{F},\mathcal{D}}(p) = \#\{m \in \mathbb{N} : p \in \mathcal{P}(\mathcal{F}_m, \mathcal{D}_m)\}.$$

Suppose that $\tau_{\mathcal{F},\mathcal{D}}(p) < \infty$ *for each prime* p. *What can one say about the behaviour of* $\sum_{p\le x} \tau_{\mathcal{F},\mathcal{D}}(p)$ *as* $x \to \infty$*?*

Subject to certain constraints on $\mathcal{D}_m$, Akbary and Ghioca [1] obtained some new results on this question.

In the present paper, we are motivated by a further generalization of the above problem. Henceforth, let $K/\mathbb{Q}$ be a fixed Galois extension of $\mathbb{Q}$ with Galois group G and absolute discriminant d_K. We let C denote a union of conjugacy classes of G, and let $\mathcal{P} = \mathcal{P}(K,C)$ be a fixed Chebotarev set.

We may formulate the following version of the generalized Titchmarsh divisor problem.

Problem (Generalized Titchmarsh divisor problem, version 2). *Define*

$$\tau^{K,C}_{\mathcal{F},\mathcal{D}}(p) = \#\{m \in \mathbb{N} : p \in \mathcal{P}_m := \mathcal{P}(\mathcal{F}_m, \mathcal{D}_m) \text{ and } p \in \mathcal{P}(K,C)\}. \tag{1.3}$$

What can one say about the behaviour of $\sum_{p\le x} \tau^{K,C}_{\mathcal{F},\mathcal{D}}(p)$ *as* $x \to \infty$*?*

Given $p \leq x$, this version of the Titchmarsh divisor problem not only counts all occurrences of p in the family $\{\mathcal{P}_m\}$ of Chebotarev sets, but also imposes the condition that p belongs to the *fixed* Chebotarev set $\mathcal{P}(K, C)$. Clearly, when $K = \mathbb{Q}$, this problem reduces to the previous version of the generalized Titchmarsh divisor problem.

The main result of our paper is related to version 2 of the generalized Titchmarsh divisor problem. In order to state our result, we set up some notation. We recall the notion of level of distribution for Chebotarev sets. Let $\mathcal{P}$ be a set of primes. We use the standard notation

$$\pi_{\mathcal{P}}(x) = \#\{p \in \mathcal{P} : p \leq x\},$$
$$\pi_{\mathcal{P}}(x; q, a) = \#\{p \in \mathcal{P} : p \leq x,\, p \equiv a \pmod{q}\}.$$

Note that $\pi_{\mathcal{P}}(x; q, a)$ is in fact $\pi_{\mathcal{P}_1}(x)$, where $\mathcal{P}_1$ denotes the set of primes in $\mathcal{P}$ which are unramified in $\mathbb{Q}(\zeta_q)/\mathbb{Q}$ with $\sigma_p = a$.

A Chebotarev set $\mathcal{P} = \mathcal{P}(K, C)$ is said to have *level of distribution* θ if there exists a natural number M such that

$$\sum_{\substack{q \leq \frac{x^\theta}{(\log x)^B} \\ (q,M)=1}} \max_{y \leq x} \max_{(a,q)=1} \left| \pi_{\mathcal{P}}(y; q, a) - \frac{\pi_{\mathcal{P}}(y)}{\phi(q)} \right| \ll_A \frac{x}{(\log x)^A} \tag{1.4}$$

for any $A > 0$. Roughly speaking, this measures the range of moduli q for which the primes in $\mathcal{P}$ are equidistributed in arithmetic progressions modulo q on average. For $K = \mathbb{Q}$, or equivalently $\mathcal{P} = \mathbb{P}$, we have $\pi_{\mathcal{P}}(x; q, a) = \pi(x; q, a)$. In this case, the well-known Bombieri–Vinogradov theorem asserts that the estimate (1.4) holds when $0 < \theta \leq 1/2$ and $M = 1$. Moreover, the Elliott–Halberstam conjecture predicts that the above estimate holds for all $0 < \theta < 1$.

We will be concerned with version 2 of the generalized Titchmarsh divisor problem for the special case when $\mathcal{F} = \{\mathcal{F}_m\}$ is a family of cyclotomic extensions of $\mathbb{Q}$ and the Chebotarev set $\mathcal{P}(K, C)$ has level of distribution $1/2$. Our main result is the following.

Theorem 1.1. *Let K be a Galois extension of $\mathbb{Q}$ with absolute discriminant d_K, let C be a union of conjugacy classes in $G = \mathrm{Gal}(K/\mathbb{Q})$, and $\mathcal{P} = \mathcal{P}(K, C)$ be the corresponding Chebotarev set of primes. Fix an integer $a \neq 0$ and let*

$$\mathcal{F} = \{\mathcal{F}_m = \mathbb{Q}(\zeta_m) : m \in \mathbb{N},\, (m, d_K) = 1\}$$

be a family of extensions of $\mathbb{Q}$. Set $\mathcal{D}_m = \{a\}$ if $(m, a) = 1$, and $\mathcal{D}_m = \emptyset$ otherwise. Suppose the Chebotarev set $\mathcal{P}(K, C)$ has level of distribution $1/2$

with $M = ad_K$. Then

$$\sum_{p\le x} \tau_{\mathcal{F},\mathcal{D}}^{K,C}(p) = \frac{|C|}{|G|} C_{ad_K} x + O\left(\frac{x\log\log x}{\log x}\right),$$

where the constant C_{ad_K} is given by the Euler product

$$C_{ad_K} = \prod_{p\nmid ad_K}\left(1+\frac{1}{p(p-1)}\right)\prod_{p\mid ad_K}\left(1-\frac{1}{p}\right). \tag{1.5}$$

We prove this result in Section 4.1 by adapting Halberstam's unconditional proof of the Titchmarsh divisor problem.

