

Base change, tensor product and the Birch-Swinnerton-Dyer conjecture

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Abstract. We prove the Rankin-Selberg convolution of two cuspidal automorphic representations are automorphic whenever one of them arises from an irreducible representation of an abelian-by-nilpotent Galois extension, which extends the previous result of Arthur-Clozel. Moreover, if one of such representations is of dimension at most 2 and another representation arises from a nearly nilpotent extension or a Galois extension of degree at most 59, the automorphy of the Rankin-Selberg convolution has been derived. As an application, we show that certain quotients of L-functions associated to non-CM elliptic curves are automorphic, which generalises a result of M. R. Murty and V. K. Murty.

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1. Introduction

Let E be an elliptic curve defined over a number field k . For every good reduction v of E/k , we let

$$Nv + 1 - a_v$$

represent the number of points of $E \pmod{v}$, where Nv stands for the absolute norm of v . The L-function $L(s, E, k)$ of E/k is defined as an Euler product:

$$L(s, E, k) = \prod_v L_v(s, E, k),$$

where v is over all finite places of k and for good reduction v of E ,

$$L_v(s, E, k) = (1 - a_v Nv^{-s} + Nv^{1-2s})^{-1}.$$

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A classical conjecture of Birch and Swinnerton-Dyer asserts that $L(s, E, k)$ extends to an entire function and that the analytic rank of E/k , i.e., the order of $L(s, E, k)$ at $s = 1$ is equal to the rank of the group $E(k)$ of k -rational points of E .

For any finite extension K/k , it is clear that

$$\text{rank } E(k) \leq \text{rank } E(K).$$

In light of the Birch and Swinnerton-Dyer conjecture, one therefore expects that

$$\text{ord}_{s=1} L(s, E, k) \leq \text{ord}_{s=1} L(s, E, K) \quad (1.1)$$

whenever $k \subseteq K$. As demonstrated in [18] (see also [25]), this problem is related to the problem of base change and tensor product in the Langlands program as well as Heilbronn characters. Indeed, M. R. Murty and V. K. Murty in [18] showed that if E has CM and K/k is contained in a solvable normal closure of k , then (1.1) holds. Moreover, they derived the holomorphy of the quotient

$$\frac{L(s, E, K)}{L(s, E, k)}. \quad (1.2)$$

This can be seen as an elliptic analogue of the classical results of Aramata-Brauer and Uchida-van der Waall (see [2] and [22,23], respectively).

For E/\mathbb{Q} non-CM, the famous conjecture of Taniyama-Shimura predicts that E arises from a modular form, which is now the celebrated modularity theorem of Wiles, Taylor *et al.* (cf. [24,4,3]). More generally, for E/k , the Langlands program suggests that E is associated to a cuspidal automorphic representation π_E of $GL_2(\mathbb{A}_k)$, where \mathbb{A}_k stands for the adèle ring of k . We shall call this the (generalised) Taniyama-Shimura conjecture (over k) and this conjecture is known in some cases. We remark that assuming the Taniyama-Shimura conjecture for E/k , M. R. Murty and V. K. Murty in [18] proved that if K/k is contained in a (finite) solvable Galois extension of k , then (1.1) is valid and the quotient (1.2) is holomorphic.

Indeed, the result of M. R. Murty and V. K. Murty is a part of the Langlands program. Let us assume K/k is a Galois extension of number fields with Galois group G and E/k is a non-CM elliptic curve. We recall that Langlands conjectured that for each irreducible representation ρ of G with character χ , χ should be associated to a cuspidal automorphic representation π_χ of $GL_{\chi(1)}(\mathbb{A}_k)$. Furthermore, the Langlands program predicts that the twisted L-function $L(s, E, \chi, k)$ (after a suitable translation) is equal to $L(s, \pi)$ for some cuspidal automorphic representation π over k . (We note that the L-function $L(s, E, k)$ is given by a family of ℓ -adic representations attached

to E , which will be discussed in more detail in Section 5, and that the twisted L-function $L(s, E, \chi, k)$ is formed by modifying the local factor by $\rho(\text{Frob}_v)$ for v unramified. For the remaining finitely many places, one needs to use a more suitable modification. In particular, one has $L(s, E, k) = L(s, E, 1, k)$.) Thus, one expects

$$\pi = \pi_E \otimes \pi_\chi.$$

In particular, this suggests that the quotient (1.2) should be automorphic and hence the result of M. R. Murty and V. K. Murty follows from the theory of automorphic representations. In fact, this is predicted by the following functoriality conjecture in the Langlands program.

