

# Zeros of Dedekind zeta functions and holomorphy of Artin L-functions

Peng-Jie Wong

*Department of Mathematics, Queen's University, Kingston, Ontario K7L 3N6, Canada  
e-mail: 13pjw@queensu.ca*

*Communicated by: Prof. Sanoli Gun*

Received: November 22, 2017

**Abstract.** For any Galois extension  $K/k$  of number fields, we show that every Artin L-function for  $\text{Gal}(K/k)$  is holomorphic at  $s = s_0 \neq 1$  whenever the quotient  $\zeta_K(s)/\zeta_k(s)$  of Dedekind zeta functions has a zero of order at most  $\max\{2, p_2 - 2\}$  at  $s = s_0$  (here  $p_2$  stands for the second smallest prime divisor of  $[K : k]$ ). This result gives a refinement of the previous work of Foote and V. K. Murty.

*2010 Mathematics Subject Classification:* 11R42, 20C15.

## 1. Introduction

More than a century ago, Dedekind studied the quotients of his zeta functions and proved that the quotient  $\zeta_M(s)/\zeta(s)$  is entire for every pure cubic extension  $M/\mathbb{Q}$ . This result suggests that the quotient  $\zeta_M(s)/\zeta_k(s)$  of Dedekind zeta functions is entire whenever  $M/k$  is an extension of number fields, what is now called the Dedekind conjecture. In a slightly different theme, for any  $G = \text{Gal}(K/k)$ , Artin conjectured that every L-function  $L(s, \chi, K/k)$  attached to a non-trivial irreducible character  $\chi$  of  $G$  extends to an entire function. According to Artin reciprocity, this conjecture is valid if  $\chi$  is a monomial character, i.e., a character induced from a 1-dimensional character. From this and his induction theorem, Brauer further showed that each  $L(s, \chi, K/k)$  extends to a meromorphic function over  $\mathbb{C}$ .

Using the theory of induced representations, Aramata and Brauer independently proved the Dedekind conjecture for every Galois extension of number fields. Furthermore, if  $M$  is contained in a solvable normal closure of  $k$ , Uchida [18] and van der Waall [19] established the Dedekind conjecture in this case. In general, the Dedekind conjecture would follow from Artin's conjecture. However, both conjectures are still open.

Nevertheless, several important progressions have been made by Langlands and many others (cf. [11–13, 15, 17]) for (primitive) irreducible representations of dimension 2 and 4. In particular, by the work of Langlands and Tunnell, Artin's conjecture holds for all 2-dimensional Galois representations with solvable image. Furthermore, the holomorphy of Artin L-functions attached to odd 2-dimensional Galois representations (over  $\mathbb{Q}$ ) follows from Serre's modularity conjecture, which is now a theorem of Khare and Wintenberger. Also, in light of the work of Artin and Brauer on Artin L-functions and induced (Galois) representations, the author [21] recently showed that Artin's conjecture holds for any *nearly supersolvable*  $G = \text{Gal}(K/k)$ , i.e.,  $G$  admits a normal subgroup  $N$  with  $G/N$  supersolvable such that all irreducible characters of  $N$  are of degree at most 2. Moreover, Artin's conjecture is established for all solvable groups of order at most 200, only one group of order 108 excepted, in [22].

The character-theoretic approach was also adapted by Heilbronn. Indeed, Heilbronn introduced what now are called Heilbronn characters and gave a simple proof of the Aramata-Brauer theorem. In the same spirit of Heilbronn, Stark proved Artin's conjecture is “locally” true if  $\text{ord}_{s=s_0} \zeta_K(s) \leq 1$ , where as later,  $s_0$  denotes a complex number not equal to 1. Inspired by these works, Foote and V. K. Murty [7] derived an inequality for all Heilbronn characters (cf. Proposition 3.2), which yields a variant of the Aramata-Brauer theorem. Moreover, they obtained a partial generalisation of Stark's result for solvable Galois extensions as follows.

**Theorem 1.1 ([7]).** *For any solvable  $G = \text{Gal}(K/k)$  with  $|G| = p_1^{n_1} \cdots p_k^{n_k}$ , where  $p_1 < \cdots < p_k$  are distinct primes, if  $\text{ord}_{s=s_0} \zeta_K(s) \leq p_2 - 2$ , then all Artin L-functions  $L(s, \chi, K/k)$  are holomorphic at  $s = s_0$ .*

We remark that this result has been improved by Foote and Wales (cf. [6,8]) as well as M. R. Murty and Raghuram [14]. In particular, Foote and Wales proved the following.