Theorem 1.1 roughly asserts that the asymptotic distribution of the divisors of $p-a$ is uniform over the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$. This direction of generalization has not been considered before. As mentioned in [14], it is expected that one can always pick $\theta = 1/2$ for the estimate (1.4), and thus Theorem 1.1 should hold for *all* Galois extensions $K/\mathbb{Q}$. However, achieving such a level of distribution is still out of reach. This brings to the forefront the question of whether equidistribution estimates of the type proved by Bombieri–Friedlander–Iwaniec [2, 3, 4] can be extended to primes satisfying Chebotarev conditions. This is a deep question which begs careful investigation, and we do not delve more into it here. However, we do demonstrate in Section 4.2 some examples of Chebotarev sets having level of distribution 1/2. In Lemma 4.2, we describe these examples in more generality in terms of the set of values of $\chi(1)$ for irreducible characters χ of $\mathrm{Gal}(K/\mathbb{Q})$. However, some specific non-abelian examples of interest for the generalized Titchmarsh divisor problem are given by the following corollary.

COROLLARY 1.2. *Fix an integer $a \neq 0$ and let the families $\{\mathcal{F}_m\}_{m\in\mathbb{N}}$ and $\{\mathcal{D}_m\}_{m\in\mathbb{N}}$ be as in Theorem 1.1. Let $K/\mathbb{Q}$ be a Galois extension with Galois group G and absolute discriminant d_K. Let C be a union of conjugacy classes in $G = \mathrm{Gal}(K/\mathbb{Q})$ and $\mathcal{P} = \mathcal{P}(K,C)$ be the corresponding Chebotarev set of primes. Assume G is isomorphic to S_3, A_4, S_4, or a generalized dihedral group* (1). *Then*

$$\sum_{\substack{p\le x\\ p\in\mathcal{P}}} \tau_{\mathcal{F},\mathcal{D}}^{K,C}(p) = \frac{|C|}{|G|} C_{ad_K} x + O\left(\frac{x\log\log x}{\log x}\right),$$

where the constant C_{ad_K} is given by the Euler product (1.5).

(1) A group G is said to be *generalized dihedral* if it admits a semidirect product $G = N \rtimes C_2$, where N is an abelian normal subgroup, and C_2 is a cyclic subgroup of order 2. In particular, if N is isomorphic to the cyclic group C_n of order n, then $G = D_{2n}$, the dihedral group of order $2n$.

In particular, in the special case when K is a cyclotomic extension $\mathbb{Q}(\zeta_r)$ of $\mathbb{Q}$, the Chebotarev condition $\sigma_p \subseteq C$ can be written as a congruence condition $p \equiv b \pmod r$ for some residue class b modulo r. Moreover, since the discriminant of the field $\mathbb{Q}(\zeta_r)$ is given by

$$d_{\mathbb{Q}(\zeta_r)} = \frac{(-1)^{\phi(r)/2} r^{\phi(r)}}{\prod_{p|r} p^{\phi(r)/p-1}}, \tag{1.6}$$

the condition $(p, d_{\mathbb{Q}(\zeta_r)}) = 1$ that is needed to ensure that the prime p is unramified reduces to the condition $(p, r) = 1$.

In this case, we obtain a stronger result than is implied by Theorem 1.1: the error terms are uniform for a certain range of the modulus r. We state this result as a corollary, since the proof is similar to the proof of Theorem 1.1.

COROLLARY 1.3. *Let $b \in \mathbb{Z}$, $r \in \mathbb{N}$, $r > 1$, with $(r, b) = 1$. Let $K = \mathbb{Q}(\zeta_r)$ and consider the conjugacy class $C = \{b\}$ in* $\mathrm{Gal}(K/\mathbb{Q})$. *Fix an integer $a \neq 0$ and let $D > 0$. Consider the family of extensions of $\mathbb{Q}$ given by*

$$\{\mathcal{F}_m = \mathbb{Q}(\zeta_m) : m \in \mathbb{N},\ (m, r) = 1\}.$$

Set $\mathcal{D}_m = \{a\}$ if $(m, a) = 1$, and $\mathcal{D}_m = \emptyset$ otherwise. Then, uniformly for $r \leq (\log x)^D$,

$$\sum_{p \leq x} \tau_{\mathcal{F},\mathcal{D}}^{K,C}(p) = \frac{C_{ar}}{\phi(r)} x + O\left(\frac{x \log\log x}{\log x}\right) \quad \text{as } x \to \infty.$$

Here C_{ar} is a constant depending only on a and r, given by replacing d_K by r in the Euler product (1.5).

This can be thought of as a Siegel–Walfisz type result for this version of the Titchmarsh divisor problem in arithmetic progressions, the uniformity of error terms being its special feature. This uniformity is a consequence of the fact that the Chebotarev conditions involved can be written as a single congruence condition using the Chinese remainder theorem. We give a sketch of the proof in Section 5.1.

