# A variant of Heilbronn characters 

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#### Abstract

In this paper, we introduce arithmetic Heilbronn characters that generalize the notion of the classical Heilbronn characters, and discuss several properties of these characters. This formalism has several arithmetic applications. For instance, we obtain the holomorphy of suitable quotients of L-functions attached to elliptic curves, which is predicted by the Birch-Swinnerton-Dyer conjecture, and the non-existence of simple zeros or poles in such quotients.


Keywords: Heilbronn characters; character theory; L-functions; automorphic representations.

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## 1. Introduction

Let $K / k$ be an extension of number fields. A celebrated conjecture of Dedekind asserts that the quotient $\frac{\zeta_{K}(s)}{\zeta_{k}(s)}$ is entire. By the work of Aramata and Brauer [4], this conjecture is valid if $K / k$ is a Galois extension. Moreover, if $K$ is contained in a solvable normal closure of $k$, Uchida [28] and van der Waall [29] independently proved Dedekind's conjecture in this case. However, this conjecture is still open in general.

In a slightly different vein, to study zeros of Dedekind zeta functions, Heilbronn [15] introduced what are now called Heilbronn characters. His innovation allowed him to give a simple proof of the Aramata-Brauer theorem. This profound idea was also used by Stark [25] to prove that if $K / k$ is Galois and $\zeta_{K}(s)$ has a simple zero at $s=s_{0}$, then it must arise from a cyclic extension $M$ of $k$. Moreover, in the spirit of Heilbronn and Stark, Foote and Murty [13] showed that if $K / k$ is a Galois extension with Galois group $G$, for fixed $s_{0} \in \mathbb{C}$,

$$
\sum_{\chi \in \operatorname{Irr}(G)} n(G, \chi)^{2} \leq\left(\operatorname{ord}_{s=s_{0}} \zeta_{K}(s)\right)^{2}
$$

where $n(G, \chi)$ denotes the order of $L(s, \chi, K / k)$ at $s=s_{0}$. Furthermore, if $G$ is solvable, this result has been improved by Murty-Raghuram in [24] and LanskyWilson in [19] later. In particular, the result of Murty and Raghuram generalizes the work of Uchida and van der Waall.

It is well known that Dedekind's conjecture will follow if Artin's (holomorphy) conjecture is true [2]. Also, these two conjectures follow from either the "Langlands program" or the "Selberg philosophy" (see, for example, [20, 21]). This connection suggests that the method of Heilbronn shall play a role in studying L-functions. Indeed, Murty and Murty [22] realized this idea to show that suitable quotients of L-functions attached to elliptic curves are entire, which proves a consequence of the Birch-Swinnerton-Dyer conjecture unconditionally.

As remarked in [24], the authors believe that Lemmata 2.1-2.5 in their paper, which concern monomial representations of finite groups, will eventually be useful in analytic number theory in much the same spirit as in Heilbronn [15] and Stark [25]. We regard this paper as one of the steps to achieve such a goal. In fact, this paper is devoted to building up a formalism setting for Heilbronn characters and demonstrating how to apply this machinery to study L-functions.

This paper will be arranged as follows. Firstly, in Sec. 2, we will introduce the notion of weak arithmetic Heilbronn characters that satisfy properties analogous to some properties of the classical Heilbronn characters known by the work of Heilbronn-Stark (Theorem 2.3), Aramata-Brauer (Corollary 2.5), Foote-Murty (Theorem 2.4), and Murty-Murty (Theorem 2.7).

In Sec. 3, more conditions will be imposed on weak arithmetic Heilbronn characters which take them closer to Heilbronn characters. These will be outlined in Theorem 3.2 (Heilbronn-Stark lemma in full strength), Theorem 3.5 (known by the work of Murty-Raghuram), and Theorem 3.6 (the Uchida-van der Waall Theorem). We will go on to derive several extensions of results of Murty and Raghuram for arithmetic Heilbronn characters. In particular, we prove the following variant (Theorem 3.18):

Let $K / k$ be a solvable Galois extension of number fields with Galois group $G$, and let $\rho$ be a 2 -dimensional representation of $G$. Then for any subgroup $H$ of $G$, the quotient (of Artin L-functions)

$$
\frac{L\left(s,\left.\operatorname{Ind}_{H}^{G} \rho\right|_{H}, K / k\right)}{L(s, \rho, K / k)}
$$

is holomorphic at $s \neq 1$. Moreover, for every 1-dimensional character $\chi_{0}$ of $G$, an inequality is derived for the sum of squares of $\operatorname{ord}_{s=s_{0}} L(s, \chi, K / k)$ where $\chi$ runs over all irreducible characters of $G$ different from $\chi_{0}$.

In Sec. 4, we will apply the results from Sec. 3 to study Artin-Hecke L-functions and L-functions of CM-elliptic curves. For instance, one such theorem is proved as Theorem 4.5.

