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Langlands reciprocity for certain Galois extensions

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ABSTRACT

In this note, we study Artin's conjecture via group theory and derive Langlands reciprocity for certain solvable Galois extensions of number fields, which extends the previous work of Arthur and Clozel. In particular, we show that all nearly nilpotent groups and all groups of order less than 60 are of automorphic type.

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1. Introduction

Nearly a century ago, Emil Artin [3] introduced a new kind of L-function, which generalises both Dirichlet L-functions and Hecke L-functions, and made the following famous conjecture.

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Conjecture 1.1 *(Artin's (holomorphy) conjecture). Let K/k be a Galois extension of number fields with Galois group G. For every non-trivial irreducible character χ of G, the Artin L-function $L(s, \chi, K/k)$ attached to χ extends to an entire function.*

Via his celebrated reciprocity law, Artin showed that all his L-functions attached to characters of degree 1 correspond to Hecke L-functions and then established his conjecture in this case. After Artin, Langlands [22] and Tunnell [30] proved Artin's conjecture for any 2-dimensional irreducible representation with solvable image, and it is a major result in the Langlands program. More recently, the case of odd 2-dimensional irreducible representations (of the absolute Galois group of $\mathbb{Q}$) with non-solvable images was settled by Khare and Wintenberger [18]. Moreover, in light of Artin's work, Langlands conjectured that all Artin L-functions are automorphic, which is sometimes called the strong Artin conjecture or the Langlands reciprocity conjecture.

In a different vein, Brauer [6] showed that every Artin L-function admits a meromorphic continuation via his induction theorem. Also, it is well-known that Artin's conjecture is true for any supersolvable Galois extension of number fields. This follows from the fact that supersolvable groups are M-groups, the groups all of whose irreducible characters are monomial. Undoubtedly, these results suggest that the group-theoretic method shall play a role in studying the (strong) Artin conjecture. In fact, by knowing that all subgroups of nilpotent groups are subnormal, the Arthur–Clozel theory, which will be discussed in Section 2, implies that the Langlands reciprocity holds for all nilpotent Galois extensions of number fields.

For non-nilpotent cases, the Langlands reciprocity has been derived for certain solvable Frobenius extensions by Zhang [32], which will be discussed in the next section. (We recall that a finite group G is said to be a Frobenius group if there is a non-trivial proper subgroup H of G such that $g^{-1}Hg \cap H = 1$ whenever $g \in G \backslash H$. In this case, H is called a Frobenius complement of G.) More recently, Langlands reciprocity was established for A_4, S_4, $SL_2(\mathbb{F}_3)$, and $GL_2(\mathbb{F}_3)$-extensions. The first two cases were proved by Cho [8,9] (although Cho in his Ph.D. thesis [8] said these two cases are well-known for experts). Indeed, Cho derived his theorem based on the work of Kim [20] on $SL_2(\mathbb{F}_3)$ and $GL_2(\mathbb{F}_3)$-extensions. Furthermore, under certain conditions, the automorphy of A_5 and S_5 was derived by Kim [19] and Calegari [7], respectively.

In this note, we will apply a method of "low-dimensional groups" developed by the author in [31] to study the strong Artin conjecture. (We call a group low-dimensional if ALL its irreducible characters are of "small degree".) We will say a finite group G is nearly nilpotent if it admits a normal subgroup N, all of whose irreducible characters are of degree less than or equal to 2, such that G/N is nilpotent. These groups will be shown to be solvable and discussed in Sections 2 and 6. Besides, as a theorem of Shafarevich asserts every finite solvable group is realisable over $\mathbb{Q}$, our results below actually present an enlargement of Galois extensions of number fields satisfying the Langlands reciprocity.

Theorem 1.2. *Let K/k be a Galois extension of number fields with Galois group G. If G is either a direct product of two nearly nilpotent groups or a group of order less than 60, then the Langlands reciprocity is true for K/k.*

Theorem 1.3. *Suppose that K/k is a solvable Galois extension with Frobenius Galois group G (with a Frobenius complement H). Then the Artin conjecture holds for K/k. Let $\mathbf{F}(H)$ be the maximal normal nilpotent subgroup of H. If either $H/\mathbf{F}(H)$ is not isomorphic to S_3 or a Sylow 2-subgroup of $\mathbf{F}(H)$ is abelian, then the Langlands reciprocity law is valid for K/k.*

2. Reciprocity law for certain solvable extensions

Throughout this note, G always denotes a finite group, and H and N denote a subgroup and a normal subgroup of G, respectively. We let $\mathbf{Z}(G)$ denote the centre of G, and set $G' = [G, G]$, $G'' = [G', G']$, and $G''' = [G'', G'']$. The direct product of k-copies of G will be denoted as G^k or $(G)^k$. The maximal normal nilpotent subgroup of G, the Fitting subgroup of G, is denoted by $\mathbf{F}(G)$. The cyclic group of order m will be denoted as C_m. We also let $Irr(G)$ be the set of irreducible characters of G, and $\mathrm{cd}(G) := \{\chi(1) | \chi \in Irr(G)\}$. The trivial group will often be denoted by 1. We will usually let p and q denote primes without mentioning.

For any $\chi \in Irr(G)$, χ is said to be of automorphic type if for every Galois extension K/k of number fields with Galois group G, Langlands' reciprocity conjecture holds for the Artin L-function $L(s, \chi, K/k)$. In addition, G is of automorphic type if every irreducible character of G is of automorphic type.

In [32], Zhang showed that certain Frobenius extensions satisfy the Langlands reciprocity conjecture as follows.

Theorem 2.1 *([32]). Let K/k be a Galois extension of number fields with Galois group G. Assume that G is a Frobenius group and H is a Frobenius complement of G. Let $\mathbf{F}(H)$ be the maximal normal nilpotent subgroup of H. If $H/\mathbf{F}(H)$ is nilpotent, then every Artin L-function attached to an irreducible representation of G are automorphic over k.*

Zhang's method employs the theory of Frobenius groups and nilpotent groups as well as Arthur–Clozel's theory of base change and automorphic induction. In light of Zhang's work, we will derive several results which enlarge the class of groups being of automorphic type. Before we state and prove our results, we shall review some concepts of SM-groups as well as the main theorem of Arthur–Clozel's theory.