We may also think of Corollary 1.3 as a variant of a related result that was obtained by Felix [6, Theorem 1.2]. We elaborate on this below. Setting

$$\tau(r, n) = \sum_{\substack{d|n \\ (d,r)=1}} 1, \tag{1.7}$$

we can restate Corollary 1.3 in a more concrete manner: uniformly for $r \leq (\log x)^D$,

$$\sum_{\substack{p \leq x \\ p \equiv b \,(\mathrm{mod}\, r)}} \tau(r, p-a) = \frac{C_{ar}}{\phi(r)} x + O\left(\frac{x \log\log x}{\log x}\right) \quad \text{as } x \to \infty. \tag{1.8}$$

Let $v_q(n)$ denote the highest power of q dividing n. If $r \mid p-a$, then it is easy to see that

$$\tau(r, p-a) = \tau\biggl(\frac{p-a}{\prod_{q \mid r} q^{v_q(p-a)}}\biggr).$$

Hence, if we put $b = a$ in (1.8), we see that for $(r, a) = 1$, we have

$$\sum_{\substack{p \le x \\ p \equiv a \,(\mathrm{mod}\, r)}} \tau\biggl(\frac{p-a}{\prod_{q \mid r} q^{v_q(p-a)}}\biggr) = \frac{C_{ar}}{\phi(r)} x + O\biggl(\frac{x \log\log x}{\log x}\biggr) \quad \text{as } x \to \infty.$$

Related sums and subsequent connections to Artin's conjecture on primitive roots were investigated by Felix [6, 7]. In particular, it is instructive to compare the above asymptotic formula with a result in [6]:

Theorem 1.4 ([6, Theorem 1.2]). *Fix an integer $a \neq 0$. Let $r \in \mathbb{N}$, $r > 1$, with $(r, a) = 1$. Let $A > 0$. Then, uniformly for $r \le (\log x)^{A+1}$,*

$$\sum_{\substack{p \le x \\ p \equiv a \,(\mathrm{mod}\, r)}} \tau\biggl(\frac{p-a}{r}\biggr) = \frac{r C_{ar}}{\phi(r)^2} x + O_{r,a}\biggl(\frac{x}{\log x}\biggr) + O_{a,A}\biggl(\frac{x}{(\log x)^A}\biggr)$$

as $x \to \infty$. Here C_{ar} is the same as in Corollary 1.3.

It is worth remarking here that in the proof of Corollary 1.3, the general case $b \neq a$ does not allow us to invoke equidistribution results of Bombieri–Friedlander–Iwaniec [2], which play a key role in the proof of [6, Theorem 1.2]. The error term in (1.8) is not optimal and is expected to be much smaller. In the absence of sharp error terms, it is natural to study the error *on average* over the modulus r. This leads us to ask whether we can have the following average result for this problem:

Problem (Bombieri–Vinogradov type estimate for a version of the Titchmarsh divisor problem in arithmetic progressions). *Fix integers $a, b \neq 0$. Does there exist $\lambda > 0$ such that*

$$\sum_{\substack{r \le x^\lambda \\ (r,b)=1}} \biggl| \sum_{\substack{p \le x \\ p \equiv b \,(\mathrm{mod}\, r)}} \tau(r, p-a) - \frac{C_{ar}}{\phi(r)} x \biggr| \ll_{a,b,\lambda,A} \frac{x}{(\log x)^A} \quad \textit{for any } A > 0?$$

This problem, as stated, seems to be still open. Motivated by the discussion preceding Theorem 1.4, one is led to consider such estimates for the related sum

$$\sum_{\substack{p \le x \\ p \equiv a \,(\mathrm{mod}\, r)}} \tau\biggl(\frac{p-a}{r}\biggr).$$

Alternatively, one may consider a version of this sum supported on prime powers, that is,

$$\sum_{\substack{n\le x\\ n\equiv a\,(\mathrm{mod}\,r)}} \Lambda(n)\tau\Big(\frac{n-a}{r}\Big).$$

A Bombieri–Vinogradov type estimate for such a sum has been proved by Fiorilli [8].

THEOREM 1.5 ([8, Theorem 2.4]). *Fix an integer $a\neq 0$ and let $\lambda<1/10$ and A be two positive real numbers. Then, for $R\le x^{\lambda}$,*

$$\sum_{\substack{r\le R\\ (r,a)=1}} \Big|\sum_{|a|/r<m\le x/r} \Lambda(rm+a)\tau(m)-\mathrm{M.T.}\Big| \ll_{a,A,\lambda} \frac{x}{(\log x)^{A}},$$

where the main term M.T. *is given by*

$$\frac{x}{r}\Big(C_1(a,r)\log x+2C_2(a,r)+C_1(a,r)\log\frac{(r')^2}{er}\Big),$$

with

$$C_1(a,r)=\frac{\zeta(2)\zeta(3)}{\zeta(6)}\prod_{p\mid a}\Big(1-\frac{p}{p^2-p+1}\Big)\prod_{p\mid r}\Big(1+\frac{p-1}{p^2-p+1}\Big),$$

$r'=\prod_{p\mid r}p$, and $C_2(a,r)$ given by

$$C_1(a,r)\Big(\gamma-\sum_{p}\frac{\log p}{p^2-p+1}+\sum_{p\mid a}\frac{p^2\log p}{(p-1)(p^2-p+1)}-\sum_{p\mid r}\frac{(p-1)p\log p}{p^2-p+1}\Big),$$

where γ is the Euler–Mascheroni constant.

In the final result of this paper, we extend the above theorem to the function $\tau_y(m)$ (the number of positive divisors d of m satisfying $d\le y$, where y is a parameter depending on the modulus r). We thus obtain the following analogue of Theorem 1.5, giving a Bombieri–Vinogradov type estimate for a modified Titchmarsh divisor problem involving a 'truncated' divisor function.