Suppose $K / k$ is a solvable Galois extension with Galois group $G$, and let $H$ be a subgroup of $G$. Let $\chi$ and $\phi$ be abelian characters of $G$ and $H$, respectively. Then
for any CM-elliptic curve $E$ over $k$, the quotient (of L-functions associated to $E$ )

$$
\frac{L\left(s, \operatorname{Ind}_{H}^{G} \phi, E, k\right)}{L(s, \chi, E, k)^{(\chi \mid H, \phi)}}
$$

is entire. Also, for every 1-dimensional character $\chi_{0}$ of $G$, an inequality is obtained for the sum of squares of $\operatorname{ord}_{s=s_{0}} L(s, \chi, E, k)$ where the sum is over all irreducible characters of $G$ different from $\chi_{0}$. From the above, an interesting result about quotients of L-functions of CM-elliptic curves is deduced in Corollary 4.6.

Furthermore, in Sec. 5, by applying the results from Sec. 3, we will study holomorphy of quotients of Rankin-Selberg L-functions arising from certain cuspidal automorphic representations that allows one to investigate holomorphy of quotients of L-functions associated to non-CM elliptic curves. Finally, in Sec. 6, we will use properties of weak arithmetic Heilbronn characters along with the celebrated result of Taylor and his school on the potential automorphy for symmetric power L-functions of non-CM elliptic curves to deduce generalizations of the results of Foote, Murty and Murty. In particular, one such consequence, Theorem 6.4, is predicted by the Birch-Swinnerton-Dyer conjecture.

## 2. Weak Arithmetic Heilbronn Characters

In this section, we will introduce weak arithmetic Heilbronn characters that generalize the classical Heilbronn characters, and we will discuss several properties of such Heilbronn characters.

From now on, $G$ always denotes a finite group. For any subgroup $H$ of $G$, we denote the trivial character and the regular representation of $H$ by $1_{H}$ and $\operatorname{Reg}_{H}$, respectively. In addition, $\langle h\rangle$ denotes the cyclic subgroup of $H$ generated by an element $h \in H$, and $e_{H}$ is the identity element of $H$.

Definition 2.1. Suppose that there is a set of integers $\{n(H, \phi)\}_{(H, \phi) \in I(G)}$, where $I(G)=\{(H, \phi) \mid H$ is a cyclic subgroup of $G$ or $H=G$, and $\phi$ is a character of $H\}$, satisfying the following three properties:

WAHC1 $n\left(H, \phi_{1}+\phi_{2}\right)=n\left(H, \phi_{1}\right)+n\left(H, \phi_{2}\right)$ for any characters $\phi_{1}$ and $\phi_{2}$ of $H$, where $H$ is a cyclic subgroup or an improper subgroup of $G$;
WAHC2 $n\left(G, \operatorname{Ind}_{H}^{G} \phi\right)=n(H, \phi)$ for every cyclic subgroup $H$ and every character $\phi$ of $H$; and
WAHC3 $n(H, \phi) \geq 0$ for all cyclic subgroups $H$ of $G$ and all characters $\phi$ of $H$.
Then the weak arithmetic Heilbronn character of a subgroup $H$ of $G$, which is either cyclic or improper, associated with such $n(H, \phi)$ 's is defined by

$$
\Theta_{H}=\sum_{\phi \in \operatorname{Irr}(H)} n(H, \phi) \phi
$$

which by condition WAHC2, is equal to $\sum_{\phi \in \operatorname{Irr}(H)} n\left(G, \operatorname{Ind}_{H}^{G} \phi\right) \phi$.

Such a formalism technique was used by Foote in [11] as well as by Murty and Murty in [22] to study certain L-functions. However, we will see such "abstract" Heilbronn characters are of interest in their own right. In fact, weak Heilbronn arithmetic characters and arithmetic Heilbronn characters, which will be discussed in the next section, inherit a lot of properties of the classical Heilbronn characters. For instance, these Heilbronn characters also admit an Artin-Takagi decomposition.

## Theorem 2.2 (Artin-Takagi decomposition).

$$
n\left(G, \operatorname{Reg}_{G}\right)=\sum_{\chi \in \operatorname{Irr}(G)} \chi(1) n(G, \chi)
$$

Proof. Since $\operatorname{Reg}_{G}=\sum_{\chi \in \operatorname{Irr}(G)} \chi(1) \chi$, the decomposition follows simply from condition WAHC1.

By conditions WAHC2 and WAHC3, one can see $n(G, \chi) \geq 0$ for any character $\chi$ of $G$ induced from a character of a cyclic subgroup of $G$. On the other hand, condition WAHC2 in fact implies a stronger condition: $n\left(\widetilde{H}, \operatorname{Ind}_{H}^{\widetilde{H}} \phi\right)=n(H, \phi)$ for any cyclic subgroup $\widetilde{H}$ of $G$ containing $H$, since

$$
n\left(\widetilde{H}, \operatorname{Ind}_{H}^{\widetilde{H}} \phi\right)=n\left(G, \operatorname{Ind}_{\widetilde{H}}^{G} \operatorname{Ind}_{H}^{\widetilde{H}} \phi\right)=n\left(G, \operatorname{Ind}_{H}^{G} \phi\right)=n(H, \phi)
$$

Now we shall state and prove several properties of weak arithmetic Heilbronn characters. Our methods are based on earlier works of Heilbronn, Stark, Foote, and Murty.