Theorem 2.2 *([2, Arthur and Clozel]). Let K/k be a cyclic Galois extension of number fields of prime degree, and π and Π denote automorphic representations induced from cuspidal of $GL_n(\mathbb{A}_k)$ and $GL_n(\mathbb{A}_K)$ respectively (or, in particular, cuspidal automorphic representations of $GL_n(\mathbb{A}_k)$ and $GL_n(\mathbb{A}_K)$ respectively). Then the base change $B(\pi)$ of π and the automorphic induction $I(\Pi)$ of Π exist. Moreover, $I(\Pi)$ is induced from cuspidal.*

Definition 1 *([12, Definition 2.3])*. Let G be a finite group, and N be a normal subgroup of G. A character χ of G is called a relative SM-character with respect to N if there exist a subnormal subgroup H with $N \leq H \leq G$ and an irreducible character $\psi \in Irr(H)$ such that $\mathrm{Ind}_H^G \psi = \chi$ and $\psi|_N \in Irr(N)$. If every irreducible character of G is a relative SM-character with respect to N, then G is said to be a relative SM-group with respect to N.

Since all subgroups of a nilpotent group are monomial and subnormal, it is clear that all nilpotent groups are relative SM-groups with respect to the trivial group. Indeed, this fact together with the above-mentioned Arthur–Clozel theorem asserts that all nilpotent groups are of automorphic type as predicted by the Langlands reciprocity conjecture. We refer the interested reader to [2] for the complete details. Nevertheless, we will give a proof later by invoking below a result of Horváth that gives a sufficient condition for groups being relative SM-groups.

Proposition 2.3 *([12, Proposition 2.7]). Let G be a finite group and N be a normal subgroup of G such that G/N is nilpotent. Then G is a relative SM-group with respect to N.*

We remark that in [12], Horváth considered the relation among subgroup-closed M-groups, the groups whose all subgroups are M-groups, SM-groups, and supersolvable groups, and showed that these classes are all distinct.

In the proof of [32, Proposition 4], Zhang used the theory of relative M-groups, which will be discussed in the next section, to show that certain solvable groups are of automorphic type. However, lacking of “subnormality”, it might not be possible to apply the Arthur–Clozel’s theory. Thanks to the above theorem of Horváth, we can overcome this obstacle. Moreover, by invoking below a result of Isaacs, we can drop the assumption on the solvability of groups in Zhang’s theorem.

Proposition 2.4 *([13, Theorems 12.5, 12.6 and 12.15]). If G is a finite group such that $|\operatorname{cd}(G)| \leq 3$, then G must be solvable.*

Theorem 2.5. *Let K/k be a Galois extension of number fields with Galois group G. If there is a normal subgroup N of G such that*

1: *G/N is nilpotent, and*
2: *all irreducible characters of N are of dimension 1 or 2,*

then the Langlands reciprocity law is valid for K/k.

Proof. First of all, by Isaacs’ result, N must be solvable. As G/N is nilpotent, G/N is also solvable. Thus, G is necessarily solvable.

We now prove this theorem by using mathematical induction on the order of G. Since G/N is nilpotent, Proposition 2.3 asserts that G is a relative SM-group with respect to N. Thus, for every $\chi \in Irr(G)$, there exist a subnormal subgroup $N \leq H$ of G and a character $\psi \in Irr(H)$ such that $\mathrm{Ind}_H^G \psi = \chi$ and $\psi|_N \in Irr(N)$.

If H is a proper subgroup of G, as H clearly satisfies conditions 1 and 2, one can apply the induction hypothesis on H. In particular, ψ is automorphic over the fixed field K^H, i.e., there is a cuspidal automorphic form Π of $GL_{\psi(1)}(\mathbb{A}_{K^H})$ such that

$$L(s, \psi, K/K^H) = L(s, \Pi).$$

On the other hand, since H is a subnormal subgroup of G, there is an invariant series

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_{k-1} \trianglelefteq H_k = G,$$

where for each i, H_i is a normal subgroup of H_{i+1}. As G is finite, we may require each H_{i+1}/H_i is a (finite) simple group. Since G is solvable, none of these quotient groups can be non-cyclic. Thus, each H_{i+1}/H_i is a cyclic group of prime order. Now applying the Arthur–Clozel theorem of automorphic induction successively, one can derive that $\mathrm{Ind}_H^G \psi$ corresponds to an automorphic form over k. More precisely, there is an automorphic form π of $GL_{\chi(1)}(\mathbb{A}_k)$ such that

$$L(s, \Pi) = L(s, \pi).$$

Finally, since χ is irreducible, a result of Jacquet and Shalika (see [17, Theorem 4.7]) asserts that π is necessarily cuspidal, which completes the proof in this case.

Otherwise, suppose $H = G$. Since $\chi|_N = \psi|_N \in Irr(N)$ and all irreducible characters of N are of dimension 1 or 2, $\chi(1)$ is at most 2. Now Artin reciprocity together with the theorem of Langlands–Tunnell ensures that χ is automorphic over k. $\square$

As one can tell, this theorem indeed implies that all nilpotent groups are of automorphic type, which covers the earlier-mentioned result of Arthur–Clozel. Now let us borrow the following results of Gow and Jacquet–Piatetski–Shapiro–Shalika that will enable us to obtain a slight improvement of Theorem 2.5.

Lemma 2.6 *([11]). Let χ be an irreducible character of odd degree of a solvable group G. If χ is real-valued, then χ is monomial and rational-valued.*

Theorem 2.7 *([16]). Let K/k be a non-normal cubic extension of number fields. Let χ be an idèle class character of K. Then the automorphic induction $I(\chi)$ of χ exists as an automorphic representation of $GL_3(\mathbb{A}_k)$.*

Combining the Arthur–Clozel theory and these two theorems together gives:

Proposition 2.8. *Let G be a solvable group, and let ρ be an irreducible representation of G of dimension 3 whose character is real-valued. Then ρ is of automorphic type.*

Using a similar argument as in the proof of Theorem 2.5, the above-mentioned result of Isaacs, and this proposition, we then derive:

Proposition 2.9. *Assume that G has a normal subgroup N satisfying*

1: *G/N is nilpotent, and*
2: *all irreducible characters of N are of dimension ≤ 3.*

Suppose, further, that all 3-dimensional irreducible characters (if any) of subnormal subgroups of G containing N are real-valued. Then G is of automorphic type.