**Theorem 1.2 ([8]).** *Let  $K/k$  be a solvable Galois extension of number fields. If  $\text{ord}_{s=s_0} \zeta_K(s)$  is at most 2, then all Artin L-functions  $L(s, \chi, K/k)$  are holomorphic at  $s = s_0$ .*

Also, M. R. Murty and Raghuram generalised the theorem of Uchida-van der Waall and then refined Foote-V. K. Murty's inequality. As we shall demonstrate in Section 3, M. R. Murty-Raghuram's inequality implies that for any solvable  $\text{Gal}(K/k)$ , Artin's conjecture is locally true at  $s = s_0$  if the order of the quotient  $\zeta_K(s)/\zeta_k(s)$  at  $s = s_0$  is at most 1. Clearly, this refines the above-mentioned result of Stark.

In light of the above discussion, it is natural to ask if one can refine Theorems 1.1 and 1.2. The object of this note is to give an affirmative answer by proving the following theorems.

**Theorem 1.3.** *Let  $K/k$  be a Galois extension of number fields with solvable Galois group  $G$ , and write  $|G| = p_1^{n_1} \cdots p_k^{n_k}$ , where  $p_1 < \cdots < p_k$  are distinct primes. Then all Artin L-functions  $L(s, \chi, K/k)$  are holomorphic at  $s = s_0$  whenever  $\text{ord}_{s=s_0}(\zeta_K(s)/\zeta_k(s)) \leq p_2 - 2$ .*

**Theorem 1.4.** *Let  $K/k$  be a solvable Galois extension of number fields. If*

$$\text{ord}_{s=s_0}(\zeta_K(s)/\zeta_k(s)) \leq 2,$$

*then all Artin L-functions  $L(s, \chi, K/k)$  are holomorphic at  $s = s_0$ .*

## 2. Preliminaries

As mentioned in the introduction, the theory of characters plays a role in helping one to study Artin L-functions. For the reader's convenience, we shall start by collecting some results from the theory of characters.

We first recall an important theorem from Clifford's theory (see, for example, [5, pp. 53-54]), which allows one to obtain information of irreducible characters from normal subgroups of prime index.

**Proposition 2.1.** *Let  $N$  be a normal subgroup of  $G$  with  $[G : N] = p$ , a prime. Then for any  $\chi \in \text{Irr}(G)$ , either*

1.  $\chi|_N$  is irreducible, or
2.  $\chi|_N = \sum_{i=1}^p \phi_i$ , where  $\phi_i$  are distinct and irreducible. Moreover,  $\chi = \text{Ind}_N^G \phi_i$ .

One also has an imprimitivity criterion due to Blichfeldt (see [4, Corollary 50.7]), which is, in fact, a consequence of Clifford's theory.

**Lemma 2.2.** *If  $G$  admits a non-central abelian normal subgroup, then all faithful irreducible characters of  $G$  are imprimitive.*

We invoke a theorem of Ito (cf. [5, p. 128]) that gives a sufficient condition for Sylow subgroups being abelian and normal.

**Lemma 2.3.** *Suppose that  $G$  is solvable, and that  $G$  has a faithful irreducible character of degree less than  $q - 1$ . Then  $G$  admits an abelian normal Sylow  $q$ -subgroup.*

To end this section, we recall a famous result of Huppert (cf. [9, Theorem 6.23]), which yields a large class of (solvable) M-groups.

**Proposition 2.4.** *Assume that  $G$  is solvable, and that  $G$  admits a normal subgroup  $N$  such that  $G/N$  is supersolvable. If all Sylow subgroups of  $N$  are abelian, then  $G$  is an M-group.*

### 3. Heilbronn characters

In this section, we shall give a brief review of the theory of Heilbronn characters. As before, let  $K/k$  be a Galois extension of number fields with Galois group  $G$ . Fix  $s_0 \in \mathbb{C} \setminus \{1\}$ . The Heilbronn character (of  $K/k$  with respect to  $s = s_0$ ) is defined as

$$\Theta_G := \sum_{\chi \in \text{Irr}(G)} n_\chi \chi,$$

where  $n_\chi := \text{ord}_{s=s_0} L(s, \chi, K/k)$ .

One has a ‘‘lemma’’ of Heilbronn-Stark as stated below.