Conjecture 1.1 (The functoriality of $GL(n) \times GL(m)$). Let π_1 and π_2 be cuspidal automorphic representations of $GL_n(\mathbb{A}_k)$ and $GL_m(\mathbb{A}_k)$, respectively. Then $\pi_1 \otimes \pi_2$ is an automorphic representation of $GL_{nm}(\mathbb{A}_k)$.

When $m = 1$, this is known since for π_1 automorphic, any ‘‘abelian twist’’ $\pi_1 \otimes \chi$ is automorphic for any (unitary) character of $k^\times \backslash \mathbb{A}_k^\times$; and the functoriality was recently established for $GL(2) \times GL(2)$ by Ramakrishnan [19] and $GL(2) \times GL(3)$ by Kim-Shahidi [14]. Also, Arthur and Clozel [1] proved the following theorem.

Theorem 1.2. *Let K/k be a (finite) nilpotent Galois extension with Galois group G . Let ρ and π be cuspidal automorphic representations of k . Suppose that ρ arises from an irreducible representation of G . Then $\pi \otimes \rho$ is automorphic.*

Via their theory of automorphic induction, Arthur and Clozel proved that the Langlands reciprocity is valid for all nilpotent Galois extensions. Thus, as a consequence, assuming the Taniyama-Shimura conjecture for E/k , the L-function

$$L(s, E, \chi, k)$$

is automorphic for any $\chi \in Irr(G)$ if K/k is a nilpotent Galois extension of number fields with Galois group G .

We recall that a group G is said to be abelian-by-nilpotent (resp., nearly nilpotent) if G admits a normal subgroup N with G/N nilpotent such that N is abelian (resp., $\text{cd}(N) \subseteq \{1, 2\}$), where $\text{cd}(N) := \{\chi(1) \mid \chi \in Irr(N)\}$. It is easy to see that all nilpotent and abelian-by-nilpotent groups are nearly nilpotent. In [26], the author derived the Langlands reciprocity for all nearly nilpotent extensions. In this note, following the path enlightened by Arthur and Clozel, we will prove the following variants, which can be seen as generalisations of results of Arthur-Clozel and the author.

Theorem 1.3. *Let K/k be an abelian-by-nilpotent Galois extension with Galois group G . Let ρ and π be cuspidal automorphic representations of k . Assume that ρ arises from an irreducible representation of G . Then $\pi \otimes \rho$ is automorphic.*

Theorem 1.4. *Let K/k be a nearly nilpotent Galois extension with Galois group G . Let ρ and π be cuspidal automorphic representations of k . If ρ arises from an irreducible representation of G and π is of degree at most 3, then $\pi \otimes \rho$ is automorphic.*

As an application, we have the following.

Corollary 1.5. *Let K/k be a Galois extension of degree < 60 with Galois group G . Let ρ and π be cuspidal automorphic representations of k . If ρ arises from an irreducible representation of G and π is of degree at most 2, then $\pi \otimes \rho$ is automorphic.*

Finally, we will give a generalisation of the earlier-mentioned result of M. R. Murty and V. K. Murty as stated below.

Corollary 1.6. *Assume that the Taniyama-Shimura conjecture holds for E/k . Then the quotient*

$$\frac{L(s, E, F)}{L(s, E, k)}$$

is automorphic whenever F/k is contained in a normal closure K/k which is nearly nilpotent or of degree less than 60. In particular, the Taniyama-Shimura conjecture is true for E/F .