Following Ramakrishnan [26], we let $GO(n, \mathbb{C})$ denote the subgroup of $GL(n, \mathbb{C})$ consisting of orthogonal similitudes, i.e., matrices M such that $M^t M = \lambda_M I$, with $\lambda_M \in \mathbb{C}$, and we will say that a $\mathbb{C}$-representation (ρ, V) (of the absolute Galois group of a number field F) is of $GO(n)$-type if and only if $\dim V = n$ and it factors as

$$\rho : \mathrm{Gal}(\overline{F}/F) \to GO(n, \mathbb{C}) \subset GL(V).$$

In his paper [26], Ramakrishnan derived the automorphy of solvable Artin representations of $GO(4)$-type as follows.

Theorem 2.10. *Let F be a number field and let ρ be a continuous, 4-dimensional representation of $\mathrm{Gal}(\overline{F}/F)$ whose image is solvable and lies in $GO(4, \mathbb{C})$. Then ρ is automorphic.*

Applying this theorem together with a similar argument as before then gives:

Proposition 2.11. *Suppose that there is a normal subgroup N of G so that*

1: *G/N is nilpotent, and*
2: *all irreducible characters of N are of dimension either 1, 2, or 4.*

If all 4-dimensional irreducible representations (if any) of subnormal subgroups of G containing N are of $GO(4)$-type. Then G is of automorphic type.

By the theory of Frobenius groups, if G is a Frobenius group with a Frobenius complement H, there exists a normal subgroup N of G such that $G = N \rtimes H$, where N is called a Frobenius kernel of G. Moreover, one has a lemma below.

Lemma 2.12. *Let N be a Frobenius kernel of a Frobenius group G. For $\chi \in Irr(G)$ with $N \nsubseteq \operatorname{Ker}\chi$, one has $\chi = \operatorname{Ind}_N^G \psi$ for some $\psi \in Irr(N)$.*

Also, all Sylow subgroups of a Frobenius complement are cyclic or generalised quaternion groups. Furthermore, a deep theorem of Thompson asserts that every Frobenius kernel must be nilpotent. For more details, we refer the interested reader to [13, Chapter 7]. From this, we have the following proposition.

Proposition 2.13. *Let K/k be a Galois extension of number fields with Galois group G. Suppose $G = N \rtimes H$ is a Frobenius group with a Frobenius kernel N and a Frobenius complement H. If H is solvable and of automorphic type, then so is G.*

Proof. Let χ be an irreducible character of G. If $\operatorname{Ker}\chi$ contains N, then χ can be seen as an irreducible character of H. As H is of automorphic type, χ is automorphic over k.

If $N \nsubseteq \operatorname{Ker}\chi$, then by Lemma 2.12, there is $\psi \in Irr(N)$ such that $\chi = \operatorname{Ind}_N^G \psi$. Since N is nilpotent, N is of automorphic type. In addition, K^N/k is a solvable Galois extension, Arthur–Clozel's theory yields that χ is automorphic over k. □

Now suppose that G is a Frobenius group, and H is a Frobenius complement of G. Assume, further, that the Fitting subgroup $\mathbf{F}(H)$ of H satisfies that $H/\mathbf{F}(H)$ is nilpotent. As every Sylow subgroup of H is either cyclic or a generalised quaternion group, all irreducible characters of $\mathbf{F}(H)$ are of degree 1 or 2. Thus, Theorem 2.5 and Proposition 2.13 assert that G is of automorphic type, which is Zhang's theorem (cf. Theorem 2.1).

We now recall a result that gives a non-trivial bound of character degrees.

Lemma 2.14 *([13, p. 28]). Let G be a finite group and $\mathbf{Z}(G)$ its centre. Then for every irreducible character χ of G, one has*

$$\chi(1)^2 \leq [G : \mathbf{Z}(G)].$$

We now give some sufficient conditions for (solvable) groups being of automorphic type. First of all, as any nilpotent group is isomorphic to a direct product of its Sylow subgroups and the derived subgroup of any supersolvable group is nilpotent, we then have the following corollary.

Corollary 2.15. *If G is a supersolvable group of order $2^n p_1^{n_1} \cdots p_k^{n_k}$ with $n_i \leq 2$ and $n \leq 4$, then G is of automorphic type.*

In addition, since all Z-groups, the groups whose all Sylow subgroups are cyclic, are supersolvable, a moment's reflection shows:

Corollary 2.16. *All Z-groups are of automorphic type. In particular, all groups of square-free order are of automorphic type.*

We recall that a group G is said to be abelian-by-nilpotent if G admits an abelian normal subgroup A with G/A nilpotent. By a result of Huppert, all abelian-by-nilpotent groups are M-groups. Thanks to Theorem 2.5, one can easily conclude that every abelian-by-nilpotent group is even of automorphic type.

This section will close with some semi-numerical theorems, which in particular, present a simple proof for Cho and Kim's automorphy results of A_4, S_4, and $SL_2(\mathbb{F}_3)$-extensions. For $GL_2(\mathbb{F}_3)$, which has been shown to be of automorphic type by Kim, we will treat it in the next section via our method of low dimensional groups.

Corollary 2.17. *Let p and q be distinct primes. If G is of order pq, p^2q, or p^2q^2, then G is of automorphic type.*

Proof. By the Sylow theorems, G must have a normal Sylow subgroup N (see, for example, [14, Theorems 1.30 and 1.31] and [28, 6.5.2]). It is clear that N must be abelian, and that G/N is either a p-group or a q-group. Now the claim follows from Theorem 2.5 immediately. □

Corollary 2.18. *Let p be an odd prime. If G is of order $8p$, then G is of automorphic type.*

Proof. Again the Sylow theorems asserts G admits a normal Sylow subgroup N unless $G \cong S_4$ (see [14, Theorems 1.32 and 1.33]). Assuming that G is not isomorphic to S_4, since all irreducible characters of N are of degree ≤ 2, and G/N is clearly nilpotent, Theorem 2.5 yields G is of automorphic type.

