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General Section Selberg's orthogonality conjecture and symmetric power L-functions



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ABSTRACT

Let π be a cuspidal representation of $\operatorname{GL}_2(\mathbb{A}_{\mathbb{Q}})$ defined by a non-CM holomorphic newform of weight $w \geq 2$, and let K/\mathbb{Q} be a totally real Galois extension with Galois group G. In this article, under Selberg's orthogonality conjecture, we show that for any irreducible character χ of G, the twisted symmetric power L-function $L(s, \operatorname{Sym}^m \pi \times \chi)$ is a primitive function in the Selberg class, and it is automorphic subject to further the solvability of K/\mathbb{Q} . The key new idea is to apply the work of Barnet-Lamb, Geraghty, Harris, and Taylor on the potential automorphy of $\operatorname{Sym}^m \pi$.

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1. Introduction

Thirty years ago, Selberg introduced a class S of *L*-functions, called the Selberg class nowadays, in [17]. The class S consists of functions F(s) of a complex variable *s* enjoying the following properties:

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(Dirichlet series and Euler product). For $\Re \mathfrak{e}(s) > 1$, one has

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s} = \prod_p \exp\left(\sum_{k=1}^{\infty} b_F(p^k) / p^{ks}\right),$$
(1.1)

for some complex numbers $a_F(n)$ and $b_F(p^k)$, where the product is over rational primes p. In addition, one has $a_F(1) = 1$ and $b_F(p^k) = O(p^{k\theta})$ for some $\theta < \frac{1}{2}$. Moreover, the Dirichlet series and Euler product in equation (1.1) converge absolutely on $\Re \mathfrak{e}(s) > 1$. (Analytic continuation and functional equation). There is a non-negative integer m_F such that the function $(s-1)^{m_F}F(s)$ extends to an entire function of finite order. Moreover, there are numbers r_F , $Q_F > 0$, $\alpha_F(j) > 0$, $\Re \mathfrak{e}(\gamma_F(j)) \ge 0$ so that the function

$$\Lambda_F(s) = Q_F^s \prod_{j=1}^{r_F} \Gamma(\alpha_F(j)s + \gamma_F(j))F(s)$$

satisfies the functional equation $\Lambda_F(s) = w_F \overline{\Lambda_F(1-\overline{s})}$ for some complex number w_F of absolute value 1.

(Ramanujan-Petersson conjecture). For any fixed $\epsilon > 0$, one has $a_F(n) = O(n^{\epsilon})$.

A function $F \in S$ is called primitive if for any $F_1, F_2 \in S$ satisfying the equation $F = F_1F_2$, one has either $F = F_1$ or $F = F_2$. The primitive functions are the building blocks of S. Indeed, the class S is multiplicatively closed, and every function $F \in S$ factorises into a product of primitive functions in S (see, e.g., [14]). In [17], Selberg made the following orthogonality conjecture.

Conjecture 1.1. For any $F \in S$, there exists a positive integer n_F such that one has

$$\sum_{p \le x} \frac{|a_F(p)|^2}{p} = n_F \log \log x + O_F(1), \tag{1.2}$$

as $x \to \infty$, where the implied constant depends on F. Moreover, for any primitive function F, one has $n_F = 1$.

For any distinct primitive functions F and G, as $x \to \infty$, one has

$$\sum_{p \le x} \frac{a_F(p)\overline{a_G(p)}}{p} = O_{F,G}(1), \tag{1.3}$$

where the implied constant depends on F and G.

Selberg's orthogonality conjecture has a strong impact on the theory of L-functions. For instance, under Selberg's orthogonality conjecture, Conrey and Ghosh [9] showed that every $F \in S$ has a *unique* factorisation into primitive functions. Moreover, Murty [14] showed that Selberg's orthogonality conjecture implies Artin's conjecture on the holomorphy of Artin L-functions.