THEOREM 1.6. *Fix an integer $a\neq 0$ and let $0<\lambda<1/10$ and D,A be positive real numbers. Let $M=M(r,x)$ be an integer such that $1\le M(r,x)\le(\log x)^{D}$. Then for any $\epsilon>0$, $x^{\lambda+1/2+\epsilon}\le Q\le x$, $R=R(x)\le x^{\lambda}$, letting $y(r)=Q/rM$, we have*

$$\sum_{\substack{r\le R\\ (r,a)=1}} \Big|\sum_{|a|/r<m\le x/r} \Lambda(rm+a)\tau_{y(r)}(m)-\mathrm{M.T.}\Big| \ll_{a,A,D,\lambda} \frac{x}{(\log x)^{A}},$$

where $\tau_{y(r)}(m)$ denotes the number of divisors of m that are smaller than $y(r)$, the main term M.T. *is given by*

$$\frac{x}{r}\left(C_1(a,r)\log x + C_2(a,r) + C_1(a,r)\log\frac{(r')^2}{er} - \sum_{\substack{1\le s\le M\frac{x}{Q}\\(s,a)=1}} \frac{r}{\phi(sr)}\left(x - \frac{sQ}{M}\right)\right),$$

and r', $C_1(a,r)$ and $C_2(a,r)$ are exactly as given in Theorem 1.5.

Note that with $M = 1$ and $Q = x$ in the result above, we recover Theorem 1.5. The proof of Theorem 1.6 relies upon a refinement of an equidistribution estimate of Fiorilli [8, Proposition 5.1]. We give more details in Section 6.1.

2. Notation. Recall that we let $K/\mathbb{Q}$ be a fixed Galois extension of number fields with Galois group G and absolute discriminant d_K. For every unramified prime p, σ_p denotes the Artin symbol at p. We let C be a union of conjugacy classes of G, and let $\mathcal{P}(K,C)$ be the corresponding fixed Chebotarev set. We define the divisor function $\tau_K(n)$ with respect to K as

$$\tau_K(n) := \sum_{\substack{de=n\\(de,d_K)=1}} 1.$$

We note that if $K = \mathbb{Q}$, then $\tau_{\mathbb{Q}}(n)$ is the classical divisor function $\tau(n)$.

3. Preliminaries. We will require the following well-known result. We refer the reader to [5, Theorem 7.3.1] for a proof.

THEOREM 3.1 (The Brun–Titchmarsh inequality). *Let a and q be coprime integers and x a positive real number such that $q \le x^\theta$ for some $\theta < 1$. Then for any $\epsilon > 0$, there exists $x_\epsilon > 0$ such that*

$$\pi(x;q,a) \le \frac{(2+\epsilon)x}{\phi(q)\log(2x/q)} \quad \text{for all } x > x_\epsilon.$$

We also recall an elementary estimate (see for instance, [11, eq. (iii)])

$$\sum_{\substack{d\le x\\(d,\alpha)=1}} \frac{1}{\phi(d)} = C_\alpha \log x + O(1), \tag{3.1}$$

where the constant C_α is given by

$$C_\alpha = \prod_{p\nmid\alpha}\left(1 + \frac{1}{p(p-1)}\right)\prod_{p\mid\alpha}\left(1 - \frac{1}{p}\right). \tag{3.2}$$

4. A number field analogue of the Titchmarsh divisor problem. In this section, we prove Theorem 1.1 and Corollary 1.2. As will be seen, our approach to proving the former closely follows Halberstam's unconditional proof for the Titchmarsh divisor problem.

4.1. Proof of Theorem 1.1. In the setup of Theorem 1.1, it is easy to verify that

$$\sum_{p\le x}\tau^{K,C}_{\mathcal{F},\mathcal{D}}(p)=\sum_{\substack{p\le x\\ p\in\mathcal{P}}}\tau_K(p-a).$$

Let

$$\delta(n)=\begin{cases}1 & \text{if } n \text{ is a square,}\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\begin{aligned}\sum_{\substack{p\le x\\ p\in\mathcal{P}}}\tau_K(p-a)&=\sum_{\substack{p\le x\\ p\in\mathcal{P}}}\Bigg(2\sum_{\substack{d\mid p-a\\ d\le\sqrt{p-a}\\ (d,d_K)=1}}1-\delta(p-a)\Bigg)\\ &=2\sum_{\substack{d\le\sqrt{x-a}\\ (d,d_K)=1}}\sum_{\substack{p\le x\\ p\equiv a\,(\mathrm{mod}\,d)\\ p\in\mathcal{P}}}1+O(\sqrt{x}).\end{aligned}$$

Let us note that if $(d,a)\neq 1$, then the inner sum above is at most 1, so that we may impose the condition $(d,a)=1$ on the outer sum taking into account an error of $O(\sqrt{x})$. Thus

$$\sum_{\substack{a<p\le x\\ p\in\mathcal{P}}}\tau_K(p-a)=2\sum_{\substack{d\le\sqrt{x-a}\\ (d,ad_K)=1}}\pi_{\mathcal{P}}(x;d,a)\;+O(\sqrt{x}).$$

We split the sum as

$$\sum_{\substack{d\le\frac{\sqrt{x}}{(\log x)^B}\\ (d,ad_K)=1}}\pi_{\mathcal{P}}(x;d,a)+\sum_{\substack{\frac{\sqrt{x}}{(\log x)^B}\le d\le\sqrt{x-a}\\ (d,ad_K)=1}}\pi_{\mathcal{P}}(x;d,a). \tag{4.1}$$

For the first range of d, applying the estimate (1.4) gives

$$\begin{aligned}\sum_{\substack{d\le\frac{\sqrt{x}}{(\log x)^B}\\ (d,ad_K)=1}}\pi_{\mathcal{P}}(x;d,a)&=\sum_{\substack{d\le\frac{\sqrt{x}}{(\log x)^B}\\ (d,ad_K)=1}}\left(\pi_{\mathcal{P}}(x;d,a)-\frac{|C|\,\mathrm{Li}(x)}{|G|\phi(d)}\right)+\sum_{\substack{d\le\frac{\sqrt{x}}{(\log x)^B}\\ (d,ad_K)=1}}\frac{|C|\,\mathrm{Li}(x)}{|G|\phi(d)}\\ &=\frac{|C|}{|G|}\,\mathrm{Li}(x)\sum_{\substack{d\le\frac{\sqrt{x}}{(\log x)^B}\\ (d,ad_K)=1}}\frac{1}{\phi(d)}+O\left(\frac{x}{(\log x)^A}\right).\end{aligned}$$