Now suppose G is isomorphic to S_4. Then $\mathrm{cd}(G) = \{1, 2, 3\}$. Since all irreducible characters of S_4 are rational-valued, the Artin–Langlands–Tunnell theorem and Proposition 2.8 assert that G is of automorphic type. □

3. Nearly supersolvable groups and nearly monomial groups

We recall that Taketa's theorem [29] asserts that (finite) M-groups are necessarily solvable. In addition, Taketa's theorem can be generalised (see, for example, [4, Chapter 14]) for groups whose irreducible characters are all induced from n-dimensional characters with $n \leq 2$. We will call these groups nearly monomial (or NM for short). Thanks to the theorem of Artin–Langlands–Tunnell and the generalisation of Taketa's theorem, Artin's conjecture holds for every Galois extension of number fields whose Galois group is an NM-group. In light of Theorem 2.5, we have the notion of nearly supersolvable groups as introduced in [31].

Definition 2. A finite group G is said to be nearly supersolvable (or NSS for short) if it has a normal subgroup $N \in \mathcal{C}$ such that G/N is supersolvable, where $\mathcal{C}$ denotes the class consisting of groups whose irreducible representations are of dimension ≤ 2.

By works of Amitsur and Isaacs (cf. [1] and Proposition 2.4), all groups belonging to $\mathcal{C}$ are necessarily solvable. Therefore, nearly supersolvable groups are indeed solvable. On the other hand, it is clear that all supersolvable groups are nearly supersolvable. For more properties of NSS-groups, we refer the interested reader to [31]. However, for the benefit of the reader, we recall some concepts of relative M-groups and a result concerning NSS-groups.

Definition 3 *([13, Definition 6.21]).* Let G be a finite group, and N be a normal subgroup of G. A character χ of G is called a relative M-character with respect to N if there exist a subgroup H with $N \leq H \leq G$ and an irreducible character $\psi \in Irr(H)$ such that $\mathrm{Ind}_H^G \psi = \chi$ and $\psi|_N \in Irr(N)$. If every irreducible character of G is a relative M-character with respect to N, then G is said to be a relative M-group with respect to N.

Proposition 3.1 *([13, Theorem 6.22]). Let G be a finite group and N be a normal subgroup of G such that G/N is supersolvable. Then G is a relative M-group with respect to N.*

Via this theorem (and a moment's reflection), the author [31] shows the following.

Theorem 3.2. *All nearly supersolvable groups are NM-groups.*

Now let us give a variant of Proposition 2.13 and borrow a classification of Frobenius complements below (see, for example, [27, Lemmata 18.3 and 18.4] or [15, Theorems 6.14 and 6.15]).

Proposition 3.3. *Let K/k be a Galois extension of number fields with Galois group G. Suppose $G = N \rtimes H$ is a Frobenius group with a Frobenius kernel N and a Frobenius complement H. If Artin's conjecture is true for K^N/k, then Artin's conjecture holds for K/k.*

Proposition 3.4. *If H is solvable Frobenius complement, then H satisfies one of the following.*

Type 1: *$H = SQ$, where S is a normal cyclic subgroup of H and Q is cyclic.*
Type 2: *$H = SQ$, where $S \trianglelefteq H$ is cyclic and Q is a generalised quaternion group.*
Type 3: *H is isomorphic to $SL_2(\mathbb{F}_3)$.*
Type 4: *$H/\mathbf{F}(H) \cong S_3$, where $\mathbf{F}(H)$ is the maximal normal nilpotent subgroup of H.*

We also remark that the above results indeed enable the author shows that the Artin conjecture is true for all solvable Frobenius extensions in [31]. Now, by Theorem 2.5, Corollary 2.17, and the fact that every Sylow subgroup of a Frobenius complement is either a cyclic or generalised quaternion group, we have the following theorem.

Theorem 3.5. *Suppose that K/k is a solvable Frobenius Galois extension with Galois group G. Then the Artin conjecture holds for K/k. Moreover, if a Frobenius complement of G is of Type* 1, 2*, or* 3*, then the Langlands reciprocity law is valid for K/k.*

Moreover, applying our method of low-dimensional groups, we still can say a little more for Frobenius complements of Type 4.

Proposition 3.6. *If G is a solvable Frobenius group G with a Frobenius kernel N and a Frobenius complement H, then any irreducible character χ of G is of automorphic type unless $N \subseteq \operatorname{Ker}\chi$, χ is of degree* 6 *and induced from a character of degree* 2*, and H is of Type* 4*.*

Proof. As we have shown before, if $N \nsubseteq \operatorname{Ker}\chi$, χ is of automorphic type. Also Theorem 3.5 asserts that if H is not of Type 4, G is of automorphic type. Thus, we may assume $H/\mathbf{F}(H) \cong S_3$ and $N \subseteq \operatorname{Ker}\chi$. In this case, χ can be seen as a character of H. Since $H/\mathbf{F}(H)$ is isomorphic to S_3, Proposition 3.1 implies that χ must be induced from an irreducible character ψ of degree ≤ 2 of a subgroup $\widetilde{H} \leq H$ of index 1, 2, 3, 6.

By the Arthur–Clozel theory and Theorem 2.7, it sufficient to show that χ is automorphic type if ψ is of degree 1 or 2 and $[H : \widetilde{H}] = 6$. However, as S_3 has a subgroup of index 2, there exists a subgroup $\widehat{H}$ with $\widetilde{H} \trianglelefteq \widehat{H} \trianglelefteq H$ and $[H : \widehat{H}] = 2$. In other words, $[\widehat{H} : \widetilde{H}] = 3$, and hence Arthur–Clozel's theory asserts $\operatorname{Ind}_{\widetilde{H}}^{\widehat{H}}\psi$ is of automorphic type. Now inducing $\operatorname{Ind}_{\widetilde{H}}^{\widehat{H}}\psi$ from $\widehat{H}$ to H and applying Arthur–Clozel's induction again completes the proof. $\square$

Corollary 3.7. *Assume that G is a solvable Frobenius group G with a Frobenius complement H. If any Sylow 2-subgroup of the Fitting subgroup $\mathbf{F}(H)$ of H is abelian, then G is of automorphic type. In particular, if* 16 *does not divide $|G|$, then G is of automorphic type.*

Proof. By Theorem 3.5, we may assume $H/\mathbf{F}(H)$ is isomorphic to S_3. Observe that if 16 does not divide $|G|$, then 8 cannot divide $\mathbf{F}(H)$. In this case, any Sylow 2-subgroup of $\mathbf{F}(H)$ is abelian. Since for every $p > 2$, all Sylow p-subgroups of H are cyclic and $\mathbf{F}(H)$ is nilpotent, $\mathbf{F}(H)$ is abelian if any Sylow 2-subgroup of $\mathbf{F}(H)$ is.