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Although Selberg introduced the class S to study the value distribution of L-functions (see [17, Sec. 2]), the study of the structural properties of the Selberg class S is also of interest (see, e.g., [9,11,12]). For example, Conrey and Ghosh [9] proved that the only $F \in S$ of degree zero is 1 and that there is no function $F \in S$ with $0 < d_F < 1$, where the degree d_F of F is defined by $d_F = 2 \sum_{j=1}^{r_F} \alpha_F(j)$. Moreover, Kaczorowski and Perelli [12] showed that there is no function $F \in S$ with $1 < d_F < 2$. In addition, if a function $F \in S$ is of degree one, then it is either the Riemann zeta function or a shifted Dirichlet L-function (see [11]).

In this article, we shall emphasise symmetric power L-functions associated to modular forms. Let $\mathbb{A}_{\mathbb{Q}}$ be the adèle ring of \mathbb{Q} . Let π be a cuspidal representation of $\operatorname{GL}_2(\mathbb{A}_{\mathbb{Q}})$ defined by a non-CM holomorphic newform of weight $w \geq 2$, and let $L(s,\pi)$ be the automorphic L-function of π . It is known that the automorphic L-function $L(s,\pi)$ is a member of the Selberg class S. Furthermore, via the symmetric power lifting, one can construct new L-functions $L(s, \operatorname{Sym}^m \pi)$, the symmetric power L-functions of π . These L-functions are expected to belong to S, and their analytic properties have many important applications to number theory. Most famously, they led to the resolution of the Sato-Tate conjecture for non-CM elliptic curves defined over totally real number fields (see [7,10,19]).

By the work of Barnet-Lamb, Geraghty, Harris, and Taylor [4], the symmetric power L-function $L(s, \operatorname{Sym}^m \pi)$ extends holomorphically to $\mathfrak{Re}(s) \geq 1$ and meromorphically to \mathbb{C} . Furthermore, the symmetric power L-function $L(s, \operatorname{Sym}^m \pi)$ satisfies the Ramanujan-Petersson conjecture and the expected functional equation. Thus, if the symmetric power L-function $L(s, \operatorname{Sym}^m \pi)$ extends holomorphically to $\mathfrak{Re}(s) < 1$, then it will belong to the Selberg class.

Throughout this article, for any finite group G, we let Irr(G) denote the set of irreducible characters of G. Our primary objective is to show that under Selberg's orthogonality conjecture, certain twisted symmetric power L-functions belong to the Selberg class as follows.

Theorem 1.2. Let π be a cuspidal representation of $\operatorname{GL}_2(\mathbb{A}_{\mathbb{Q}})$ defined by a non-CM holomorphic newform of weight $w \geq 2$. Let K/\mathbb{Q} be a totally real Galois extension with Galois group G, and let $\chi \in \operatorname{Irr}(G)$ be an irreducible character of G. Then Conjecture 1.1 implies that the L-functions $L(s, \operatorname{Sym}^m \pi)$ and $L(s, \operatorname{Sym}^m \pi \times \chi)$ are primitive functions in S.

Moreover, we have the following conditional automorphy result.

Theorem 1.3. Let π , K, and G be as in Theorem 1.2. Suppose that the Galois group G is solvable. Assume that the mth symmetric power $\operatorname{Sym}^m \pi$ of π is automorphic over \mathbb{Q} . Then subject to Conjecture 1.1, for any $\chi \in \operatorname{Irr}(G)$, the L-function $L(s, \operatorname{Sym}^m \pi \times \chi)$ is automorphic over \mathbb{Q} . Consequently, under Selberg's orthogonality conjecture, if $m \leq 8$, then the L-function $L(s, \operatorname{Sym}^m \pi \times \chi)$ is automorphic over \mathbb{Q} . Note added on November 17, 2021. Since this article was prepared, the author has been pleased to learn of the recent work by Newton and Thorne [15], proving the automorphy of the symmetric powers $\text{Sym}^m \pi$ and therefore removing the automorphy assumption in the above theorem.