For the second sum in (4.1), as $\log x\ll\log(2x/d)\ll\log x$ in this range of d, the Brun–Titchmarsh inequality yields

$$\sum_{\substack{\frac{\sqrt{x}}{(\log x)^B}\le d\le\sqrt{x}\\(d,ad_K)=1}}\pi_{\mathcal{P}}(x;d,a)\ll\frac{x}{\log x}\sum_{\substack{\frac{\sqrt{x}}{(\log x)^B}\le d\le\sqrt{x}\\(d,ad_K)=1}}\frac{1}{\phi(d)}\ll\frac{x\log\log x}{\log x}.$$

Thus, by the estimate (3.1), we have

$$\sum_{\substack{a<p\le x\\ \sigma_p\subseteq C\\(p,d_K)=1}}\tau_K(p-a)=\frac{|C|}{|G|}C_{ad_K}x+O\left(\frac{x\log\log x}{\log x}\right),$$

which concludes the proof.

4.2. Proof of Corollary 1.2. Our starting point is the following variant of the Bombieri–Vinogradov theorem, which was proved by M. R. Murty and V. K. Murty [14].

Theorem 4.1 ([14, Theorem 7.3]). *Let K be a Galois extension of $\mathbb{Q}$ with absolute discriminant d_K, let C be a union of conjugacy classes in $G=\mathrm{Gal}(K/\mathbb{Q})$, and $\mathcal{P}=\mathcal{P}(K,C)$ be the corresponding Chebotarev set of primes. Let*

$$d^*=\min_H\max_\chi\,[G:H]\chi(1),$$

where the minimum is over all subgroups H of G satisfying

- *$H\cap C\neq\emptyset$, and*
- (AC) *is true for H, i.e., all non-trivial L-functions attached to abelian twists of any irreducible character of H are entire,*

while the maximum runs over all irreducible characters of H. Define

$$\eta^*=\begin{cases}d^*-2 & \text{if } d^*\ge 4,\\ 2 & \text{if } d^*<4.\end{cases}$$

Then the average result (1.4) *holds for $M=d_K$ and $0<\theta\le 1/\eta^*$.*

For any finite group G, we let $\mathrm{cd}(G)=\{\chi(1):\chi\in\mathrm{Irr}(G)\}$. It is clear that if G is abelian, then $\mathrm{cd}(G)=\{1\}$. By Artin reciprocity, the mean estimate (1.4) holds for all $0<\theta\le 1/2$ if $K/\mathbb{Q}$ is an abelian Galois extension. In the light of this, we present a lemma.

Lemma 4.2. *With the same notation as above, if $\mathrm{cd}(G)=\{1,3\}$ or $\mathrm{cd}(G)\subseteq\{1,2,4\}$, then the average result* (1.4) *holds for all $0<\theta\le 1/2$ (with $M=d_K$). In particular, if G is a generalized dihedral group, then $K/\mathbb{Q}$ has level of distribution $1/2$.*

Proof. Assume that $\mathrm{cd}(G)=\{1,3\}$ or $\mathrm{cd}(G)\subseteq\{1,2,4\}$. In each case, we can choose p equal to 2 or 3 such that $\chi(1)$ is a power of p for every non-trivial character $\chi \in \operatorname{Irr}(G)$. By [12, Theorem 6.9], G admits an abelian normal p-complement N. This means that G/N is a p-group, and in particular is nilpotent. One can show that G is nearly nilpotent, that is, it admits a normal subgroup N, all of whose irreducible characters are of degree less than or equal to 2, such that G/N is nilpotent. By a result of the second-named author [17, Theorem 1.2], Langlands reciprocity holds for $K/\mathbb{Q}$. This means that for any irreducible character $\chi \in \operatorname{Irr}(G)$, there is a cuspidal automorphic representation π_χ such that the L-function attached to the abelian twist of χ by ψ is equal to the Rankin–Selberg L-function $L(s, \pi_\chi \times \psi)$, which is entire by Rankin–Selberg theory (unless the L-function is the Riemann zeta function).

Thus, $G = \operatorname{Gal}(K/\mathbb{Q})$ satisfies the two conditions required of the subgroup H in Theorem 4.1. This allows us to conclude that

$$d^* \leq \max_{\chi \in \operatorname{Irr}(G)} [G : G]\chi(1).$$

Since $\operatorname{cd}(G) = \{1, 3\}$ or $\operatorname{cd}(G) \subseteq \{1, 2, 4\}$, we obtain $d^* \leq 4$, which gives $\eta^* = 2$. Applying Theorem 4.1, we obtain level of distribution $1/2$ as needed.

Furthermore, we recall that if G is a generalized dihedral group, then it admits an abelian normal subgroup N of index 2. Thus, from [12, Theorem 6.22], we know $\operatorname{cd}(G) \subseteq \{1, 2\}$ and conclude the proof. ∎

We also remark that S_4 is of automorphic type, and that $\operatorname{cd}(S_4) = \{1, 2, 3\}$ (see, e.g, [17, Corollary 2.18]). Thus, we know that all S_3-, A_4-, and S_4-extensions have level of distribution $1/2$.

Applying Theorems 1.1 and 4.1, together with Lemma 4.2 and the above discussion, we deduce Corollary 1.2 immediately.