Theorem 2.3 (Heilbronn-Stark Lemma). For every cyclic subgroup $H$ of $G$, one has

$$
\left.\Theta_{G}\right|_{H}=\Theta_{H}
$$

Proof. By the definition, the first orthogonality property of irreducible characters, and Frobenius reciprocity, we have

$$
\begin{aligned}
\left.\Theta_{G}\right|_{H}=\left.\sum_{\chi \in \operatorname{Irr}(G)} n(G, \chi) \chi\right|_{H} & =\sum_{\chi \in \operatorname{Irr}(G)} n(G, \chi) \sum_{\phi \in \operatorname{Irr}(H)}\left(\left.\chi\right|_{H}, \phi\right) \phi \\
& =\sum_{\chi \in \operatorname{Irr}(G)} n(G, \chi) \sum_{\phi \in \operatorname{Irr}(H)}\left(\chi, \operatorname{Ind}_{H}^{G} \phi\right) \phi \\
& =\sum_{\phi \in \operatorname{Irr}(H)}\left(\sum_{\chi \in \operatorname{Irr}(G)}\left(\chi, \operatorname{Ind}_{H}^{G} \phi\right) n(G, \chi)\right) \phi .
\end{aligned}
$$

Now we use conditions WAHC1 and WAHC2, and the first orthogonality property
of irreducible characters again to get

$$
\begin{aligned}
\left.\Theta_{G}\right|_{H} & =\sum_{\phi \in \operatorname{Irr}(H)} n\left(G, \sum_{\chi \in \operatorname{Irr}(G)}\left(\chi, \operatorname{Ind}_{H}^{G} \phi\right) \chi\right) \phi \\
& =\sum_{\phi \in \operatorname{Irr}(H)} n\left(G, \operatorname{Ind}_{H}^{G} \phi\right) \phi=\Theta_{H} .
\end{aligned}
$$

Like the classical Heilbronn-Stark lemma, the above lemma enables us to bound the coefficients of our Heilbronn characters.

## Theorem 2.4.

$$
\sum_{\chi \in \operatorname{Irr}(G)} n(G, \chi)^{2} \leq n\left(G, \operatorname{Reg}_{G}\right)^{2}
$$

Proof. We will give a proof based on the method developed in [13, 22]. By the first orthogonality property and the definition of the (usual) inner product of class functions of $G$, one has

$$
\sum_{\chi \in \operatorname{Irr}(G)} n(G, \chi)^{2}=\left(\Theta_{G}, \Theta_{G}\right)=\frac{1}{|G|} \sum_{g \in G}\left|\Theta_{G}(g)\right|^{2}
$$

Applying the Heilbronn-Stark lemma, for any $g \in G$, one has

$$
\Theta_{G}(g)=\Theta_{\langle g\rangle}(g)=\sum_{\phi \in \operatorname{Irr}(\langle g\rangle)} n(\langle g\rangle, \phi) \phi(g) .
$$

Since $\langle g\rangle$ is cyclic, the triangle inequality and conditions WAHC2 and WAHC3 yield

$$
\begin{aligned}
\left|\Theta_{G}(g)\right| & =\left|\Theta_{\langle g\rangle}(g)\right| \leq \sum_{\phi \in \operatorname{Irr}(\langle g\rangle)} n(\langle g\rangle, \phi) \\
& =n\left(\langle g\rangle, \sum_{\phi \in \operatorname{Irr}(\langle g\rangle)} \phi\right)=n\left(\langle g\rangle, \operatorname{Reg}_{\langle g\rangle}\right)=n\left(G, \operatorname{Reg}_{G}\right) .
\end{aligned}
$$

Therefore, the theorem follows.
Using this theorem and the fact that $n\left(G, \operatorname{Reg}_{G}\right)=n\left(G, \operatorname{Ind}_{\left\langle e_{G}\right\rangle}^{G} 1_{\left\langle e_{G}\right\rangle}\right) \geq 0$, one can immediately obtain the following analogues of several famous theorems.

Corollary 2.5 (Aramata-Brauer Theorem). $n\left(G, \operatorname{Reg}_{G}\right) \pm n\left(G, 1_{G}\right) \geq 0$.
Corollary 2.6 (Stark Lemma). If $n\left(G, \operatorname{Reg}_{G}\right) \leq 1$, then $n(G, \chi) \geq 0$ for all irreducible characters $\chi$ of $G$.

Proof. If $n\left(G, \operatorname{Reg}_{G}\right)=0$, then the corollary follows from the above theorem immediately. Otherwise, for $n\left(G, \operatorname{Reg}_{G}\right)=1$, by Artin-Takagi decomposition, one has

$$
\sum_{\chi \in \operatorname{Irr}(G)} \chi(1) n(G, \chi)=n\left(G, \operatorname{Reg}_{G}\right)=1
$$

In addition, Theorem 2.4 forces that all integers $n(G, \chi)$ are bounded by 1 . Thus, we can conclude that there is exactly one abelian character $\chi_{0}$ of $G$ such that $\chi_{0}(1)=1$ and $n\left(G, \chi_{0}\right)=1$. In other words, for any irreducible character $\chi \neq \chi_{0}$ of $G, n(G, \chi)=0$.