2. Induced characters and automorphic representations

Let G be a finite group, and N be a normal subgroup of G . A character χ of G is called a relative M-character (resp., a relative SM-character) with respect to N if there exist a subgroup (resp., a subnormal subgroup) H with $N \leq H \leq G$ and an irreducible character $\psi \in Irr(H)$ such that $\text{Ind}_H^G \psi = \chi$ and $\psi|_N \in Irr(N)$. If every irreducible character of G is a relative M-character (resp., a relative SM-character) with respect to N , then G is said to be a relative M-group (resp., a relative SM-group) with respect to N . We note that if N is normal in G and G/N is supersolvable, then G is a relative M-group with respect to N (cf. [8, Chapter 6]). Moreover, Horváth gave a sufficient condition for groups being relative SM-groups as follows.

Proposition 2.1 ([7, Proposition 2.7]). *Let G be a finite group and N be a normal subgroup of G such that G/N is nilpotent. Then G is a relative SM-group with respect to N .*

We now state Arthur-Clozel's theory of base change.

Theorem 2.2 ([1, Arthur and Clozel]). *Let K/k be a Galois extension of prime degree. Then for every (isobaric) representation π of $GL_n(\mathbb{A}_k)$, there exists a unique (isobaric) automorphic representation $\pi|_K$ of $GL_n(\mathbb{A}_K)$, called the base change of π to K , such that*

- 1: *a cuspidal representation Π of $GL_n(\mathbb{A}_K)$ is the base change $\pi|_K$ of π if and only if Π is Galois invariant (in particular, if Π is associated to $\rho|_K$ for some Galois representation ρ over k);*
- 2: *for any (isobaric) π' over k , $\pi'|_K = \pi|_K$ if and only if $\pi' = \pi \otimes \chi$ for some idèle class character χ of k ;*
- 3: *for every Galois representation ρ over k associated to π , $\rho|_K$ is associated to $\pi|_K$; and*
- 4: *if χ is an idèle class character of k , then $(\pi \otimes \chi)|_K = \pi|_K \otimes \chi|_K$.*

Moreover, one has the adjoint map to base change, called automorphic induction, which corresponds to induction of Galois representations as stated in the following theorem.

Theorem 2.3 ([1, Arthur and Clozel]). *Let K/k be a Galois extension of number fields of prime degree, and Π denote an automorphic representation induced from cuspidal of $GL_n(\mathbb{A}_K)$ (or, in particular, a cuspidal automorphic representation of $GL_n(\mathbb{A}_K)$). Then there is an automorphic representation $I(\Pi)$ of $GL_{np}(\mathbb{A}_k)$, called automorphic induction of Π , such that $L(s, \Pi) = L(s, I(\Pi))$; and $I(\Pi)$ is also induced from cuspidal. Moreover, if ρ is a Galois representation corresponding to Π , then $\text{Ind}_K^k \rho$ corresponds to $I(\Pi)$.*

For non-normal extensions, one has a theorem below due to Jacquet, Piatetski-Shapiro and Shalika.

Theorem 2.4 ([10]). *Let K/k be a non-normal cubic extension of number fields. Let χ be an idèle class character of K and π an automorphic representation of $GL_2(\mathbb{A}_k)$. Then the automorphic induction $I(\chi)$ of χ and the base change $\pi|_K$ exist as automorphic representations of $GL_3(\mathbb{A}_k)$ and $GL_2(\mathbb{A}_K)$, respectively.*

We remark that Theorems 2.3 and 2.4 yield that all monomial characters of degree 3 are of automorphic type. On the other hand, one has the following result of Jacquet.

Lemma 2.5 ([11, Jacquet]). *Let K/k be a Galois extension of number fields of prime degree, and π and σ be two (unitary) cuspidal automorphic representations of $GL_n(\mathbb{A}_k)$ and $GL_m(\mathbb{A}_K)$, respectively. Then the Rankin-Selberg L -functions satisfy the following formal identity:*

$$L(s, \pi|_K \otimes \sigma) = L(s, \pi \otimes I(\sigma)).$$

Now, we put our attention to the key ingredient of the earlier-mentioned result of Arthur and Clozel.