Remarks. (i) Theorem 1.2 does *not* affect the Selberg class S, but verifies expected members $L(s, \operatorname{Sym}^m \pi)$ and $L(s, \operatorname{Sym}^m \pi \times \chi)$ of S under Selberg's orthogonality conjecture. Also, the conditional primitivity of $L(s, \operatorname{Sym}^m \pi)$ may be seen as a kind of analytic version of the conjectural cuspidality of $\operatorname{Sym}^m \pi$ predicted by the Langlands programme. Indeed, the Langlands functoriality conjecture asserts that all the $\operatorname{Sym}^m \pi$ are automorphic. This implies that every $\operatorname{Sym}^m \pi$ is cuspidal by the work of Ramakrishnan [16, Theorem A(d)].

(ii) As pointed out by the referee, in [13], it is shown that conditional on the cuspidality of $L(s, \operatorname{Sym}^m \pi \times \chi)$, the *L*-function $L(s, \operatorname{Sym}^m \pi \times \chi)$ cannot be factored further into automorphic *L*-functions of smaller degree. Theorem 1.2 shows that the orthogonality implies primitivity of $L(s, \operatorname{Sym}^m \pi \times \chi)$, without passing through automorphy. The key is to apply the work of Barnet-Lamb, Geraghty, Harris, and Taylor on the potential automorphy of $\operatorname{Sym}^m \pi$. Moreover, under the automorphy of the symmetric powers and the Artin representation associated to χ , the *functoriality of the tensor product* implies that the *L*-function $L(s, \operatorname{Sym}^m \pi \times \chi)$ is automorphic. From these, Theorems 1.2 and 1.3 show the consistency between the Langlands programme and Selberg's orthogonality conjecture.

(iii) To prove Theorems 1.2 and 1.3, one does not need Conjecture 1.1 in full generality. Indeed, a weaker orthogonality conjecture, Conjecture 2.1, for L-functions appearing in the factorisations (2.4) and (2.6) is sufficient.

Notation. Given two complex-valued functions f(x) and g(x) and objects $\omega_1, \ldots, \omega_\ell$, we shall use the notation $f(x) = O_{\omega_1,\ldots,\omega_\ell}(g(x))$ to mean that there exists $M_{\omega_1,\ldots,\omega_\ell} > 0$, depending on $\omega_1,\ldots,\omega_\ell$, such that $|f(x)| \leq M_{\omega_1,\ldots,\omega_\ell}|g(x)|$ for all sufficiently large x.

The letters K and L consistently mean totally real Galois extensions of \mathbb{Q} , with L always taken to be an extension of K. For any subgroup H of the Galois group $\operatorname{Gal}(L/\mathbb{Q})$, we will let L^H denote the fixed field of H.

2. Proofs of Theorems 1.2 and 1.3

2.1. Remark on the use of Selberg's orthogonality conjecture

As remarked by Smajlovic, invoking Conjecture 1.1 for the whole Selberg class S to study certain *L*-functions is not necessary. Indeed, one may instead consider the following weak version of Selberg's orthogonality conjecture.

Conjecture 2.1 (SOC(F)). Let F be a member of the Selberg class S. Then F itself and every primitive function appearing in a factorisation of F satisfies (1.2). Moreover, any pair of primitive functions appearing in factorisations of F satisfy (1.3).

Under Conjecture 1.1, Conrey and Ghosh [9] showed that the factorisation of $F \in S$ is unique. As a demonstration of the use of Conjecture 2.1, we shall show that such a uniqueness theorem follows from SOC(F). Let $F \in S$ and assume SOC(F). Following the argument of Conrey and Ghosh [9], we suppose that the function F admitted two different factorisations into primitive functions

$$F = F_1 \cdots F_r = G_1 \cdots G_t.$$

Write $F_j(s) = \sum_{n=1}^{\infty} a_{F_j}(n)n^{-s}$ and $G_{\ell}(s) = \sum_{n=1}^{\infty} a_{G_{\ell}}(n)n^{-s}$. Without loss of generality, we may assume that no G_{ℓ} is an F_1 . Comparing the *p*th coefficients of the Dirichlet series of both sides of $F_1 \cdots F_r = G_1 \cdots G_t$, we have

$$a_{F_1}(p) + \dots + a_{F_r}(p) = a_{G_1}(p) + \dots + a_{G_t}(p).$$

Thus, we derive

$$\sum_{p \le x} \frac{\overline{a_{F_1}(p)}(a_{F_1}(p) + \dots + a_{F_r}(p))}{p} = \sum_{p \le x} \frac{\overline{a_{F_1}(p)}(a_{G_1}(p) + \dots + a_{G_t}(p))}{p}$$
(2.1)

for $x \ge 0$. However, under SOC(F), as $x \to \infty$, the sum on the left of (2.1) tends to infinity while the sum on the right of (2.1) is $O_{F_1,G_1,\ldots,G_t}(1)$, which is impossible.