In [22], Murty and Murty showed the following "twisting" result by using a formalism technique. We shall give a proof below by just checking that such "twisting" indeed defines a set of integers satisfying conditions WAHC1 to WAHC3.

Theorem 2.7. Let $n(H, \phi)$ 's be integers defining a weak Heilbronn character, i.e., these integers satisfy conditions WAHC1 to WAHC3. Let $\rho$ be an arbitrary character of $G$. Suppose that for every cyclic subgroup $H$ of $G$ and irreducible character $\phi$ of $H$, we have $n\left(H,\left.\rho\right|_{H} \otimes \phi\right) \geq 0$, then

$$
\sum_{\chi \in \operatorname{Irr}(G)} n(G, \rho \otimes \chi)^{2} \leq n\left(G, \rho \otimes \operatorname{Reg}_{G}\right)^{2}
$$

Proof. For every cyclic subgroup $H$ of $G$ (or $H=G$ ) and every character $\phi$ of $H$, let $n^{\prime}(H, \phi)=n\left(H,\left.\rho\right|_{H} \otimes \phi\right)$. By the linearity of tensor product and the assumption of this theorem, it is easy to see that $n^{\prime}(H, \phi)$ 's satisfy conditions WAHC1 and WAHC3. On the other hand, since tensoring "commutes" with induction, we have

$$
\begin{aligned}
n^{\prime}(H, \phi)=n\left(H,\left.\rho\right|_{H} \otimes \phi\right) & =n\left(G, \operatorname{Ind}_{H}^{G}\left(\left.\rho\right|_{H} \otimes \phi\right)\right) \\
& =n\left(G, \rho \otimes \operatorname{Ind}_{H}^{G} \phi\right)=n^{\prime}\left(G, \operatorname{Ind}_{H}^{G} \phi\right)
\end{aligned}
$$

Therefore, this theorem follows from Theorem 2.4 immediately.

## 3. Arithmetic Heilbronn Characters

In this section, we will put more conditions on $n(H, \phi)$ 's, which make weak arithmetic Heilbronn characters catch almost all properties that we know for the classical Heilbronn characters.

Definition 3.1. Suppose that there is a set of integers $\{n(H, \phi)\}_{(H, \phi) \in I(G)}$, where

$$
I(G)=\{(H, \phi) \mid H \text { is a subgroup of } G, \text { and } \phi \text { is a character of } H\}
$$

satisfying the following three properties:
AHC1 $n\left(H, \phi_{1}+\phi_{2}\right)=n\left(H, \phi_{1}\right)+n\left(H, \phi_{2}\right)$ for any subgroup $H$ of $G$ and any characters $\phi_{1}$ and $\phi_{2}$ of $H$;
AHC2 $n\left(G, \operatorname{Ind}_{H}^{G} \phi\right)=n(H, \phi)$ for every subgroup $H$ and every character $\phi$ of $H$; and
AHC3 $n(H, \phi) \geq 0$ for all 1-dimensional characters $\phi$ of subgroups $H$ of $G$.

Then the arithmetic Heilbronn character of a subgroup $H$ of $G$ associated with such $n(H, \phi)$ 's is defined as

$$
\Theta_{H}=\sum_{\phi \in \operatorname{Irr}(H)} n(H, \phi) \phi
$$

which by condition AHC2, is equal to $\sum_{\phi \in \operatorname{Irr}(H)} n\left(G, \operatorname{Ind}_{H}^{G} \phi\right) \phi$.
It is clear that all arithmetic Heilbronn characters have properties discussed in the previous section. Moreover, since $n(H, \phi)$ 's are now defined for all subgroups $H$ of $G$, we have the following full-powered Heilbronn-Stark Lemma.

Theorem 3.2 (Heilbronn-Stark Lemma). For every subgroup H of $G$, one has

$$
\left.\Theta_{G}\right|_{H}=\Theta_{H} .
$$

From now on, $\Theta_{G}$ always denotes an arithmetic Heilbronn character of $G$. Furthermore, we assume $G$ is solvable. The following powerful lemma is essential due to the work of Uchida and van der Waall, which is used by Murty-Murty [22] implicitly, and is stated precisely by Murty and Raghuram in [24].

Lemma 3.3 ([24, Lemma 2.4]). Let $G$ be a finite solvable group, and let $H$ be a subgroup of $G$. Then

$$
\operatorname{Ind}_{H}^{G} 1_{H}=1_{G}+\sum \operatorname{Ind}_{H_{i}}^{G} \phi_{i},
$$

where $\phi_{i}$ 's are non-trivial 1-dimensional characters of some subgroups $H_{i}$ 's of $G$.
Following [24], we let $G^{0}=G$, and define $G^{i}$ to be $\left[G^{i-1}, G^{i-1}\right.$ ] for all $i \geq 1$. The series $\left\{G^{i}\right\}$ is called the derived series of $G$. Since $G$ is solvable, such a series is eventually trivial. Using this series, one may define the level of an irreducible character $\chi$ of $G$, denoted $l(\chi)$, as the least non-negative integer $n$ such that $\chi$ is trivial on $G^{n}$. For instance, the level one characters are exactly the non-trivial 1dimensional characters of $G$. In addition, Murty and Raghuram showed a stronger version of the above lemma.