Lemma 2.6 ([1, pp. 222–223]). *Let K/k be a solvable Galois extension of number fields with Galois group G . Let H be a subnormal subgroup of G . Suppose that χ and ψ are irreducible representations of G and H , respectively, such that*

$$\chi = \text{Ind}_H^G \psi.$$

Assume further that ψ corresponds to a cuspidal automorphic representation π_ψ . Then for any cuspidal automorphic representation π of k such that $\pi|_{K^H} \otimes \pi_\psi$ is induced from cuspidal, the Rankin-Selberg convolution $\pi \otimes I_{K^H}^k(\pi_\psi)$ is equal to $I_{K^H}^k(\pi|_{K^H} \otimes \pi_\psi)$ and automorphic. In particular, χ is associated to $I_{K^H}^k(\pi_\psi)$.

For the sake of completeness, we sketch the proof here. First of all, as H is subnormal in G , there is an invariant series

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots H_{m-1} \trianglelefteq H_m = G,$$

where for each i , H_i is a normal subgroup of H_{i+1} . As G is solvable, we may require each H_{i+1}/H_i is of prime degree. Thus, one has a tower of Galois extensions of prime degree

$$K^H \supset K^{H_1} \supset \cdots K^{H_{m-1}} \supset k.$$

Applying the Arthur-Clozel theory of base change successively, $\pi|_{K^H}$ exists as an automorphic representation over K^H . Keeping in mind that we assume that the convolution $\pi|_{K^H} \otimes \pi_\psi$ is induced from cuspidal, we can apply Arthur-Clozel's automorphic induction successively to derive the automorphy of $I_{K^H}^k(\pi|_{K^H} \otimes \pi_\psi)$ (here $I_{K^H}^k$ stands for the automorphic induction map from K^H to k). Finally, by Jacquet's result, Lemma 2.5, $\pi \otimes I_{K^H}^k(\pi_\psi)$ is equal to $I_{K^H}^k(\pi|_{K^H} \otimes \pi_\psi)$ and hence automorphic (cf. [1, Lemma 7.4]).

Remark. As mentioned by Arthur and Clozel, if π also arises from a Galois representation ρ , then the above identification translates to

$$\rho \otimes \text{Ind}_H^G \psi = \text{Ind}_H^G(\rho|_H \otimes \psi).$$

In particular, the automorphy of $\rho \otimes \text{Ind}_H^G \psi$ will follow from the automorphy of $\rho|_H \otimes \psi$ and Arthur-Clozel's automorphic induction.

3. Proof of main theorems

Let K/k be a solvable Galois extension with Galois group G . Let π_1 and π_2 be cuspidal automorphic representations of k . Assume that π_2 arises from an irreducible character χ of G and that there exist a subnormal subgroup H of G and $\psi \in \text{Irr}(H)$ such that

$$\chi = \text{Ind}_H^G \psi.$$

Proof of Theorem 1.3. As in [1], if ψ is of degree 1, then Artin reciprocity asserts that ψ can be seen as an idèle class character. Moreover, the base change $\pi_1|_{K^H}$ exists. Furthermore, the functoriality of $GL(n) \times GL(1)$ tells us that

$$\pi_1|_{K^H} \otimes \psi$$

is automorphic. Now Lemma 2.6 asserts that $\pi_1 \otimes \pi_2$ is automorphic.

In particular, if K/k be an abelian-by-nilpotent Galois extension with Galois group G , then there is a normal abelian subgroup N in G such that G/N is nilpotent. By Horváth's result, for every $\chi \in \text{Irr}(G)$ there exist a subnormal subgroup H with $N \leq H \leq G$ and an irreducible character $\psi \in \text{Irr}(H)$ such that $\text{Ind}_H^G \psi = \chi$ and $\psi|_N \in \text{Irr}(N)$. As N is abelian, ψ is of degree 1, which together with the above discussion proves Theorem 1.3. \square

Proof of Theorem 1.4. Now, let us assume that ψ is of degree 2 and that π_1 has degree at most 3. Then the Langlands-Tunnell theorem asserts that ψ is associated to a cuspidal automorphic representation Π of $GL_2(\mathbb{A}_k)$. Also, the base change $\pi_1|_{K^H}$ exists and is of degree ≤ 3 . Thus, the functoriality of tensor product (thanks to Ramakrishnan and Kim-Shahidi) asserts that $\pi_1|_{K^H} \otimes \Pi$ is automorphic, and so is $\pi_1 \otimes \pi_2$.