Now assuming $\mathbf{F}(H)$ is abelian, the theory of relative M-groups tells us that all irreducible characters of H are monomial. Thus, Proposition 3.6 (together with its proof) implies H is of automorphic type. $\square$

As discussed in the proof of Proposition 3.6, we cannot derive the automorphy for irreducible characters of degree 6, induced from a character of degree 2. Nevertheless, if the existence of the automorphic induction is assumed, one will have the following.

Theorem 3.8 *(Conditional). If the non-normal cubic automorphic induction exists for all 2-dimensional cuspidal automorphic representations, then all solvable Frobenius groups are of automorphic type.*

To end this section, we will apply Theorem 2.7 to derive some sufficient conditions for NSS groups being of automorphic type.

Proposition 3.9. *Suppose G is NSS. If G has a normal subgroup N, whose irreducible characters are of degree ≤ 3, such that G/N is nilpotent, then G is of automorphic type.*

Proof. We induct on the order of $|G|$. By Proposition 2.3, G is a relative SM-group with respect to N. Thus, for every irreducible character χ of G, there exist a subnormal subgroup H with $N \leq H \leq G$ and an irreducible character $\psi \in Irr(H)$ such that $\mathrm{Ind}_H^G \psi = \chi$ and $\psi|_N \in Irr(N)$. If $H \neq G$, then the induction hypothesis assures that H is of automorphic type, and so applying Arthur–Clozel's theory completes the proof in this case.

Now assume that $H = G$. Since G is NSS, G is an NM-group, and χ must be induced from a character of degree 1 or 2. On the other hand, as $\chi|_N = \psi|_N$ is an irreducible character of N, χ is of degree ≤ 3. If $\chi(1) \leq 2$, then Artin reciprocity and the Langlands–Tunnell theorem assert that χ is of automorphic type. Otherwise, for χ of degree 3, χ must be a monomial character. Now applying Arthur–Clozel's theory and Theorem 2.7 completes the proof. □

Corollary 3.10. *If G is a finite group of order 54 or 162, then G is of automorphic type.*

Proof. By [28, 7.2.15], G is a supersolvable group. Since any Sylow 3-subgroup P of G has index 2, P is a normal subgroup. As all non-trivial p-groups have non-trivial centre, $[P : \mathbf{Z}(P)] \leq 27$. Thus, Lemma 2.14 yields that $\mathrm{cd}(P) \subseteq \{1, 3\}$. Since G/P is cyclic, the corollary follows from Proposition 3.9 immediately. □

Corollary 3.11. *Assume that G admits an abelian normal subgroup N with $G/N \cong Q$, where S is a normal subgroup of Q of order 3. If Q/S is nilpotent then G is of automorphic type. In particular, if G has an abelian normal subgroup N with $G/N \cong S_3$, then G is of automorphic type.*

Proof. First, we observe that as Q/S is nilpotent, Q/S is supersolvable. Thus, Q is also supersolvable. Since N is abelian and $G/N \cong Q$, we conclude that G is NSS.

Now lifting the invariant series $1 \trianglelefteq S \trianglelefteq Q$ gives a subgroup H with $N \trianglelefteq H \trianglelefteq G$ and $[H : N] = 3$. Moreover, G/H is nilpotent. Now Proposition 3.1 implies that all irreducible characters of H are of degree 1 or 3, and hence applying Proposition 3.9 completes the proof. □

4. Little groups

In this section, we will apply the machinery developed in the previous sections to derive the automorphy for groups of order less than 60. Clearly, the trivial group is always of automorphic type. On the other hand, by the theorem of Arthur and Clozel, we know that all p-groups are of automorphic type. Hence, if $|G|$ belongs to

$$\{1,2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59\} \cup \{4,8,16,32,9,27,25,49\},$$

then G is of automorphic type. There are 26 classes of groups.

According to Corollaries 2.17 and 2.18, any group of order pq, pq^2, p^2q^2, or $8p$ for some primes p and q is of automorphic type (thanks to Artin reciprocity, the Langlands–Tunnell theorem, and Arthur–Clozel's theory). Thus, if G has order 6, 10, 12, 14, 15, 18, 20, 21, 22, 24, 26, 28, 33, 34, 35, 36, 38, 39, 40, 44, 45, 46, 50, 51, 52, 55, 56, 57, or 58, then G is of automorphic type. Here we have 29 classes of groups.

Now, there are only 4 remaining cases, namely, the groups of order 30, 42, 48, or 54. If G is of order 30, 42 or 54, G is of automorphic type by Corollaries 2.16 and 3.10.

For G of order 48, G has a normal subgroup N of order 8 or 16. According to Lemma 2.14, all irreducible characters of N are of degree ≤ 2. Since G/N is either of order 3 or 6, G/N must be supersolvable. Thus, G is clearly NSS and NM. In addition, if $|G/N| = 3$, Theorem 2.5 asserts that G is of automorphic type.

Now assume $|N| = 8$. As G is an NM-group, Artin reciprocity, the Langlands–Tunnell theorem, and Theorem 2.7 ensure that every irreducible character of G of degree ≤ 3 is of automorphic type. On the other hand, we note that all irreducible representations of G are of dimension ≤ 4, which can easily be checked via GAP [10] for instance. By the fact that G is nearly supersolvable, and $[G : N] = 6$, we conclude that if χ is an irreducible character of degree 4, it must be induced from a 2-dimensional character of a subgroup H of G containing N. As $[G : H] = 2$, H is a normal subgroup, and so χ is of automorphic type.

To end this section, we note that our method can be applied to groups of order ≥ 60 in some cases. For example, if G is of order 60, as a consequence of the Sylow theorem, G is either isomorphic to A_5, $A_4 \times C_5$, or $C_{15} \rtimes T$ where $T = C_4$ or $T = (C_2)^2$.

Since A_4 is of automorphic type, and C_5 is abelian, Artin reciprocity and the functoriality of $GL(n) \times GL(1)$ assert that $A_4 \times C_5$ is of automorphic type. (We, however, will still give another proof in the last section by developing general criteria.) On the other hand, for the third case, G is clearly of automorphic type thanks to Theorem 2.5. Therefore, we have the following.