Lemma 3.4 ([24, Lemma 2.5]). Let $G$ be a finite solvable group having more than one element, and let $H$ be a subgroup of $G$. Let $\left\{G^{i}\right\}$ denote the derived series of $G$, and let $m$ be the least non-negative integer such that $G^{m+1}=\left\langle e_{G}\right\rangle$. Then for all $i \geq 1$,

$$
\operatorname{Ind}_{H}^{G} 1_{H}=\operatorname{Ind}_{H \cdot G^{i}}^{G} 1_{H \cdot G^{i}}+\sum \operatorname{Ind}_{H_{j}}^{G} \phi_{j},
$$

where $\phi_{j}$ 's are non-trivial 1-dimensional characters of some subgroups $H_{j}$ 's of $G$, and the sum might be empty.

Using these lemmas and the method developed in [24], we are able to prove the following sequence of properties for arithmetic Heilbronn characters.

Theorem 3.5. Let $H$ be a subgroup of $G$. Let $\chi$ and $\phi$ be 1-dimensional characters of $G$ and $H$, respectively. Then

$$
n\left(G, \operatorname{Ind}_{H}^{G} \phi\right)-\left(\left.\chi\right|_{H}, \phi\right) n(G, \chi) \geq 0
$$

Proof. Note that if $\left(\left.\chi\right|_{H}, \phi\right)=0$, the theorem is clearly true by conditions AHC2 and AHC3. Suppose that $\left(\left.\chi\right|_{H}, \phi\right)>0$. Since both $\chi$ and $\phi$ are 1-dimensional, we obtain $\left.\chi\right|_{H}=\phi$ and $\left(\left.\chi\right|_{H}, \phi\right)=1$. Following the proof of [24, Theorem 4.1], by Lemma 3.3, one has

$$
\operatorname{Ind}_{H}^{G} 1_{H}=1_{G}+\sum \operatorname{Ind}_{H_{i}}^{G} \phi_{i},
$$

where $\phi_{i}$ 's are non-trivial 1-dimensional characters of some subgroups $H_{i}$ 's of $G$. Since tensoring and induction "commute", by tensoring $\chi$ on the both sides of the above equation, we then get

$$
\left.\operatorname{Ind}_{H}^{G} \chi\right|_{H}=\chi+\sum \operatorname{Ind}_{H_{i}}^{G}\left(\left.\chi\right|_{H_{i}} \phi_{i}\right)
$$

As $\left.\chi\right|_{H_{i}} \phi_{i}$ 's are still 1-dimensional, by condition AHC3, $n\left(H,\left.\chi\right|_{H_{i}} \phi_{i}\right) \geq 0$ for all $i$. Hence, the theorem follows from condition AHC2 and the fact that $\left(\left.\chi\right|_{H}, \phi\right)=1$ and $\left.\chi\right|_{H}=\phi$.

For any subgroup $H$ of $G$, by taking $\chi=1_{G}$ and $\phi=1_{H}$, one can deduce an analogue of the Uchida-van der Waall theorem as below.

Theorem 3.6 (Uchida-van der Waall Theorem). Let $G$ be a solvable group, and $H$ a subgroup. One has

$$
n\left(G, \operatorname{Ind}_{H}^{G} 1_{H}\right)-n\left(G, 1_{G}\right) \geq 0
$$

Moreover, by applying Lemma 3.4 and Theorem 3.5, it is possible to derive several analogues of Murty and Raghuram's results for arithmetic Heilbronn characters.

Theorem 3.7. Let $\chi_{0}$ be a 1-dimensional character of $G$. Then

$$
\sum_{\chi \in \operatorname{Irr}(G) \backslash\left\{\chi_{0}\right\}} n(G, \chi)^{2} \leq\left(n\left(G, \operatorname{Reg}_{G}\right)-n\left(G, \chi_{0}\right)\right)^{2} .
$$

Proof. In light of the proof of [24, Theorem 4.4], we define a "truncated" (arithmetic) Heilbronn character with respect to $\chi_{0}$ as

$$
\Theta_{G}^{\chi_{0}}=\sum_{\chi \in \operatorname{Irr}(G) \backslash\left\{\chi_{0}\right\}} n(G, \chi) \chi .
$$

Taking norms on both sides of the above equation, one has

$$
\left|\Theta_{G}^{\chi_{0}}\right|^{2}=\frac{1}{|G|} \sum_{g \in G}\left|\Theta_{G}^{\chi_{0}}(g)\right|^{2}=\sum_{\chi \in \operatorname{Irr}(G) \backslash\left\{\chi_{0}\right\}} n(G, \chi)^{2} .
$$