**Proposition 3.1 (Heilbronn-Stark Lemma).** *For every subgroup  $H$  of  $G$ ,*

$$\Theta_G|_H = \Theta_H.$$

From this, Foote and V. K. Murty [7] derived:

**Proposition 3.2.** *Let  $K/k$  and  $G$  be as above. Then one has*

$$\sum_{\chi \in \text{Irr}(G)} (\text{ord}_{s=s_0} L(s, \chi, K/k))^2 \leq (\text{ord}_{s=s_0} \zeta_K(s))^2.$$

Thus, for any  $\chi \in \text{Irr}(G)$ , the quotient

$$\frac{\zeta_K(s)}{L(s, \chi, K/k)}$$

is holomorphic at  $s = s_0 \neq 1$ . In particular, by choosing  $\chi = 1_G$ , the trivial character of  $G$ , one then deduces the Aramata-Brauer theorem as mentioned earlier. Besides, this implies the following result of Stark.

**Proposition 3.3.** *If  $\text{ord}_{s=s_0} \zeta_K(s) \leq 1$ , all Artin L-functions  $L(s, \chi, K/k)$  are holomorphic at  $s = s_0$ .*

This result was improved by Foote and V. K. Murty (cf. Theorem 1.1) for solvable extensions. Moreover, from the classification of finite simple groups, Foote and Wales [8] derived the following result, which together with the Langlands-Tunnell theorem yields Theorem 1.2.

**Proposition 3.4.** *Suppose that  $G$  is a finite group with a virtual character  $\theta$  satisfying:*

1.  $\theta(1) \leq 2$ ,
2.  $\theta$  is not a character of  $G$  but its restriction to every proper subgroup of  $G$  is a character, and
3. if  $\chi \in \text{Irr}(G)$  appears in  $\theta$  with  $(\chi, \theta) < 0$ , then  $\chi$  is faithful, non-linear, and primitive.

*Then  $G$  is either a semidirect product of the quaternion group of order 8 and a cyclic 3-group, or  $SL(2, p)$  for some prime  $p \geq 5$ .*

In a different vein, M. R. Murty and Raghuram derived the following variant of the Uchida-van der Waall theorem, which gives an improvement of Proposition 3.2.

**Theorem 3.5 ([14]).** *Let  $K/k$  be a solvable Galois extension of number fields with Galois group  $G$ . Then for every subgroup  $H$  of  $G$  and every  $\phi \in \text{Irr}(H)$  of degree one,*

$$\frac{L(s, \text{Ind}_H^G \phi, K/k)}{L(s, \chi, K/k)^{(\chi|_H, \phi)}}$$

*is holomorphic at  $s = s_0 \neq 1$  whenever  $\chi(1) = 1$ . Moreover, for any  $\chi_0 \in \text{Irr}(G)$  of degree one, one has*

$$\sum_{\chi \neq \chi_0} (\text{ord}_{s=s_0} L(s, \chi, K/k))^2 \leq \left( \text{ord}_{s=s_0} \frac{\zeta_K(s)}{L(s, \chi_0, k/K)} \right)^2.$$

We note that assuming  $M/k$  is an extension of number fields contained in a solvable Galois extension  $K/k$  with Galois group  $G$ , there exists a subgroup  $H$  of  $G$  fixing  $M$ . By applying this theorem with  $\chi = 1_G$  and  $\phi = 1_H$ , it is easy to see that  $\zeta_M(s)/\zeta_k(s)$  is entire, which is the Uchida-van der Waall theorem. Also, via this theorem, one can partially improve Stark's result as follows.

**Corollary 3.6.** *For solvable  $\text{Gal}(K/k)$ , all Artin L-functions  $L(s, \chi, K/k)$  are holomorphic at  $s = s_0 \neq 1$  if  $\chi_0(1) = 1$  and the order of  $\zeta_K(s)/L(s, \chi_0, k/K)$  at  $s = s_0$  is at most 1.*

*Proof.* If  $\text{ord}_{s=s_0}(\zeta_K(s)/L(s, \chi_0, k/K)) = 0$ , then the corollary follows from the above theorem immediately. Otherwise, for  $\text{ord}_{s=s_0}(\zeta_K(s)/L(s, \chi_0, k/K)) = 1$ , by the Artin-Takagi decomposition

$$\zeta_K(s) = L(s, \chi_0, K/k) \prod_{\chi \neq \chi_0} L(s, \chi, K/k)^{\chi(1)},$$

one can conclude that there exists an irreducible character  $\chi' \neq \chi_0$  (say) of  $G$  such that  $\text{ord}_{s=s_0} L(s, \chi', k/K) \neq 0$ . Also, Theorem 3.5 forces that there is at most one such an irreducible character. Thus, for every  $\chi \notin \{\chi_0, \chi'\}$ ,  $n_\chi = 0$ . Now the Artin-Takagi decomposition assures that both  $\chi'(1)$  and  $n_{\chi'}$  must be equal to 1, which completes the proof.  $\square$