2.2. Chebotarev-Sato-Tate distribution and its consequence

In this section, we shall give a refined version of the "Chebotarev-Sato-Tate distribution" established in [20]. As in [20], we will follow the strategy of Serre [18].

Let π be a cuspidal representation of $\operatorname{GL}_2(\mathbb{A}_{\mathbb{Q}})$ defined by a non-CM holomorphic newform of weight $w \geq 2$. For any unramified p of π , let $\alpha_{\pi}(p)$ and $\beta_{\pi}(p)$ denote the Satake parameters of π at p. Since the Satake parameters $\alpha_{\pi}(p)$ and $\beta_{\pi}(p)$ are conjugate with absolute value 1, there exists $\theta_p \in [0, \pi]$ such that $\alpha_{\pi}(p) = e^{i\theta_p}$ and $\beta_{\pi}(p) = e^{-i\theta_p}$.

Let K/\mathbb{Q} be a totally real Galois extension with Galois group G. Following [20], we consider the compact group $\mathcal{G} = \mathrm{SU}(2) \times G$. Let \mathcal{X} be the space of conjugacy classes of \mathcal{G} , and let μ denote the push-forward measure of \mathcal{X} obtained from the normalised Haar measure of \mathcal{G} . Let ρ_0 (resp. 1_G) denote the trivial representation of $\mathrm{SU}(2)$ (resp. G), and, for $m \geq 1$, let ρ_m denote the *m*th symmetric power of the natural representation ρ_1 of degree two for $\mathrm{SU}(2)$. It is clear that any non-trivial irreducible representation of $\mathcal{G} = \mathrm{SU}(2) \times G$ is of the form $\rho_m \otimes \tau \neq \rho_0 \otimes 1_G$ for some $m \geq 0$ and irreducible representation τ of G, and the tensor product $\rho_0 \otimes 1_G$ is the trivial representation of $\mathcal{G} = \mathrm{SU}(2) \times G$. Let τ be an irreducible representation of G and χ be its character. For $\theta \in [0, \pi]$, we set

$$M_{\theta} = \begin{pmatrix} e^{i\theta} & 0\\ 0 & e^{-i\theta} \end{pmatrix}.$$

For every unramified prime p of both π and K/\mathbb{Q} , we choose $x_p = (M_{\theta_p}, \sigma_p) \in \mathcal{X}$, where the symbol σ_p denotes the Artin symbol at p. Otherwise, we set $x_p = (M_0, e_G)$, where the notation e_G denotes the identity of G. Following Serre [18, Ch. I, A.2], we then form the *L*-function $L(s, \rho_m \otimes \tau)$ by

$$L(s,\rho_m\otimes\tau)=\prod_p\det(I-\rho_m(M_{\theta_p})\otimes\tau(\sigma_p)p^{-s})^{-1},$$

for $\Re \mathfrak{e}(s) > 1$. It can be checked that the *L*-function $L(s, \rho_m \otimes \tau)$ is the same as $L(s, \operatorname{Sym}^m \pi \times \chi)$ up to a finite number of terms in their Euler products. Moreover, it was shown in [20, Proof of Theorem 1.1] that each $L(s, \operatorname{Sym}^m \pi \times \chi)$ extends to a non-vanishing holomorphic function on $\Re \mathfrak{e}(s) \geq 1$ unless one has m = 0 and $\chi = 1_G$, where the notation $\operatorname{Sym}^0 \pi$ denotes the trivial representation of $\operatorname{GL}_1(\mathbb{A}_Q)$. Hence, by Serre's equidistribution criterion [18, I-21-23], the elements x_p are μ -equidistributed. More precisely, this means that for any complex-valued continuous function f on \mathcal{X} , one has

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \le x} f(x_p) = \int_{\mathcal{X}} f d\mu.$$

Here, the function $\pi(x)$ is the usual prime-counting function.