On the other hand, by the Heilbronn-Stark lemma, we have

$$
\begin{aligned}
\Theta_{G}^{\chi_{0}}(g) & =\Theta_{G}(g)-n\left(G, \chi_{0}\right) \chi_{0}(g)=\Theta_{\langle g\rangle}(g)-n\left(G, \chi_{0}\right) \chi_{0}(g) \\
& =\sum_{\phi \in \operatorname{Irr}(\langle g\rangle)} n(\langle g\rangle, \phi) \phi(g)-n\left(G, \chi_{0}\right) \sum_{\phi \in \operatorname{Irr}(\langle g\rangle)}\left(\left.\chi_{0}\right|_{\langle g\rangle}, \phi\right) \phi(g) \\
& =\sum_{\phi \in \operatorname{Irr}(\langle g\rangle)}\left(n(\langle g\rangle, \phi)-n\left(G, \chi_{0}\right)\left(\left.\chi_{0}\right|_{\langle g\rangle}, \phi\right)\right) \phi(g) .
\end{aligned}
$$

Applying Theorem 3.5 with $H=\langle g\rangle$ and $\phi \in \operatorname{Irr}(\langle g\rangle)$, we get

$$
n(\langle g\rangle, \phi)-n\left(G, \chi_{0}\right)\left(\left.\chi_{0}\right|_{\langle g\rangle}, \phi\right) \geq 0
$$

which combining with the triangle inequality gives

$$
\begin{aligned}
\left|\Theta_{G}^{\chi_{0}}(g)\right| & \leq \sum_{\phi \in \operatorname{Irr}(\langle g\rangle)}\left(n(\langle g\rangle, \phi)-n\left(G, \chi_{0}\right)\left(\left.\chi_{0}\right|_{\langle g\rangle}, \phi\right)\right) \\
& =n\left(\langle g\rangle, \sum_{\phi \in \operatorname{Irr}(\langle g\rangle)} \phi\right)-n\left(G, \sum_{\phi \in \operatorname{Irr}(\langle g\rangle)}\left(\left.\chi_{0}\right|_{\langle g\rangle}, \phi\right) \chi_{0}\right) \\
& =n\left(\langle g\rangle, \operatorname{Reg}_{\langle g\rangle}\right)-n\left(G,\left(\left.\chi_{0}\right|_{\langle g\rangle}, \operatorname{Reg}_{\langle g\rangle}\right) \chi_{0}\right)=n\left(G, \operatorname{Reg}_{G}\right)-n\left(G, \chi_{0}\right),
\end{aligned}
$$

where the last equality holds provided that $\left(\left.\chi_{0}\right|_{\langle g\rangle}, \operatorname{Reg}_{\langle g\rangle}\right)=\left.\chi_{0}\right|_{\langle g\rangle}(1)=1$.
Theorem 3.8. Let $H$ be a subgroup of $G$, and let $\phi$ be any 1-dimensional character of $H$. Let $S_{\phi}$ denote the set of all 1-dimensional characters of $G$ whose restrictions on $H$ are $\phi$. Then

$$
n\left(G, \operatorname{Ind}_{H}^{G} \phi\right)-\sum_{\chi \in S_{\phi}} n(G, \chi) \geq 0
$$

Proof. Note that if $S_{\phi}$ is empty, then the theorem is obviously true by conditions AHC2 and AHC3. Now we may assume $S_{\phi}$ is non-empty, and take $\chi_{0} \in S_{\phi}$. Applying Lemma 3.4 with $i=1$, we have

$$
\operatorname{Ind}_{H}^{G} 1_{H}=\operatorname{Ind}_{H \cdot G^{1}}^{G} 1_{H \cdot G^{1}}+\sum \operatorname{Ind}_{H_{j}}^{G} \phi_{j},
$$

where for each $j, \phi_{j}$ is a non-trivial 1-dimensional character of a subgroup $H_{j}$ of $G$, and the sum might be empty. Again, twisting the above equation by $\chi_{0}$, we have

$$
\operatorname{Ind}_{H}^{G} \phi=\left.\operatorname{Ind}_{H \cdot G^{1}}^{G} \chi_{0}\right|_{H \cdot G^{1}}+\sum \operatorname{Ind}_{H_{i}}^{G}\left(\left.\chi_{0}\right|_{H_{i}} \phi_{i}\right)
$$

Since $\left.\chi_{0}\right|_{H_{i}} \phi_{i}$ 's are still 1-dimensional and $\left.\operatorname{Ind}_{H \cdot G^{1}}^{G} \chi_{0}\right|_{H \cdot G^{1}}$ is exactly $\sum_{\chi \in S_{\phi}} \chi$, the theorem follows.

Theorem 3.9. Let $S$ be the set of all 1-dimensional characters of $G$. Then

$$
\sum_{\chi \in \operatorname{Irr}(G) \backslash S} n(G, \chi)^{2} \leq\left(n\left(G, \operatorname{Reg}_{G}\right)-n\left(G, \operatorname{Ind}_{G^{1}}^{G} 1_{G^{1}}\right)\right)^{2} .
$$

Proof. Following the proof of [24, Theorem 5.3], we define a truncated arithmetic Heilbronn character with respect to $S$ as

$$
\Theta_{G}^{S}=\sum_{\chi \in \operatorname{Irr}(G) \backslash S} n(G, \chi) \chi
$$

Taking norms on both sides of the above equation gives

$$
\frac{1}{|G|} \sum_{g \in G}\left|\Theta_{G}^{S}(g)\right|^{2}=\sum_{\chi \in \operatorname{Irr}(G) \backslash S} n(G, \chi)^{2} .
$$