By Proposition 2.1, if G is nearly nilpotent, i.e., G admits a normal subgroup N with $\text{cd}(N) \subseteq \{1, 2\}$ and G/N nilpotent. Hence, for any $\chi \in \text{Irr}(G)$ there exist a subnormal subgroup H with $N \leq H \leq G$ and an irreducible character $\psi \in \text{Irr}(H)$ such that $\text{Ind}_H^G \psi = \chi$ and $\psi(1) \leq 2$. Therefore, Theorem 1.4 follows. \square

Remark. If ψ is of degree 3 and can be associated to a cuspidal automorphic representation Π of $GL_3(\mathbb{A}_{K^H})$, then for any 2-dimensional cuspidal automorphic representation π_1 over k , $\pi_1|_{K^H} \otimes \Pi$ is automorphic thanks to the functoriality of $GL(2) \times GL(3)$ due to Kim and Shahidi. Again, Lemma 2.6 yields the automorphy of $\pi_1 \otimes \pi_2$.

4. Small extensions

In this section, we will apply the machinery developed in the previous section to prove Corollary 1.5.

Firstly, the theorem of Arthur and Clozel applies if G is nilpotent. In particular, if G is abelian or a p -group, then the statement follows. Hence, if $|G|$ belongs to

$$\begin{aligned} & \{1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59\} \\ & \cup \{4, 8, 16, 32, 9, 27, 25, 49\}, \end{aligned}$$

then the statement is valid. There are 26 classes of groups.

As discussed in [26], any non- S_4 group G of order pq , pq^2 , p^2q^2 , or $8p$ for some primes p and q is nearly nilpotent. Indeed, G has a normal Sylow subgroup unless G is isomorphic to S_4 (cf. [9, Chapter 1]). We further note that all irreducible characters of S_4 are of degree at most 3 and induced from characters of degree at most 2. Thus, if G has order 6, 10, 12, 14, 15, 18, 20, 21, 22, 24, 26, 28, 33, 34, 35, 36, 38, 39, 40, 44, 45, 46, 50, 51, 52, 55, 56, 57, or 58, then that the assertion holds (cf. Proof of Theorem 1.4). Here we have 29 classes of groups.

Moreover, a result of Hölder asserts that any group G of square-free order must be meta-cyclic, i.e., G has a cyclic normal subgroup with G/N cyclic. Hence, the groups of order 30 or 42 are meta-cyclic, which are clearly abelian-by-nilpotent.

Now, there are only 2 cases left, namely, the groups of order 48 or 54.

4.1 The case $|G| = 48$

For G of order 48, G has a normal subgroup N of order 8 or 16. Clearly, all irreducible characters of N are of degree ≤ 2 . Since G/N is either of order 3 or 6, G/N must be supersolvable. In addition, if $|G/N| = 3$, then G is nearly nilpotent.

Now assume $|N| = 8$. We note that any irreducible character χ of G is of degree ≤ 4 , which can easily be checked via GAP [6] for instance. Also, as G/N is supersolvable, G is a relative M-group with respect to N . Thus, χ is induced from an irreducible character of degree at most 2. If $\chi(1) = 3$, it is monomial and automorphic. For χ 4-dimensional, it must be induced from an irreducible character of degree 2 of a normal subgroup H (indeed, $[G : H] = 2$). In other words, all irreducible characters of G are induced from characters of degree at most 3, which are all automorphic. Thus, the statement follows from the remark in the previous section.