Corollary 4.1. *If G is a non-simple group of order 60, then G is of automorphic type.*

Also, a straightforward application of Sylow's theory yields that every group of order $16p$ has a normal Sylow subgroup unless $p = 3$. As a consequence, Lemma 2.14 and

Theorem 2.5 assert every group of order $16p$ is of automorphic type unless $p = 3$. As shown above, for G of order 48, G is of automorphic type, and we hence have:

Corollary 4.2. *If G is of order $16p$, then G is of automorphic type.*

Similarly, the Sylow theory yields that every group G of order $27p$ has a normal Sylow subgroup (cf. [14, Theorem 1.32]). Moreover, if $p \neq 2$ or $p \neq 13$, G has a normal Sylow p-subgroup. Therefore, if G is a group of order $27p$ with $p \neq 13$, Theorem 2.5 and Corollary 3.10 assert that G is of automorphic type.

For $p = 13$, by Burnside's $p^a q^b$-theorem (or the previous discussion), G must be solvable. In particular, G admits a non-trivial abelian normal subgroup A. Thus, G/A is either a 3-group or of order $3^a 13$ for some $a \leq 2$. Again, Sylow's theorem asserts that all groups of order $3^a 13$ with $a \leq 2$ must have normal Sylow 13-subgroups, which implies that these groups are supersolvable. Thus, G/A is necessarily supersolvable, and so G is NNS. Now as mentioned above, G has a normal Sylow subgroup P (say). In particular, all irreducible characters of P are of degree ≤ 3. Proposition 3.9 then gives:

Corollary 4.3. *If G is of order $27p$, then G is of automorphic type.*

5. Groups with few non-linear irreducible characters

As discussed in the previous sections, if a p-group G is "small" or with a "big" abelian normal subgroup, $\mathrm{cd}(G)$ will have only two elements, namely, 1 and p. Meanwhile, as G is a p-group, it is of automorphic type. One may wonder if groups G with $|\mathrm{cd}(G)| = 2$ are of automorphic type in general. It can be answered by knowing below a result due to Amitsur (for $m = 2$) and Isaacs–Passman.

Proposition 5.1 *([13, Theorem 12.5 and Corollary 12.6]). Let G be a finite group. Assume that $\mathrm{cd}(G) = \{1, m\}$. Then the derived subgroup G' of G is abelian.*

Corollary 5.2. *If $|\mathrm{cd}(G)| = 2$, then G is of automorphic type.*

Proof. By the above theorem, G' is abelian. As G/G' is also abelian, Theorem 2.5 asserts that G is of automorphic type immediately. $\square$

As all irreducible characters of an abelian group have degree 1, Artin reciprocity asserts that all abelian groups are of automorphic type. In this spirit, one may read the above result as "if a group G has only one non-linear irreducible character, then G is of automorphic type", and wonder if a similar assertion holds for groups having only two non-linear irreducible characters. Thanks to the following result of Berkovich, we will give an affirmative answer.

Proposition 5.3 *([4, Chapter 31, Theorem 6]). Let G be a finite group with only two non-linear irreducible characters. Then G is one of the following groups:*

1: *extra-special groups of order 3^{1+2m};*
2: *Frobenius groups $(C_p)^a \rtimes C_{\frac{1}{2}(p^a-1)}$;*
3: *the Frobenius group $(C_3)^2 \rtimes Q_8$;*
4: $|G| = 2m$, $|\mathbf{Z}(G)| = 4$, $|G'| = 2$; *or*
5: $(C_p)^m \rtimes C_{2p^m-2}$,

where Q_8 is the quaternion group of order 8, and the fifth case is due to [4, Theorem 24.7 (g)].

Corollary 5.4. *Let G be a finite group with only two non-linear irreducible characters. Then G is of automorphic type.*

Proof. For case 1, G is a 3-group and hence is of automorphic type. For the remaining cases, it is clear that G always has an abelian normal subgroup N (say) so that G/N is nilpotent. Thus, the corollary follows from Theorem 2.5 immediately. □

For the case that groups have only three non-linear irreducible characters, which is also classified by Berkovich (see [4, Chapter 31, Theorem 9]), it is possible to examine whether such groups are of automorphic type by applying Theorem 2.5. However, for the sake of conceptual clarity, we will not do it here. Instead, we will consider the other extreme, namely, finite groups in which all the non-linear irreducible characters have distinct degrees. This was considered by Berkovich, Chillag, and Herzog.

Proposition 5.5 *([5]). Let G be a non-abelian finite group whose all non-linear irreducible characters have distinct degrees. Then one of the following holds:*

1: *G is an extra-special 2-group;*
2: *G is a Frobenius group of order $p^n(p^n-1)$ for some prime p with an abelian Frobenius kernel of order p^n and a cyclic Frobenius complement; or*
3: *G is a Frobenius group of order 72 whose Frobenius complement is isomorphic to the quaternion group of order 8.*

Applying this proposition and Proposition 2.13, we then obtain:

Corollary 5.6. *If G is a non-abelian finite group whose all non-linear irreducible characters have distinct degrees, then G is of automorphic type.*

6. The principle of functoriality and nearly nilpotent groups

As we mentioned in the very beginning, Langlands conjectured that every (irreducible) Galois representation arises from a (cuspidal) automorphic representation. This is, in fact, a part of what is called the principle of functoriality. The principle of functoriality is the core of the Langlands program and has many remarkable consequences. For instance, it is well-known that Artin's conjecture follows from this principle. One may regard this principle as the Holy Grail of number theory. However, since we could not afford to give all the necessary definitions to state the principle of functoriality, we direct the serious reader to Langlands' inspiring article [23], and just discuss a small piece of this beautiful principle.