5. A version of the Titchmarsh divisor problem in arithmetic progression. In the setup of Corollary 1.3, one obtains the following sum which can be thought of as a version of the Titchmarsh divisor problem in arithmetic progressions:

$$\sum_{p \leq x} \tau_{\mathcal{F},\mathcal{D}}^{K,C}(p) = \sum_{\substack{p \leq x \\ \sigma_p \subseteq C \\ (p, d_{\mathbb{Q}(\zeta_r)}) = 1}} \tau_{\mathbb{Q}(\zeta_r)}(p - a) = \sum_{\substack{p \leq x \\ p \equiv b \,(\mathrm{mod}\, r)}} \tau_{\mathbb{Q}(\zeta_r)}(p - a). \tag{5.1}$$

Here we have used (1.6) to replace the condition $(p, d_{\mathbb{Q}(\zeta_r)}) = 1$ by $(p, r) = 1$. With this in hand, we may now commence the proof of Corollary 1.3.

5.1. Proof of Corollary 1.3. As in the proof of Theorem 1.1, we obtain

$$\sum_{\substack{p \leq x \\ p \equiv b \,(\mathrm{mod}\, r)}} \tau_{\mathbb{Q}(\zeta_r)}(p - a) = 2 \sum_{\substack{d \leq \sqrt{x - a} \\ (d, ar) = 1}} \pi(x; rd, c) + O\left(\frac{x}{\log x}\right),$$

where $c \pmod{rd}$ is uniquely determined by the congruence conditions $c \equiv a \pmod d$ and $c \equiv b \pmod r$. The residue class c is now no longer fixed and varies with d in the above sum. It is this subtle distinction that hinders us from invoking stronger equidistribution results that are available when c is fixed (cf. [2]). Instead, we now rely on the classical Bombieri–Vinogradov theorem. As done in the proof of Theorem 1.1, we split the above sum over d into a sum over $d \le \sqrt{x-a}/(\log x)^B$ and the remainder, for some $B > 0$ sufficiently large. Again, as in the proof of Theorem 1.1, the Brun–Titchmarsh theorem implies that the remainder sum is $\ll x(\log\log x)/\log x$. For the first sum, we have

$$\sum_{\substack{d \le \frac{\sqrt{x-a}}{(\log x)^B} \\ (d,ar)=1}} \pi(x; rd, c) = \sum_{\substack{d \le \frac{\sqrt{x-a}}{(\log x)^B} \\ (d,ar)=1}} \frac{\mathrm{Li}(x)}{\phi(rd)} + O\Bigg(\sum_{d \le \frac{\sqrt{x-a}}{(\log x)^B}} \left| \pi(x; rd, c) - \frac{\mathrm{Li}(x)}{\phi(rd)} \right| \Bigg).$$

Since $r \le (\log x)^D$, we choose $B = B(D)$ sufficiently large, so that the Bombieri–Vinogradov theorem shows that the error above is $\ll x/(\log x)^A$ for any $A > 0$. This is the crucial step where we derive the uniformity of r. Putting everything together and using (3.1) and (3.2), we obtain

$$\sum_{\substack{p \le x \\ p \equiv b \,(\mathrm{mod}\, r)}} \tau_{\mathbb{Q}(\zeta_r)}(p-a) = 2\frac{\mathrm{Li}(x)}{\phi(r)} \big(C_{ar}(\log \sqrt{x} - B \log\log x) + O(1)\big) + O\bigg(\frac{x \log\log x}{\log x}\bigg),$$

which after a straightforward computation completes the proof.

6. An average result for a variant of the Titchmarsh divisor problem.

Bombieri, Friedlander and Iwaniec proved the following result.

Theorem 6.1 (Bombieri–Friedlander–Iwaniec [2, 3, 4]). *Let $a \ne 0$, $\lambda < 1/10$ and $R < x^\lambda$. For any $A > 0$ there exists $B = B(A)$ such that if $QR < x/(\log x)^B$, then*

$$\sum_{\substack{r \le R \\ (r,a)=1}} \Bigg| \sum_{\substack{q \le Q \\ (q,a)=1}} \bigg(\psi(x; qr, a) - \Lambda(a) - \frac{x}{\phi(qr)} \bigg) \Bigg| \ll_{a,A,\lambda} \frac{x}{(\log x)^A}.$$

A more precise variant of this result was obtained by Fiorilli [8, Proposition 5.1]. In this section, we obtain the following refinement; by taking $Q = x$, one recovers Fiorilli's result.

Proposition 6.2. *Fix an integer $a \ne 0$ and let $\lambda < 1/10$ and D be positive real numbers. Let $M = M(r,x)$ be an integer such that $1 \le M(r,x) \le (\log x)^D$. Then for any $\epsilon > 0$, $x^{\lambda+1/2+\epsilon} \le Q \le x$, and $R = R(x) \le x^\lambda$,*

$$\sum_{\substack{R/2<r\le R\\(r,a)=1}}\left|\sum_{\substack{q\le\frac{Q}{rM}\\(q,a)=1}}\left(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\right)-\mathrm{M.T.}\right|=O_{a,A,D,\lambda}\left(\frac{x}{(\log x)^A}\right),$$

where the main term M.T. *is given by*

$$\mathrm{M.T.}=x\frac{C_1(a,r)}{r}\log\frac{r'Mx}{Q}+x\frac{C_2(a,r)-C_1(a,r)}{r}-\sum_{\substack{1\le s\le M\frac{x}{Q}\\(s,a)=1}}\frac{1}{\phi(sr)}\left(x-\frac{sQ}{M}\right)$$

and r', $C_1(a,r)$ *and* $C_2(a,r)$ *are exactly as given in Theorem* 1.5.