#### 4. Lemmata

Let  $\mathcal{P}$  be a collection of subgroup-closed and quotient-closed properties of finite groups, namely, if  $G$  has the property  $\mathcal{P}$  and  $H$  is either a subgroup or a quotient group of  $G$ , then  $H$  also satisfies  $\mathcal{P}$ . Let  $\mathcal{S}$  be the collection of isomorphic classes of finite groups with the property  $\mathcal{P}$ , and let  $\{r_G\}_{G \in \mathcal{S}}$  be a set of integers such that if  $H$  is either a subgroup or a quotient group of  $G$ , then  $r_G \leq r_H$ .

Assume that  $\mathcal{P}$  contains the property of solvability. Fix  $s_0 \in \mathbb{C} \setminus \{1\}$ . Let  $\mathcal{S}_f$  be the collection consisting of groups  $G \in \mathcal{S}$  that fail the following statement.

*Let  $K/k$  be a Galois extension of number fields with Galois group  $G$ . All Artin L-functions  $L(s, \chi, K/k)$  are holomorphic at  $s = s_0$  if  $\text{ord}_{s=s_0}(\zeta_K(s)/\zeta_k(s)) \leq r_G$ .*

We then pick a minimal element  $G_{mc}$  from  $\mathcal{S}_f$  in the sense that for any  $H \in \mathcal{S}$  with  $|H| < |G_{mc}|$ , one has  $H \notin \mathcal{S}_f$ . We shall call  $G_{mc}$  a minimal counterexample (for the above statement with respect to  $\{r_G\}_{G \in \mathcal{S}}$ ). For the sake of convenience, we shall write  $G$  for  $G_{mc}$  and let  $Q_k^M(s)$  stand for the quotient  $\zeta_M(s)/\zeta_k(s)$  of Dedekind zeta functions.

Adapting the method developed in [7], we have the following sequence of lemmata concerning the properties of the Heilbronn character  $\Theta_G$ .

**Lemma 4.1.**  *$\Theta_G$  is not a character of  $G$ , but  $\Theta_G|_H$  is a character of any proper subgroup  $H$  of  $G$ .*

*Proof.* By the assumption on  $G$ , there is an irreducible character  $\chi$  of  $G$  such that  $n_\chi$  is negative. Since  $n_\chi$  is the coefficient of  $\chi$  in the Heilbronn character  $\Theta_G$ ,  $\Theta_G$  is not a character of  $G$ .

Now let  $H$  be a proper subgroup of  $G$ . Since  $r_G \leq r_H$ ,  $\text{ord}_{s=s_0} Q_k^K(s) \leq r_H$ . According to the Aramata-Brauer theorem, both  $Q_k^K(s)$  and  $Q_M^K(s)$  are entire, where  $M$  denotes the subfield of  $K/k$  fixed by  $H$ . On the other hand, the Uchida-van der Waall theorem asserts that  $Q_k^M(s)$  is also entire. Thus,  $Q_M^K(s)$  divides  $Q_k^K(s)$  as entire functions, and so  $\text{ord}_{s=s_0} Q_M^K(s) \leq \text{ord}_{s=s_0} Q_k^K(s)$ . Consequently,  $\text{ord}_{s=s_0} Q_M^K(s) \leq r_H$ . Now by the minimality of  $G$ ,  $n_\psi$  is non-negative for any irreducible character  $\psi$  of  $H$ , i.e.,  $\Theta_H$  is a character of  $H$ .  $\square$

**Lemma 4.2.** *If  $\chi$  is an irreducible character of  $G$  such that  $n_\chi < 0$ , then  $\chi$  must be faithful, i.e.,  $\text{Ker } \chi$  is trivial.*

*Proof.* Suppose, on the contrary, that  $\text{Ker } \chi$  is non-trivial, and let  $M$  denote its fixed field. Since  $\text{Ker } \chi$  is a normal subgroup of  $G$ , by the fundamental theorem of Galois theory and the inflation property of Artin L-functions,  $L(s, \chi, k)$  is the same whether viewed as an Artin L-function of  $K/k$  or an Artin L-function of  $M/k$ . Let  $N$  denote the Galois group of the field extension  $M/k$ . Since the Aramata-Brauer theorem assures that all the quotients  $Q_k^K(s)$ ,  $Q_M^K(s)$ , and  $Q_k^M(s)$  are entire, one can conclude that  $\text{ord}_{s=s_0} Q_k^M(s) \leq \text{ord}_{s=s_0} Q_k^K(s)$ . Thus,

$$\text{ord}_{s=s_0} Q_k^M(s) \leq r_G \leq r_N.$$

However, if we regard  $\chi$  as a character of  $N$ , the minimality of  $G$  implies that  $n_\chi$  is non-negative, which contradicts the assumption that  $n_\chi < 0$ .  $\square$