Thanks to the Heilbronn-Stark lemma, we have

$$
\begin{aligned}
\Theta_{G}^{S}(g) & =\Theta_{G}(g)-\sum_{\chi \in S} n(G, \chi) \chi(g)=\Theta_{\langle g\rangle}(g)-\sum_{\chi \in S} n(G, \chi) \chi(g) \\
& =\sum_{\phi \in \operatorname{Irr}(\langle g\rangle)}\left(n(\langle g\rangle, \phi)-\sum_{\chi \in S} n(G, \chi)\left(\chi, \operatorname{Ind}_{\langle g\rangle}^{G} \phi\right)\right) \phi(g) .
\end{aligned}
$$

Using Theorem 3.8 with $H=\langle g\rangle$ and $\phi \in \operatorname{Irr}(\langle g\rangle)$, we then obtain

$$
n\left(G, \operatorname{Ind}_{\langle g\rangle}^{G} \phi\right)-\sum_{\chi \in S_{\phi}} n(G, \chi) \geq 0
$$

Observe that for every $\chi \in S,\left(\chi, \operatorname{Ind}_{\langle g\rangle}^{G} \phi\right)$ is either 0 or 1 , and that $\left(\chi, \operatorname{Ind}_{\langle g\rangle}^{G} \phi\right)$ is equal to 1 if and only if $\chi \in S_{\phi}$. Thus, by condition AHC2, we may rewrite the above inequality as

$$
n(\langle g\rangle, \phi)-\sum_{\chi \in S} n(G, \chi)\left(\chi, \operatorname{Ind}_{\langle g\rangle}^{G} \phi\right) \geq 0 .
$$

Finally, by the triangle inequality and the fact that for $\chi \in S,\left(\left.\chi\right|_{\langle g\rangle}, \operatorname{Reg}_{\langle g\rangle}\right)=1$, and $\operatorname{Ind}_{G^{1}}^{G} 1_{G^{1}}=\sum_{\chi \in S} \chi$, one can deduce

$$
\begin{aligned}
\left|\Theta_{G}^{\chi_{0}}(g)\right| & \leq \sum_{\phi \in \operatorname{Irr}(\langle g\rangle)}\left(n(\langle g\rangle, \phi)-\sum_{\chi \in S} n(G, \chi)\left(\chi, \operatorname{Ind}_{\langle g\rangle}^{G} \phi\right)\right) \\
& =n\left(\langle g\rangle, \operatorname{Reg}_{\langle g\rangle}\right)-\sum_{\chi \in S} n(G, \chi)\left(\left.\chi\right|_{\langle g\rangle}, \operatorname{Reg}_{\langle g\rangle}\right) \\
& =n\left(G, \operatorname{Reg}_{G}\right)-n\left(G, \operatorname{Ind}_{G^{1}}^{G} 1_{G^{1}}\right),
\end{aligned}
$$

which completes the proof.
Corollary 3.10. Let $G$ be a solvable group. Then $n\left(G, \operatorname{Reg}_{G}\right)-n\left(G, \operatorname{Ind}_{G^{1}}^{G} 1_{G^{1}}\right)$ cannot be 1 .

Proof. Observe that $\operatorname{Reg}_{G}=\operatorname{Ind}_{G^{1}}^{G} 1_{G^{1}}+\sum_{\chi \notin S} \chi(1) \chi$ where $S$ denotes the set of all 1-dimensional characters of $G$. If $n\left(G, \operatorname{Reg}_{G}\right)-n\left(G, \operatorname{Ind}_{G^{1}}^{G} 1_{G^{1}}\right)$ was equal to 1 , then conditions AHC1 and AHC2 tell us that $\sum_{\chi \notin S} \chi(1) n(G, \chi)=1$. However, Theorem 3.9 forces that there is at most one character $\chi^{\prime} \notin S$ of $G$ such that $n\left(G, \chi^{\prime}\right)$ is nonzero. In addition, the Artin-Takagi decomposition asserts that there should be a character $\chi^{\prime} \notin S$ such that $n\left(G, \chi^{\prime}\right)$ is nonzero. But $\chi^{\prime}(1) \geq 2$, which contradicts to the fact that $\chi^{\prime}(1) n\left(G, \chi^{\prime}\right)=\sum_{\chi \notin S} \chi(1) n(G, \chi)=1$.

In [19], Lansky and Wilson generalized results of Murty and Raghuram by proving the following lemma.

Lemma 3.11 ([19, Lemma 3.2]). Let $G$ be a finite solvable group, and let $H$ be a subgroup of $G$. Let $\phi$ be a 1-dimensional character of $H$ such that $\left.\phi\right|_{H \cap G^{i}}$ is trivial,
and let $\phi^{\prime}$ be the unique extension of $\phi$ to a character of $H \cdot G^{i}$ that is trivial on $G^{i}$. Then for any irreducible character $\chi$ of $G$, one has

$$
\left(\chi, \operatorname{Ind}_{H \cdot G^{i}}^{G} \phi^{\prime}\right)= \begin{cases}\left(\chi, \operatorname{Ind}_{H}^{G} \phi\right), & \text { if } l(\chi) \leq i \\ 0, & \text { if } l(\chi)>i\end{cases}
$$

Adapting the method developed by Lansky and Wilson, it is possible now to obtain a generalization of Murty and Raghuram's work in the setting of arithmetic Heilbronn characters as follows.