Now, let $\psi = \operatorname{tr}(\rho_m \otimes \tau)$ be the character of $\rho_m \otimes \tau$. Note that the function $|\psi|^2$ is continuous on \mathcal{X} . As the elements $x_p = (M_{\theta_p}, \sigma_p)$ are μ -equidistributed, we have

$$\sum_{p \le x} |\psi(x_p)|^2 = \left(\int_{\mathcal{X}} |\psi|^2 d\mu\right) \frac{x}{\log x} + o\left(\frac{x}{\log x}\right) = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right),$$

as $x \to \infty$, where the last equality is due to the orthogonality of irreducible characters of compact groups.

Denote the *p*th coefficients of Dirichlet series of $L(s, \operatorname{Sym}^m \pi)$ and $L(s, \operatorname{Sym}^m \pi \times \chi)$ by $\lambda_{\operatorname{Sym}^m \pi}(p)$ and $\lambda_{\operatorname{Sym}^m \pi \times \chi}(p)$, respectively. Since we know

$$\psi = \operatorname{tr}(\rho_m \otimes \tau) = \operatorname{tr}(\rho_m) \operatorname{tr}(\tau) = \operatorname{tr}(\rho_m) \chi,$$

we obtain

$$\psi(x_p) = \psi((M_{\theta_p}, \sigma_p)) = \frac{\sin(m+1)\theta_p}{\sin\theta_p} \chi(\sigma_p) = \lambda_{\operatorname{Sym}^m \pi}(p)\chi(\sigma_p) = \lambda_{\operatorname{Sym}^m \pi \times \chi}(p)$$

for any unramified p of both π and K/\mathbb{Q} . Hence, unconditionally, we have

$$\sum_{p \le x} \frac{|\lambda_{\text{Sym}^m \, \pi \times \chi}(p)|^2}{p} = \sum_{p \le x} \frac{|\psi(x_p)|^2}{p} + O_{m,\pi,\chi}(1) = \log\log x + o(\log\log x), \qquad (2.2)$$

as $x \to \infty$.

2.3. Proof of Theorem 1.2

Lemma 2.2. Let F belong to the Selberg class and suppose that one knows $F = \prod_{j=1}^{r} F_j^{e_j}$ for some distinct primitive functions $F_j \in S$. Then subject to SOC(F), one has

$$n_F = \sum_{j=1}^r e_j^2.$$

In addition, under SOC(F), the function $F \in S$ is primitive if and only if one has $n_F = 1$.

Proof. As before, we write $F_j(s) = \sum_{n=1}^{\infty} a_{F_j}(n) n^{-s}$. From the Dirichlet series of both sides of $F = \prod_{j=1}^{r} F_j^{e_j}$, we have

$$a_F(p) = \sum_{j=1}^r e_j a_{F_j}(p).$$

Thus, we obtain

$$\sum_{p \le x} \frac{|a_F(p)|^2}{p} = \sum_{p \le x} \frac{|\sum_{j=1}^r e_j a_{F_j}(p)|^2}{p}$$

for $x \ge 0$. Now, applying SOC(F) to both sides of the above identity, we deduce $n_F = \sum_{j=1}^{r} e_j^2$, as desired.

For the second part of the lemma, the "only if" part is clear. On the other hand, for the case that we have $n_F = 1$, we know $|e_1| = 1$ (say) and the remaining e_j are zero. In other words, the function F(s) is equal to either $F_1(s)$ or $F_1(s)^{-1}$. However, the latter instance is impossible since it would give some poles for F(s) on the half-plane $\Re \mathfrak{e}(s) < 0$, which completes the proof. \Box

Lemma 2.3. Let Π be a cuspidal representation of $\operatorname{GL}_n(\mathbb{A}_{\mathbb{Q}})$ satisfying the Ramanujan-Petersson conjecture. Then under $\operatorname{SOC}(\Pi)$, the automorphic L-function $L(s,\Pi)$ is primitive.