**Lemma 4.3.** *If  $\chi$  is an irreducible character of  $G$  such that  $n_\chi < 0$ , then  $\chi$  is not induced from any proper subgroup of  $G$ .*

*Proof.* If there was a character  $\psi$  of a proper subgroup  $H$  of  $G$  such that  $\chi = \text{Ind}_H^G \psi$ , then the induction-invariance property of Artin L-functions implies that

$$n_\chi = \text{ord}_{s=s_0} L(s, \psi, K/K^H),$$

where  $K^H$  is the fixed field of  $H$ . However, the latter value is non-negative by Lemma 4.1, a contradiction.  $\square$

Following [7], we now decompose  $\Theta_G$  into three parts  $\theta_1, \theta_2$ , and  $\theta_3$  as follows. Let  $\theta_3$  be the sum of all terms  $n_\lambda \lambda$  so that  $\lambda$  is not faithful, and let  $-\theta_2$  be the sum of all terms  $n_\chi \chi$  for which  $n_\chi$  is negative. According to Lemma 4.2, one has

$$(\theta_2, \theta_3) = 0.$$

Finally,  $\theta_1$  is defined by

$$\theta_1 := \Theta_G + \theta_2 - \theta_3.$$

It is clear that  $\theta_1$  is the sum of all terms  $n_\psi \psi$  where  $\psi$  is a faithful irreducible character with  $n_\psi > 0$ , and that  $\theta_1$  is orthogonal to  $\theta_2$  and  $\theta_3$ .

We will derive further information about  $\theta_2$  and  $\theta_3$  by restricting to appropriate subgroups of  $G$ . As one shall see, the main tool is Clifford's theorem. Observe that since  $G$  is solvable and non-abelian, it admits a normal subgroup  $N$  of prime index  $p$  (say) that contains the centre  $\mathbf{Z}(G)$  of  $G$ .

**Lemma 4.4.** *Let  $N$  be a normal subgroup of  $G$  of prime index  $p$  containing the centre  $\mathbf{Z}(G)$  of  $G$ . Then one has*

$$(\theta_2|_N, \theta_3|_N) = 0.$$

Moreover,  $\theta_1|_N - \theta_2|_N$  is either zero or equal to a character  $\phi$  of  $N$ .

*Proof.* First of all, as  $[G : N]$  is prime, Clifford's theorem (cf. Proposition 2.1) asserts that for every  $\chi \in \text{Irr}(G)$ , either  $\chi|_N$  is irreducible or  $\chi$  is induced from  $N$ . In particular, if  $\chi$  is a constituent of  $\theta_2$ , Lemma 4.3 forces that  $\chi|_N$  must be irreducible.

Secondly, a theorem of Blichfeldt, Lemma 2.2, tells us that if  $G$  had a non-central abelian normal subgroup, then every faithful irreducible character of  $G$  would be induced from a proper subgroup. However, Lemma 4.3 forces that it cannot happen. In other words, all abelian normal subgroups of  $G$  must be central, i.e., contained in the centre  $\mathbf{Z}(G)$ .

Now as every non-trivial normal subgroup of  $G$  possesses a non-trivial abelian subgroup that is normal in  $G$ , it follows that for every non-faithful  $\lambda$  of  $G$ ,

$$\text{Ker } \lambda \cap \mathbf{Z}(G)$$

is non-trivial. In other words, none of irreducible constituents of  $\theta_3$  is faithful on  $\mathbf{Z}(G)$ . Since  $\mathbf{Z}(G) \subseteq N$ , one then derives

$$(\theta_2|_N, \theta_3|_N) = 0. \tag{4.1}$$

Finally, the Heilbronn-Stark lemma and Lemma 4.1 tell us that

$$\Theta_N = \theta_1|_N - \theta_2|_N + \theta_3|_N$$

is a character of  $N$ , which together with (4.1) completes the proof.  $\square$