Proof. The proof is essentially the same as that of [8, Proposition 5.1]. The difference is that we utilize the full strength of Theorem 6.1 in Q-aspect. More precisely, in contrast to the proof in [8], where $Q=x$ was fixed, we allow Q to vary in $[x^{\lambda+1/2+\epsilon},x]$. Now following Fiorilli, we first split the inner sum over q as follows:

$$\sum_{\substack{q\le\frac{Q}{rM}\\(q,a)=1}}=\sum_{\substack{q\le\frac{Q}{RL}\\(q,a)=1}}+\sum_{\substack{\frac{Q}{RL}<q\le\frac{Q}{r}\\(q,a)=1}}-\sum_{\substack{\frac{Q}{rM}<q\le\frac{Q}{r}\\(q,a)=1}}. \tag{6.1}$$

If we choose $L=(\log x)^{A+B+D+4}$ with $B=B(A)$ as in Theorem 6.1, the first term is controlled by that theorem to give

$$\sum_{\substack{R/2<r\le R\\(r,a)=1}}\left|\sum_{\substack{q\le\frac{Q}{RL}\\(q,a)=1}}\left(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\right)\right|\ll_{a,A,D,\lambda}\frac{x}{(\log x)^A}.$$

For the remaining sums, we write

$$\begin{aligned}\sum_{\substack{\frac{Q}{rP}<q\le\frac{Q}{r}\\(q,a)=1}}\left(\psi(x;qr,a)-\Lambda(a)-\frac{x}{\phi(qr)}\right)&=\sum_{\substack{\frac{Q}{rP}<q\le\frac{Q}{r}\\(q,a)=1}}\ \sum_{\substack{|a|<n\le x\\n\equiv a\,(\mathrm{mod}\,qr)}}\Lambda(n)\\&\quad-x\sum_{\substack{\frac{Q}{rP}<q\le\frac{Q}{r}\\(q,a)=1}}\frac{1}{\phi(qr)},\end{aligned}\tag{6.2}$$

where $P\le 2L$ will be either M or RL/r. By [8, Lemma 4.3], the second sum on the right hand side of (6.2) equals

$$\frac{C_1(a,r)}{r}\log P+O\left(3^{\omega(ar)}\frac{P\log Q}{Q}\right). \tag{6.3}$$

As done by Fiorilli [8, proof of Proposition 5.1], to treat the first term on the right side of (6.2) we use [9, Lemma 5.3], which states that

$$\sum_{\substack{q\le x\\(q,a)=1}}\Bigg(\sum_{\substack{|a|<n\le x\\ n\equiv a\,(\mathrm{mod}\,q)}}\Lambda(n)-\sum_{\substack{|a|<p\le x\\ p\equiv a\,(\mathrm{mod}\,q)}}\log p\Bigg)\ll_\epsilon x^{1/2+\epsilon}.$$

Thus, we can ignore the contribution of prime powers in the aforementioned term by introducing an error of the order of $x^{1/2+\epsilon}$. Moreover, writing $p=a+qrs$, we employ Hooley's variant of the divisor switching technique as done in [8, p. 1029], to deduce that

$$\begin{aligned}
(6.4)\qquad &\sum_{\substack{\frac{Q}{rP}<q\le\frac{Q}{r}\\(q,a)=1}}\ \sum_{\substack{|a|<n\le x\\ n\equiv a\,(\mathrm{mod}\,qr)}}\Lambda(n)\\
&=\sum_{\substack{1\le s<\frac{Px}{Q}-\frac{aP}{Q}\\(s,a)=1}}\ \sum_{\substack{\frac{sQ}{P}+a<p\le x\\ p\equiv a\,(\mathrm{mod}\,sr)}}\log p+O_\epsilon(x^{1/2+\epsilon})+O(\log x)\\
&=\sum_{\substack{1\le s<\frac{Px}{Q}-\frac{aP}{Q}\\(s,a)=1}}\Bigg(\theta(x;sr,a)-\theta\Big(\frac{sQ}{P}+a;sr,a\Big)\Bigg)+O_\epsilon(x^{1/2+\epsilon})+O(\log x)\\
&=\sum_{\substack{1\le s<\frac{Px}{Q}-\frac{aP}{Q}\\(s,a)=1}}\frac{1}{\phi(sr)}\Big(x-\frac{sQ}{P}\Big)+E(r,a)+O_\epsilon(x^{1/2+\epsilon})+O(\log x),
\end{aligned}$$

where

$$E(r,a)=\sum_{\substack{1\le s<\frac{Px}{Q}-\frac{aP}{Q}\\(s,a)=1}}\Bigg(\theta(x;sr,a)-\theta\Big(\frac{sQ}{P}+a;sr,a\Big)-\frac{1}{\phi(sr)}\Big(x-\frac{sQ}{P}\Big)\Bigg).$$

Upon writing $q=sr$, we have

$$\begin{aligned}
\sum_{\substack{R/2<r\le R\\(r,a)=1}}|E(r,a)|&\ll\sum_{\substack{r\le R\\(r,a)=1}}\ \sum_{\substack{s\le\frac{Px}{Q}\\(s,a)=1}}\max_{y\le x}\left|\theta(y;sr,a)-\frac{y}{\phi(sr)}\right|\\
&\ll\sum_{q\le\frac{PRx}{Q}}\tau(q)\max_{y\le x}\left|\theta(y;q,a)-\frac{y}{\phi(q)}\right|.
\end{aligned}$$

From the Cauchy–Schwarz inequality and the trivial estimate for $\theta(x;q,a)$, it follows that the above expression is

$$\ll\Bigg(\sum_{q\le\frac{PRx}{Q}}\tau(q)^2(\log x)\Big(\frac{x}{q}+1\Big)\Bigg)^{1/2}\Bigg(\sum_{q\le\frac{PRx}{Q}}\max_{y\le x}\left|\theta(y;q,a)-\frac{y}{\phi(q)}\right|\Bigg)^{1/2}.$$