Here, as later, the notation $SOC(\Pi)$ is a shorthand for $SOC(L(s, \Pi))$.

Proof of Theorem 1.2. By the work of Barnet-Lamb, Gee, Geraghty, and Taylor [2,3], there is a totally real number field L such that the base change $(\text{Sym}^m \pi)|_L$ is cuspidal, and the extension L/\mathbb{Q} is a finite Galois extension containing K/\mathbb{Q} . Hence, one may regard χ as a character of $\tilde{G} = \text{Gal}(L/\mathbb{Q})$. By Brauer's induction theorem [5], there are $n_i \in \mathbb{Z}$, nilpotent subgroups H_i of \tilde{G} , and characters ψ_i of H_i of degree one such that

$$\chi = \sum_{i} n_i \operatorname{Ind}_{H_i}^{\widetilde{G}} \psi_i.$$

By Artin reciprocity, each ψ_i can be seen as a 1-dimensional cuspidal representation over the fixed field L^{H_i} of H_i . We then have the following factorisation

$$L(s, \operatorname{Sym}^{m} \pi \times \chi) = \prod_{i} L(s, (\operatorname{Sym}^{m} \pi)|_{L^{H_{i}}} \times \psi_{i})^{n_{i}}.$$
(2.3)

As each H_i is solvable, every extension L/L^{H_i} is solvable. By [4, Lemma 1.3], we derive that the base change $(\operatorname{Sym}^m \pi)|_{L^{H_i}}$ is cuspidal over L^{H_i} . Thus, each $L(s, (\operatorname{Sym}^m \pi)|_{L^{H_i}} \times \psi_i)$ is a Rankin-Selberg *L*-function. In addition, as both $L(s, (\operatorname{Sym}^m \pi)|_{L^{H_i}})$ and $L(s, \psi_i)$ satisfy the Ramanujan-Petersson conjecture, the Rankin-Selberg *L*-function $L(s, (\operatorname{Sym}^m \pi)|_{L^{H_i}} \times \psi_i)$ satisfies the Ramanujan-Petersson conjecture. Thus, the *L*function $L(s, (\operatorname{Sym}^m \pi)|_{L^{H_i}} \times \psi_i)$ is a member of the Selberg class S. (For the properties of Rankin-Selberg *L*-functions used here, see [6].) Therefore, by (2.3), there are functions $F_j \in S$ and integers e_j such that one has

$$L(s, \operatorname{Sym}^m \pi \times \chi) = \prod_{j=1}^{\ell} F_j(s)^{e_j}.$$
(2.4)

Since every element in S has a factorisation into primitive functions, we may assume that all functions F_j are primitive and distinct. Similar to the proof of Lemma 2.2, by (2.4), we have

$$\sum_{p \le x} \frac{|\lambda_{\text{Sym}^m \, \pi \times \chi}(p)|^2}{p} = \sum_{p \le x} \frac{1}{p} \Big| \sum_{j=1}^{\ell} e_j a_{F_j}(p) \Big|^2 \tag{2.5}$$

for $x \ge 0$. Moreover, assuming $\text{SOC}(F_j)$ for all j and using (2.2) and (2.5), we obtain $1 = \sum_{j=1}^{\ell} e_j^2$. Thus, as argued in the last part of the proof of Lemma 2.2, we have $L(s, \text{Sym}^m \pi \times \chi) = F_{j_0}$ for some j_0 . In other words, the *L*-function $L(s, \text{Sym}^m \pi \times \chi)$ is primitive. \Box

2.4. Proof of Theorem 1.3

We begin with the following proposition.