4.2 The case $|G| = 54$

If G has order 54, then G must be supersolvable (see, for example, [20, 7.2.15]). Also, the Sylow 3-subgroup P (say) is normal in G , and it is clear that $\text{cd}(P) \subseteq \{1, 3\}$. As $|G/P| = 2$, all irreducible characters of G are induced from characters of degree 1 or 3 of normal subgroups. Thus, it suffices to show any 3-dimensional irreducible character ψ of a subgroup of G is automorphic. Since G is supersolvable, all subgroups of G are supersolvable. Therefore, ψ must be monomial and automorphic.

5. Applications to L-functions of non-CM elliptic curves

As before, let E be an elliptic curve defined over k . For every finite extension F/k , E can be seen as an elliptic curve defined over F . By the works of Serre and Tate, one can associate a compatible system of ℓ -adic representations of E over F , i.e., for each prime ℓ ,

$$\rho_F := \rho_{\ell, F} : \text{Gal}(\bar{k}/F) \rightarrow \text{Aut}(T_{\ell}(E, F)),$$

where $T_{\ell}(E, F)$ denotes (ℓ -adic) Tate module of E/F . Furthermore, the L-function $L(s, E, F)$ of E/F is given by this family of ℓ -adic representations of E over F , i.e., $L(s, E, F) = L(s, \rho_F)$. Since $T_{\ell}(E, F) = T_{\ell}(E, k)$ as $\text{Gal}(\bar{k}/F)$ -modules, ρ_F is the restriction of ρ_k and

$$L(s, \rho_F) = L(s, \rho_k|_{\text{Gal}(\bar{k}/F)}).$$

Now let us fix a Galois extension K/k (containing F) with Galois group G . Since

$$\text{Ind}_{\text{Gal}(\bar{k}/F)}^{\text{Gal}(\bar{k}/k)} (\rho_k|_{\text{Gal}(\bar{k}/F)}) = \rho_k \otimes \text{Ind}_{\text{Gal}(\bar{k}/F)}^{\text{Gal}(\bar{k}/k)} 1,$$

we then have

$$\begin{aligned} L(s, \rho_F) &= L(s, \rho_k|_{\text{Gal}(\bar{k}/F)}) \\ &= L(s, \rho_k \otimes \text{Ind}_{H_F}^G 1), \end{aligned} \tag{5.1}$$

where H_F is a subgroup of G such that $K^{H_F} = F$. We remark that if $F = K$, then H_F is the trivial group and the formula (5.1) gives the Artin-Takagi decomposition for L-functions associated to elliptic curves. We also note that the twisted L-function $L(s, E, \chi, k)$ is equal to $L(s, \rho_k \otimes \tau)$ whenever τ is a representation of G with character χ . Now by Theorems 1.4 and 1.5, if ρ_k is automorphic and G is nearly nilpotent or of degree < 60 , then each $L(s, \rho_k \otimes \tau)$ is automorphic (and so is $L(s, E, \chi, k)$).

On the other hand, by Frobenius reciprocity, one has

$$\text{Ind}_{H_F}^G 1 = 1 + \sum_i a_i \chi_i,$$

where the sum is finite and for each i , $a_i \in \mathbb{N}$ and $\chi_i \in \text{Irr}(G)$ is non-trivial. Thus,

$$L(s, \rho_F) = L(s, \rho_k) \prod_i L(s, \rho_k \otimes \chi_i)^{a_i}.$$

Hence, assuming the Taniyama-Shimura conjecture is valid for E/k , the quotient

$$\frac{L(s, \rho_F)}{L(s, \rho_k)}$$

is automorphic whenever G is of degree less than 60 or nearly nilpotent, which proves Corollary 1.6.

Remark. This method is capable of further applications. For example, by the celebrated theorem of Khare and Wintenberger [13] on Serre's modularity conjecture, all abelian varieties over \mathbb{Q} of $GL(2)$ -type are modular (in other words, they are associated to 2-dimensional automorphic representations of $GL_2(\mathbb{A}_{\mathbb{Q}})$). Also, the modularity of non-CM elliptic curves over any real quadratic field is derived by Freitas, Le Hung and Siksek [5]. In both instances, one may obtain the automorphy for L-functions associated to these arithmetic objects by an analogous argument.

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