Since $x^{\lambda+1/2+\epsilon} \le Q \le x$ with $\epsilon > 0$, we can use the Bombieri–Vinogradov theorem to bound the second term in parenthesis by $\ll_A x/(\log x)^A$ for any $A > 0$. Elementary estimates imply that the first term in parenthesis is of the order $x(\log x)^{O(1)}$. Thus, $E(r,a) \ll_A x/(\log x)^A$ for any $A > 0$. Putting together (6.3) and (6.4), we see that the main term of the left hand side of (6.2) is

$$-x\frac{C_1(a,r)}{r}\log P + O\left(x3^{\omega(ar)}\frac{P\log Q}{Q}\right) + \sum_{\substack{1\le s\le \frac{Px}{Q}\\(s,a)=1}}\frac{1}{\phi(sr)}\left(x - \frac{sQ}{P}\right).$$

Putting P to be RL/r and then M, we see that the main term for (6.1) is

$$-x\frac{C_1(a,r)}{r}\log\frac{RL}{r} + O\left(x3^{\omega(ar)}\frac{\frac{RL}{r}\log Q}{Q}\right) + \sum_{\substack{1\le s\le \frac{RLx}{rQ}\\(s,a)=1}}\frac{1}{\phi(sr)}\left(x - \frac{sQ}{RL/r}\right)$$
$$-\left(-x\frac{C_1(a,r)}{r}\log M + O\left(x3^{\omega(ar)}\frac{M\log Q}{Q}\right) + \sum_{\substack{1\le s\le \frac{Mx}{Q}\\(s,a)=1}}\frac{1}{\phi(sr)}\left(x - \frac{sQ}{M}\right)\right).$$

From [8, Lemma 4.3], we have

$$\sum_{\substack{1\le s\le \frac{RLx}{rQ}\\(s,a)=1}}\frac{1}{\phi(sr)}\left(x - \frac{sQ}{RL/r}\right)$$

equals

$$x\left(\frac{C_1(a,r)}{r}\log\frac{r'RLx}{rQ} + \frac{C_2(a,r)}{r} + O\left(3^{\omega(ar)}\frac{\log\frac{r'RLx}{rQ}}{\frac{rRLx}{rQ}}\right)\right)$$
$$-\frac{Q}{RL}\left(C_1(a,r)\frac{RLx}{rQ} + O\left(3^{\omega(ar)}\log\frac{r'RLx}{rQ}\right)\right).$$

Simplifying the expression for the above main term gives

$$x\frac{C_1(a,r)}{r}\log\frac{r'Mx}{Q} + x\frac{C_2(a,r)-C_1(a,r)}{r} - \sum_{\substack{1\le s\le M\frac{x}{Q}\\(s,a)=1}}\frac{1}{\phi(sr)}\left(x - \frac{sQ}{M}\right),$$

as required. ∎

We will now apply the equidistribution estimate given by Proposition 6.2 above to prove Theorem 1.6.

6.1. Proof of Theorem 1.6. Using Proposition 6.2 together with [8, Lemma 4.3], and the triangle inequality, we have

$$\sum_{\substack{R/2<r\leq R\\(r,a)=1}}\Big|\sum_{\substack{q\leq\frac{Q}{rM}\\(q,a)=1}}(\psi(x;qr,a)-\Lambda(a))-\text{M.T.}\Big|\ll_{a,A,D,\lambda}\frac{x}{(\log x)^{A+1}}.$$

By a dyadic interval consideration, the whole sum over $r\leq R$ is $\ll_{a,A,D,\lambda}$ $x/(\log x)^A$. Assuming $a>0$ and exchanging the order of summation gives

$$\sum_{\substack{q\leq\frac{Q}{rM}\\(q,a)=1}}\sum_{\substack{a<n\leq x\\n\equiv a\,(\mathrm{mod}\,qr)}}\Lambda(n)=\sum_{\substack{a<n\leq x\\n\equiv a\,(\mathrm{mod}\,r)}}\Lambda(n)\sum_{\substack{q\leq\frac{Q}{rM}\\qr|n-a\\(q,a)=1}}1. \tag{6.5}$$

When n equals some prime p, the condition $p=a+mrq$ with m a positive integer implies that q must be coprime to a. Hence, we may drop the condition $(q,a)=1$ in this case. The contribution of prime powers p^k with $k\geq 2$ to the above sum can be estimated as follows:

$$\begin{aligned}\sum_{\substack{a<n\leq x\\n\equiv a\,(\mathrm{mod}\,r)\\n=p^k,k\geq 2}}\Lambda(n)\sum_{\substack{q\leq\frac{Q}{rM}\\qr|n-a\\(q,a)=1}}1&\ll\log x\sum_{2\leq k\leq\log x}\sum_{\substack{a<p^k\leq x\\p^k\equiv a\,(\mathrm{mod}\,r)}}\tau(p^k-a)\\&\ll_{\epsilon,a}\log x\sum_{2\leq k\leq\log x}x^{1/k+\epsilon}\ll_{\epsilon,a}(\log x)^2x^{1/2+\epsilon}\end{aligned}$$

for any $\epsilon>0$. Thus the condition $(q,a)=1$ may be dropped in (6.5), with the resulting error bounded by $x/(\log x)^A$ when summed over r. Once this coprimality condition is dropped, the inner sum in (6.5) is exactly the number of divisors of $(n-a)/r$ that are at most $Q/(rM)$. This completes the proof for $a>0$.

The case $a<0$ can be handled similarly, as in [8, proof of Theorem 2.4].

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