**Lemma 4.5.** *Suppose that  $\theta_1|_N - \theta_2|_N$  is a character  $\phi$  of  $N$ . Then there is an irreducible character  $\psi$  appearing in  $\theta_1 - \theta_2$  such that*

$$\psi(1) \leq r_G.$$

*Proof.* Choosing an irreducible constituent  $\phi_1$  of  $\phi := \theta_1|_N - \theta_2|_N$ , let  $\psi$  be an irreducible constituent of  $\theta_1 - \theta_2$  such that  $\psi|_N$  contains  $\phi_1$ . If  $\psi|_N = \phi_1$ , then it is clear that

$$\psi(1) = \phi_1(1) \leq \phi(1).$$

Otherwise, since  $[G : N] = p$ , Proposition 2.1 asserts that

$$\psi|_N = \sum_{i=1}^p \phi_i,$$

where  $\phi_i$ 's are distinct  $G$ -conjugate irreducible characters. Furthermore, as  $\phi_1$  appears in  $\phi = \theta_1|_N - \theta_2|_N$  and  $\phi$  is a  $G$ -stable character of  $N$ , each  $\phi_i$  must appear in  $\phi$ . In particular,

$$\psi(1) = \sum_{i=1}^p \phi_i(1) \leq \phi(1).$$

Together with our assumption and the fact that the trivial character is not faithful, the Heilbronn-Stark lemma then yields

$$\begin{aligned} r_G &\geq \text{ord}_{s=s_0} \frac{\zeta_K(s)}{\zeta_k(s)} \\ &= \Theta_G(1) - n_{1_G} \\ &= \Theta_N(1) - n_{1_G} 1_N(1) \\ &= \phi(1) + \theta'_3(1) \\ &\geq \phi(1), \end{aligned}$$

where the last inequality holds provided that  $\theta'_3 := \theta_3|_N - n_{1_G} 1_N$  is either zero or a character. Hence, the lemma follows.  $\square$

**Lemma 4.6.** *Let  $H$  be the subgroup generated by  $\mathbf{Z}(G)$  and an element  $x \in G \setminus N$ . Then*

$$(\theta_2|_H, \theta_3|_H) = 0.$$

*In addition,  $\theta_1|_H - \theta_2|_H$  is either zero or equal to a character of  $H$ .*

*Proof.* Since  $H$  is abelian, it must be a proper subgroup of  $G$ . As argued earlier, for every irreducible constituent  $\lambda$  of  $\theta_3$ ,  $\text{Ker } \lambda \cap \mathbf{Z}(G)$  is non-trivial. Thus, the same holds for  $\text{Ind}_H^G(\lambda|_H)$ . As any irreducible constituent  $\chi$  of  $\theta_2$  is faithful, one has

$$(\chi, \text{Ind}_H^G(\lambda|_H)) = 0.$$

Now Frobenius reciprocity tells us that  $(\chi|_H, \lambda|_H) = 0$ , and thus

$$(\theta_2|_H, \theta_3|_H) = 0.$$

By an analogous argument as before, the lemma follows.  $\square$

**Lemma 4.7.** *If  $\theta_1|_N = \theta_2|_N$ , then  $\theta_1 = \theta_2$ . In other words, if the minimal counterexample  $G = G_{mc}$  exists, then  $\theta_1|_N \neq \theta_2|_N$ .*

*Proof.* Assuming that  $\theta_1|_N = \theta_2|_N$ , one has  $\theta_1(1) = \theta_2(1)$ , which combining with Lemma 4.6 asserts that

$$\theta_1|_H = \theta_2|_H$$

for any subgroup generated  $H$  by  $\mathbf{Z}(G)$  and an element  $x \in G \setminus N$ . In particular,

$$\theta_1(x) = \theta_2(x)$$

for all  $x \in G \setminus N$ . Now as  $\theta_1|_N = \theta_2|_N$ , we then derive  $\theta_1 = \theta_2$ .  $\square$

## 5. Proofs of the main results

As before, we fix  $s_0 \in \mathbb{C} \setminus \{1\}$ . Also, we shall keep using notation introduced in Section 4. Now we are in the position to prove Theorem 1.3.

**Proof of Theorem 1.3** First of all, we let  $\mathcal{P}$  be the property that groups are solvable. For each  $G$  satisfying the property  $\mathcal{P}$ , we set  $r_G$  to be equal to  $p_2 - 2$ , where  $p_2$  denotes the second smallest prime divisor of  $|G|$ .