Let K/k be a Galois extension of number fields with Galois group $G = G_1 \times G_2$. On one hand, for every irreducible character χ of G, there exist irreducible characters χ_1 and χ_2 of G_1 and G_2, respectively, such that $\chi = \chi_1 \times \chi_2$. On the other hand, assuming G_1, G_2, and G are all of automorphic type. Then for any Galois extension K/k of number fields with Galois group G, there is a cuspidal automorphic representation π of $GL_{\chi(1)}(\mathbb{A}_k)$ such that

$$L(s, \chi, K/k) = L(s, \pi).$$

Since for each i, G_i is of automorphic type, χ_i can be seen as a cuspidal automorphic representation π_i of $GL_{\chi_i(1)}(\mathbb{A}_k)$ as well. Furthermore, as $\chi = \chi_1 \times \chi_2$, we then derive

$$L(s, \pi) = L(s, \pi_1 \otimes \pi_2).$$

In this spirit, the Langlands program then predicts:

Conjecture 6.1 *(The functoriality of $GL(n) \times GL(m)$). Let π_1 and π_2 be cuspidal automorphic representations of $GL_n(\mathbb{A}_k)$ and $GL_m(\mathbb{A}_k)$, respectively. Then $\pi_1 \otimes \pi_2$ is an automorphic representation of $GL_{nm}(\mathbb{A}_k)$.*

This implies that the direct product of any two groups of automorphic type is of automorphic type; and it was recently proved in the case of $GL(2) \times GL(2)$ by Ramakrishnan in [25] and $GL(2) \times GL(3)$ by Kim and Shahidi in [21].

Now let us consider a direct product $G = G_1 \times G_2$, where G_1 and G_2 are NM-groups. As discussed above, for every irreducible character χ of G, there exist irreducible characters χ_1 and χ_2 of G_1 and G_2, respectively, such that $\chi = \chi_1 \times \chi_2$. Since both G_1 and G_2 are NM-groups, for each i, there exist a subgroup H_i of G_i and an irreducible character $\psi_i \in Irr(H_i)$ of degree ≤ 2 such that $\chi_i = \operatorname{Ind}_{H_i}^{G_i} \psi_i$. Thus,

$$\chi = \operatorname{Ind}_{H_1 \times H_2}^{G_1 \times G_2} (\psi_1 \times \psi_2).$$

However, now one can see that χ might not be induced from an irreducible character of degree ≤ 2. As a consequence, we cannot apply the Langlands–Tunnell theorem to deduce Artin's conjecture directly. But as ψ_i is still of automorphic type (thanks to Artin reciprocity and the Langlands–Tunnell theorem), if we invoke the functoriality of $GL(n) \times GL(1)$ and $GL(2) \times GL(2)$, we then still are able to derive the automorphy of $\psi_1 \times \psi_2$. Thus, we have the following.

Proposition 6.2. *If K/k is a Galois extension of number fields whose Galois group is a direct product of two NM-groups, then Artin's conjecture is true for K/k.*

Moreover, by applying the Rankin–Selberg theory developed by Jacquet–Piatetski–Shapiro–Shalika, the above discussion then yields:

Proposition 6.3. *If K/k is a Galois extension of number fields whose Galois group is a direct product of three (or four) NM-groups, then Artin's conjecture is true for K/k.*

In a slightly different vein, since any (finite) direct product of nilpotent groups is nilpotent, the Arthur–Clozel theory implies that the principle of functoriality is valid in this case. Naturally, one may be desired to find some "non-nilpotent" examples. At the end of this note, we will give such examples of the principle of functoriality by showing that certain direct products of groups of automorphic type are again of automorphic type. In light of the discussion in the previous section, we define nearly nilpotent groups as follows.

Definition 4. A finite group G is called nearly nilpotent if it has a normal subgroup $N \in \mathcal{C}$ such that G/N is nilpotent, where $\mathcal{C}$ denotes the class consisting of groups whose irreducible representations are of dimension less than or equal to 2.

Since all subgroups and homomorphic images of a nilpotent group are nilpotent, a moment's thought shows that all subgroups and homomorphic images of any nearly nilpotent group are also nearly nilpotent. As all nilpotent groups are supersolvable, all nearly nilpotent groups form a "closed" subclass of the class of NSS groups. On the other hand, one can read Theorem 2.5 as below.

Theorem 6.4. *If G is a nearly nilpotent group, then all subgroups and homomorphic images of G are of automorphic type.*

Unlike nilpotent groups, the direct product of two nearly nilpotent groups might not be nearly nilpotent. In fact, by the previous discussion, one even cannot expect this would be an NM-group. We however still have the following substitute.

Proposition 6.5. *If G_1 is a nearly nilpotent group and G_2 is an abelian-by-nilpotent group, then $G_1 \times G_2$ is nearly nilpotent group and so is of automorphic type.*

Proof. Since G_1 is a nearly nilpotent group, there is a normal subgroup N_1 of G_1 belonging to $\mathcal{C}$ such that G_1/N_1 is nilpotent. On the other hand, G_2 has an abelian normal subgroup N_2 such that G_2/N_2 is nilpotent. Thus, we have an invariant series

$$1 \trianglelefteq N_1 \times N_2 \trianglelefteq G_1 \times G_2.$$

Since $(G_1 \times G_2)/(N_1 \times N_2) \cong (G_1/N_1) \times (G_2/N_2)$, which is a direct product of nilpotent groups, $(G_1 \times G_2)/(N_1 \times N_2)$ is nilpotent. Moreover, as all irreducible characters of $N_1 \times N_2$ are clearly of degree ≤ 2, $N_1 \times N_2 \in \mathcal{C}$. Thus, $G_1 \times G_2$ is nearly nilpotent. $\square$

Moreover, by invoking Ramakrishnan's functoriality of $GL(2) \times GL(2)$, one can show the direct product of two nearly nilpotent groups is still of automorphic type.

Theorem 6.6. *If G_1 and G_2 are nearly nilpotent, then $G_1 \times G_2$ is of automorphic type.*

Proof. Assume that K/k is a Galois extension of number fields with Galois group $G_1 \times G_2$. Since both G_1 and G_2 are nearly nilpotent, for each i, there exists N_i such that G_i/N_i is nilpotent, and Proposition 2.3 then asserts that G_i is a relative SM-group with respect to N_i. As discussed above, for each irreducible character χ of $G_1 \times G_2$, there are irreducible characters χ_1 and χ_2 of G_1 and G_2, respectively, such that

$$\chi = \chi_1 \times \chi_2.$$

Moreover, for each i, there exist a subnormal subgroup H_i (containing N_i) of G_i and an irreducible character $\psi_i \in Irr(H_i)$ such that $\chi_i = \operatorname{Ind}_{H_i}^{G_i} \psi_i$ and $\psi_i|_{N_i} \in Irr(N_i)$. Thus,

$$\chi = \operatorname{Ind}_{H_1 \times H_2}^{G_1 \times G_2}(\psi_1 \times \psi_2).$$