Proposition 2.4. Let π , K, and G be as in Theorem 1.2. Suppose that the L-function $L(s, (\text{Sym}^m \pi)|_K)$ admits a factorisation of the form

$$L(s, (\text{Sym}^m \pi)|_K) = \prod_j L(s, \pi_j)^{e_j},$$
(2.6)

for some cuspidal representations π_j over \mathbb{Q} , and $e_j \in \mathbb{N}$. Assume that the conjecture $SOC(F_j)$ holds for each F_j appearing in (2.4) and that the conjectures $SOC((Sym^m \pi)|_K)$ and $SOC(\pi_j)$ are all valid. Then for every $\chi \in Irr(G)$, the L-function $L(s, Sym^m \pi \times \chi)$ is automorphic over \mathbb{Q} .

Proof. By the work of Deligne, the Ramanujan-Petersson conjecture holds for π . Hence, comparing the local Euler factors on both sides of the decomposition (2.6), we deduce that each $L(s, \pi_j)$ satisfies the Ramanujan-Petersson conjecture. Thus, under $SOC(\pi_j)$, Lemma 2.3 yields that each $L(s, \pi_j)$ is primitive. On the other hand, we have the classical factorisation

$$L(s, (\operatorname{Sym}^m \pi)|_K) = \prod_{\chi \in \operatorname{Irr}(\operatorname{Gal}(K/\mathbb{Q}))} L(s, \operatorname{Sym}^m \pi \times \chi)^{\chi(1)}.$$
 (2.7)

By Theorem 1.2, each $L(s, \operatorname{Sym}^m \pi \times \chi)$ is a primitive function in the Selberg class S. Note that the *L*-function $L(s, (\operatorname{Sym}^m \pi)|_K)$ factorises into a product of primitive functions *uniquely* under SOC(((\operatorname{Sym}^m \pi)|_K)). Hence, by (2.6) and (2.7), each $L(s, \operatorname{Sym}^m \pi \times \chi)$ must be $L(s, \pi_j)$ for some j, which concludes the proof. \Box

Although the assumption (2.6) seems quite strong, it does not require the automorphy of $\operatorname{Sym}^m \pi$ over \mathbb{Q} or the automorphy of $(\operatorname{Sym}^m \pi)|_K$ over K. What the assumption (2.6) requires is that the formal *L*-function $L(s, (\operatorname{Sym}^m \pi)|_K)$ factors into a product of automorphic *L*-functions over \mathbb{Q} . This may be regarded as a kind of potential automorphy. As can be seen in the proof below, such a potential automorphy follows from the automorphy of $\operatorname{Sym}^m \pi$ over \mathbb{Q} if the extension K/\mathbb{Q} is solvable.

We now prove Theorem 1.3.

Proof of Theorem 1.3. We may assume $K \neq \mathbb{Q}$. Since the Galois extension K/\mathbb{Q} is solvable, there is a chain of cyclic extensions $\mathbb{Q} = K_0 \subset K_1 \subset \cdots \subset K_\ell = K$, for some $\ell \in \mathbb{N}$, so that each K_i/K_{i-1} is of prime degree. By Arthur-Clozel's base change [1, Sec. 3.4 and 3.5], the base change $(\text{Sym}^m \pi)|_K$ exists as an automorphic representation over K. Moreover, the automorphic induction of $(\text{Sym}^m \pi)|_K$ from K to \mathbb{Q} exists as an automorphic representation over \mathbb{Q} . Thus, the L-function $L(s, (\text{Sym}^m \pi)|_K)$ factorises into a product of cuspidal L-functions over \mathbb{Q} . Hence, by Proposition 2.4, we prove the first assertion of the theorem. Finally, the last part of the theorem follows from the work of Clozel and Thorne [8] on the automorphy of $\text{Sym}^m \pi$ for $m \leq 8$. \Box As may be noticed, in the proof of Theorem 1.3, the focus is on $L(s, (\text{Sym}^m \pi)|_K)$, and the *L*-functions $L(s, \text{Sym}^m \pi)$ and $L(s, \text{Sym}^m \pi \times \chi)$ do not appear. It is because the *L*-function $L(s, \text{Sym}^m \pi)$ and its twists were already used to prove Proposition 2.4, which the proof of Theorem 1.3 subtly relies on.

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