If the theorem was false, then we take  $G$  as a minimal counterexample. In the notation of the previous section, we may assume that  $\theta_1|_N - \theta_2|_N$  is a character  $\phi$  of  $N$ . Then Lemma 4.5 asserts that there is a faithful irreducible character  $\psi$  appearing in  $\theta_1 - \theta_2$  such that

$$\psi(1) \leq p_2 - 2.$$

Since  $G$  is solvable, Lemma 2.3 tells us that if  $G$  has a faithful irreducible character of degree less than  $q - 1$ , then any Sylow  $q$ -subgroup  $G$  is abelian and normal. As  $\psi$  is faithful,  $G$  admits an abelian normal subgroup  $A$  such that  $G/A$  is of order  $p_1^{n_1}$ . However, by Huppert's result, Proposition 2.4, it would force  $G$  to be an M-group, a contradiction. Therefore,  $\theta_1|_N = \theta_2|_N$ . However, now Lemma 4.7 would lead to another contradiction. In other words, such a minimal counterexample cannot exist, and hence the theorem follows.  $\square$

By the celebrated theorem of Feit and Thompson, all groups of odd order are solvable. We then immediately obtain:

**Corollary 5.1.** *Assume that  $K/k$  is Galois and of odd degree. If  $\text{ord}_{s=s_0}(\zeta_K(s)/\zeta_k(s)) \leq 3$ , then every Artin L-function  $L(s, \chi, K/k)$  is holomorphic at  $s = s_0$ .*

We note that if  $|G|$  is divisible by 6, then Theorem 1.3 follows from Theorem 3.5 (cf. Corollary 3.6). On the other hand, one can see that the proof relies on the “Sylow structure” of  $G$ . In light of this observation, we give some variants below.

**Theorem 5.2.** *Let  $K/k$  be a solvable Galois extension of number fields with Galois group  $G$ . Assume that either*

1.  $|G| = q^\ell p^n p_1^{n_1} \cdots p_k^{n_k}$ , where  $q < p < p_1 < \cdots < p_k$  are distinct primes with  $q \mid (p-1)$  and  $\ell \leq 1$ , or
2.  $|G| = q^\ell p^n p_1^{n_1} \cdots p_k^{n_k}$ , where  $q < p < p_1 < \cdots < p_k$  are distinct primes with  $q^2 \mid (p-1)$  and  $\ell \leq 2$ .

*If  $\text{ord}_{s=s_0}(\zeta_K(s)/\zeta_k(s)) \leq p_1 - 2$ , then all Artin L-functions  $L(s, \chi, K/k)$  are holomorphic at  $s = s_0$ .*

*Proof.* This time we may let  $\mathcal{P}$  be the property that either

1. For fixed primes  $q < p$  such that  $q \mid (p-1)$ ,  $G$  is a solvable group of order  $q^\ell p^n p_1^{n_1} \cdots p_k^{n_k}$ , where  $p < p_1 < \cdots < p_k$  are fixed distinct primes and  $\ell \leq 1$ , or
2. For fixed primes  $q < p$  such that  $q^2 \mid (p-1)$ ,  $G$  is a solvable group of order  $q^\ell p^n p_1^{n_1} \cdots p_k^{n_k}$ , where  $p < p_1 < \cdots < p_k$  are fixed distinct primes and  $\ell \leq 2$ .

It is clear that these properties are subgroup-closed and quotient-closed. We then set  $r_G = p_1 - 2$ .

Now assume the claim was false and let  $G$  be a minimal counterexample. Under the same notation as before, we again suppose that  $\theta_1|_N - \theta_2|_N$  is a character  $\phi$  of  $N$ . Then Lemma 4.5 tells us that there exists an irreducible character  $\psi$  appearing in  $\theta_1 - \theta_2$  such that

$$\psi(1) \leq p_1 - 2.$$

Now Ito's theorem assures that all Sylow  $p_j$ -subgroups of  $G$  are abelian and normal.

Thus,  $G$  admits an abelian normal subgroup  $A$  of index  $q^\ell p^n$  (with  $\ell \leq 1$  and  $q \mid (p-1)$ ) or  $q^\ell p^n$  (with  $\ell \leq 2$  and  $q^2 \mid (p-1)$ ), which means that  $G/A$  is necessarily supersolvable (cf. [20]). Now  $G$  is an M-group (à la Huppert), a contradiction. Therefore,  $\theta_1|_N = \theta_2|_N$ , which together with Lemma 4.7 leads to the final contradiction.  $\square$