Now on one hand, as ψ_1 and ψ_2 are of degree ≤ 2, Artin reciprocity and Langlands–Tunnell's theorem assert that for each i, (by regrading ψ_i as an irreducible character of $H_1 \times H_2$) ψ_i corresponds to a cuspidal automorphic representation of dimension $\psi_i(1)$ over $K^{H_1 \times H_2}$. Thus, the functoriality of $GL(n) \times GL(1)$ and $GL(2) \times GL(2)$ imply that $\psi_1 \times \psi_2$ corresponds to a cuspidal automorphic representation (of dimension $\psi_1(1)\psi_2(1)$) over $K^{H_1 \times H_2}$. On the other hand, as $H_1 \times H_2$ is subnormal in $G_1 \times H_2$, and $G_1 \times H_2$ is subnormal in $G_1 \times G_2$, we can conclude that $H_1 \times H_2$ is subnormal in $G_1 \times G_2$. Putting everything together, the above-mentioned theorems of Arthur–Clozel and Jacquet–Shalika yield χ is of automorphic type. $\square$

Proposition 6.7. *Assume G_1 is abelian-by-nilpotent and G_2 is nearly supersolvable. Suppose, further, that G_2 has a normal subgroup N, whose irreducible characters are of degree ≤ 3, such that G_2/N is nilpotent. Then $G_1 \times G_2$ is NSS of automorphic type.*

Proof. Since G_1 is abelian-by-nilpotent, it admits an abelian normal subgroup N_1 such that G_1/N_1 is nilpotent. On the other hand, as there is $N_2 \in \mathcal{C}$ such that G_2/N_2 is supersolvable, we then have an invariant series

$$1 \trianglelefteq N_1 \times N_2 \trianglelefteq G_1 \times G_2,$$

where $N_1 \times N_2$ belongs to $\mathcal{C}$. Since $(G_1 \times G_2)/(N_1 \times N_2) \cong (G_1/N_1) \times (G_2/N_2)$ is a direct product of supersolvable groups, $(G_1 \times G_2)/(N_1 \times N_2)$ is supersolvable. Thus, $G_1 \times G_2$ is NSS. Finally, observing that $G_1 \times G_2$ has a normal subgroup $N_1 \times N$, whose all irreducible characters are of degree ≤ 3, such that $(G_1 \times G_2)/(N_1 \times N)$ is nilpotent, Theorem 3.9 asserts that $G_1 \times G_2$ is of automorphic type. $\square$

As one can tell, we in the above proof showed that G_2 and $G_1 \times G_2$ are of the same type, i.e., they are both NNS groups that have normal subgroups whose irreducible characters are of degree ≤ 3. However, these groups are essentially NM-groups so that there is nothing new in the view of character theory. The reader may wonder if the above theorem can be improved in the same spirit of Theorem 6.6. To end this note, we present the following generalisation of Propositions 2.9, 3.9, and 6.7 by invoking the functionality due to Ramakrishnan and Kim–Shahidi.

Proposition 6.8. *Assume that G_1 is a nearly nilpotent group and that G_2 has a normal subgroup N_2, whose irreducible characters are of dimension ≤ 3, such that G_2/N_2 is nilpotent. Suppose either one of the following conditions is satisfied:*

1: *All 3-dimensional irreducible characters of every subnormal subgroup of G_2 containing N_2 are real-valued.*

2: *G_2 is NSS.*

Then $G_1 \times G_2$ is of automorphic type.

Proof. As before, there exists $N_1 \in \mathcal{C}$ such that G_1/N_1 is nilpotent, and Proposition 2.3 then asserts that G_1 is a relative SM-group with respect to N_1. Also, for each irreducible character χ of $G_1 \times G_2$, there are irreducible characters χ_1 and χ_2 of G_1 and G_2, respectively, such that $\chi = \chi_1 \times \chi_2$. By our assumption on G_1 and G_2, Horváth's theorem tells us that for each i, there exist a subnormal subgroup H_i (containing N_i) of G_i and an irreducible character $\psi_i \in Irr(H_i)$ such that $\chi_i = \mathrm{Ind}_{H_i}^{G_i} \psi_i$ and $\psi_i|_{N_i} \in Irr(N_i)$. Thus,

$$\chi = \mathrm{Ind}_{H_1 \times H_2}^{G_1 \times G_2}(\psi_1 \times \psi_2),$$

where $\psi_1(1) \leq 2$ and $\psi_2(1) \leq 3$. Thus, $\psi_1 \times 1$ and $1 \times \psi_2$ are of degree less than or equal to 2 and 3, respectively. Since both conditions 1 and 2 imply that ψ_2 is a monomial character if $\psi_2(1) = 3$, Arthur–Clozel's theory and Theorem 2.7 yield that $1 \times \psi_2$ is of automorphic type in this case. This fact together with Artin reciprocity and the

Langlands–Tunnell theorem asserts that both $\psi_1 \times 1$ and $1 \times \psi_2$ must be of automorphic type. Now observe that $\psi_1 \times \psi_2 = (\psi_1 \times 1) \otimes (1 \times \psi_2)$. The above discussion and the functoriality of $GL(n) \times GL(1)$, $GL(2) \times GL(2)$ and $GL(2) \times GL(3)$ assert that $\psi_1 \times \psi_2$ is also of automorphic type. Finally, as $H_1 \times H_2$ is subnormal in $G_1 \times G_2$, applying Arthur–Clozel's theorem completes the proof. $\square$

7. Concluding remarks

Our argument allows one to study the Langlands reciprocity conjecture for solvable Galois extensions via elementary group theory (e.g. Sylow's theorems). Indeed, for solvable G, one can also argue using the derived subgroup G'. More precisely, as G/G' is abelian, the results obtained in previous sections enable us to investigate the automorphy of G by simply considering $\mathrm{cd}(G')$, the set of character degrees of G', which can be easily computed via the computer algebra package [10].

We also remark that our results have several arithmetic applications. For instance, in the sieve theory, to study primes satisfying Chebotarev conditions, one of the main tools is a variant of the Bombieri–Vinogradov theorem due to M. Ram Murty and V. Kumar Murty [24], and the Langlands reciprocity plays the crucial role in obtaining a better "level of distribution" in their theorem.

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