**Corollary 5.3.** *Let  $K/k$  be a Galois extension of number fields with Galois group  $G$ . If  $4 \nmid |G|$  and  $\text{ord}_{s=s_0}(\zeta_K(s)/\zeta_k(s)) \leq 3$ , then all Artin L-functions  $L(s, \chi, K/k)$  are holomorphic at  $s = s_0$ .*

*Proof.* We recall that a theorem of Cayley and Burnside asserts that if a Sylow 2-subgroup  $P$  of  $G$  is cyclic, then  $G$  has a normal 2-complement, i.e., there exists a normal subgroup  $N$  of  $G$  such that  $G$  is the semidirect product of  $N$  and  $P$ . From this result and the Feit-Thompson theorem, if  $G$  admits a cyclic Sylow 2-subgroup, then  $G$  is solvable. In particular, if  $4 \nmid |G|$ , then  $G$  is solvable. This fact together with the first part of the above theorem finishes the proof.  $\square$

**Theorem 5.4.** *Let  $K/k$  be a solvable Galois extension of number fields with Galois group  $G$ . Suppose that*

$$|G| = q_1^{m_1} \cdots q_l^{m_l} p_1^{n_1} \cdots p_k^{n_k},$$

where  $q_1 < \cdots < q_l < p_1 < \cdots < p_k$  are distinct primes and all Sylow  $q_i$ -subgroups are abelian for all  $i$ . If  $\text{ord}_{s=s_0}(\zeta_K(s)/\zeta_k(s)) \leq p_1 - 1$ , then all Artin L-functions  $L(s, \chi, K/k)$  are holomorphic at  $s = s_0$ .

*Proof.* Fix primes  $q_1 < \cdots < q_l$  and choose  $\mathcal{P}$  to be the property that

$$|G| = q_1^{m_1} \cdots q_l^{m_l} p_1^{n_1} \cdots p_k^{n_k},$$

where  $q_l < p_1 < \cdots < p_k$  are fixed distinct primes and all Sylow  $q_i$ -subgroups are abelian for all  $i$ . We then pick  $r_G = p_1 - 1$ .

Now suppose that there was a minimal counterexample  $G$ . As before, Lemma 4.7 forces that  $\theta_1|_N - \theta_2|_N$  had to be a character  $\phi$  of  $N$ , and by Lemma 4.5, there was a faithful irreducible character  $\psi$  with  $\psi(1) \leq p_1 - 1$ . Now every Sylow  $p_j$ -subgroup  $P$  of  $G$  must be abelian since  $\psi|_P$  is faithful and of degree less than  $p_j$ . Therefore, all Sylow subgroups of  $G$  are abelian. Now Proposition 2.4 implies that  $G$  is an M-group, a contradiction.  $\square$

To end this note, we will utilise our lemmata proved in the previous section and the result of Foote and Wales, Proposition 3.4, to derive Theorem 1.4.

**Proof of Theorem 1.4** Let  $\mathcal{P}$  only require the solvability of  $G$  and  $r_G = 2$ . Suppose that the theorem was invalid. Then we may pick a minimal counterexample  $G$  (say). Clearly,  $\theta := \Theta_G - n_{1_G} 1_G$  is a virtual character with  $\theta(1) \leq 2$ . Furthermore, Lemmata 4.1–4.3 and Artin reciprocity assure that  $\theta$  satisfies all conditions required in Proposition 3.4. (We note that for any subgroup  $H$  of  $G$ , the Uchida-van der Waall theorem tells us that

$$n_{1_H} - n_{1_G} = \text{ord}_{s=s_0} L(s, 1_H, K/K^H) - \text{ord}_{s=s_0} L(s, 1_G, K/k) = \text{ord}_{s=s_0} \frac{\zeta_{K^H}(s)}{\zeta_k(s)}$$

is non-negative. A moment's reflection shows that  $\theta|_H = \Theta_H - n_{1_G} 1_H$  is still a character.) Thus, as  $G$  is solvable, Proposition 3.4 implies that  $G$  is a semidirect product of the quaternion group of order 8 and a cyclic 3-group. However, the theory of relative M-groups (see, for example, [9, Chapter 6]) tells us that every irreducible character of  $G$  is induced from a character of degree at most 2. Now together with the induction-invariance property of Artin L-functions, Artin reciprocity and the Langlands-Tunnell theorem assure that all Artin L-functions (of  $G = \text{Gal}(K/k)$ ) are holomorphic at  $s = s_0 \neq 1$ , a contradiction.  $\square$

### Acknowledgments

The author would like to thank Professor Ram Murty and the referee for their careful reading as well as making many helpful remarks on previous versions of this manuscript.

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