Corollary 3.12. Let $d$ be the greatest common divisor of the degrees of the characters in $\operatorname{Irr}(G)) \backslash S^{i}$, where $S^{i}$ denotes the set of irreducible characters of $G$ of level less than or equal to $i$. Then $n\left(G, \operatorname{Reg}_{G}\right)-n\left(G, \operatorname{Ind}_{G^{i}}^{G} 1_{G^{i}}\right)=k d$ for some non-negative integer $k$.

Proof. By conditions AHC1 and AHC2, and Lemma 3.11 with $H=\left\langle e_{G}\right\rangle$, we have

$$
\begin{aligned}
n\left(G, \operatorname{Reg}_{G}\right)-n\left(G, \operatorname{Ind}_{G^{i}}^{G} 1_{G^{i}}\right) & =n\left(G, \operatorname{Reg}_{G}\right)-\sum_{\chi \in S^{i}} \chi(1) n(G, \chi) \\
& =\sum_{\chi \in \operatorname{Irr}(G) \backslash S^{i}} \chi(1) n(G, \chi),
\end{aligned}
$$

which is a multiple of the greatest common divisor of the degrees of the characters $\chi$ of $G$ with $l(\chi)>i$. Since the Aramata-Brauer Theorem asserts that

$$
n\left(G^{i}, \operatorname{Reg}_{G^{i}}\right)-n\left(G^{i}, 1_{G^{i}}\right) \geq 0
$$

by condition AHC2, we obtain $n\left(G, \operatorname{Reg}_{G}\right)-n\left(G, \operatorname{Ind}_{G^{i}}^{G} 1_{G^{i}}\right) \geq 0$ which completes the proof.

Theorem 3.13. Let $\phi$ be a 1-dimensional character of a subgroup $H$ of $G$. Then

$$
n\left(G, \operatorname{Ind}_{H}^{G} \phi\right)-\sum_{\chi \in S^{i}}\left(\chi, \operatorname{Ind}_{H}^{G} \phi\right) n(G, \chi) \geq 0
$$

where $S^{i}$ denotes the set of irreducible characters of $G$ of level at most $i$.

Proof. The proof is exactly the same as the proof in [19], but for the sake of completeness and clarity, we shall reproduce a proof in our setting. Firstly, we assume $\phi$ is trivial on $H \cap G^{i}$, then $\phi$ extends uniquely to a character $\phi^{\prime}$ of $H \cdot G^{i}$. Now Lemma 3.11 implies that

$$
\begin{aligned}
\sum_{\chi \in S^{i}}\left(\chi, \operatorname{Ind}_{H}^{G} \phi\right) n(G, \chi) & =\sum_{\chi \in \operatorname{Irr}(G)}\left(\chi, \operatorname{Ind}_{H \cdot G^{i}}^{G} \phi^{\prime}\right) n(G, \chi) \\
& =n\left(G, \operatorname{Ind}_{H \cdot G^{i}}^{G} \phi^{\prime}\right) .
\end{aligned}
$$

By Lemma 3.3, we have

$$
\operatorname{Ind}_{H}^{H \cdot G^{i}} 1_{H}=1_{H \cdot G^{i}}+\sum \operatorname{Ind}_{H_{j}}^{H \cdot G^{i}} \phi_{j}
$$

where $\phi_{j}$ 's are non-trivial 1-dimensional characters of some subgroups $H_{j}$ 's of $H \cdot G^{i}$, and the sum might be empty. By twisting the above equation by $\phi^{\prime}$, using the fact that tensoring and induction commute, and inducing everything to $G$, one has

$$
\operatorname{Ind}_{H}^{G} \phi=\operatorname{Ind}_{H \cdot G^{i}}^{G} \phi^{\prime}+\left.\sum \operatorname{Ind}_{H_{j}}^{G} \phi^{\prime}\right|_{H_{j}} \phi_{j} .
$$

Thus, the theorem follows in this case that $\phi$ is trivial on $H \cap G^{i}$.
We remark that none of $\left.\phi^{\prime}\right|_{H_{j}} \phi_{j}$ 's is trivial. If $\phi \neq 1_{H}$, then $\left(1_{G}, \operatorname{Ind}_{H}^{G} \phi\right)=0$, and thus $1_{G}$ does not occur. On the other hand, if $\phi=1_{H}$, then Lemma 3.11 and Frobenius reciprocity imply that $\left(1_{G}, \operatorname{Ind}_{H \cdot G^{i}}^{G} \phi^{\prime}\right)=\left(1_{G}, \operatorname{Ind}_{H}^{G} \phi\right)=1$, and thus $1_{G}$ cannot occur in the summation in the above equation.

For the case that $\phi$ is non-trivial on $H \cap G^{i}$, Mackey's theorem and Frobenius reciprocity tell us that

$$
\begin{aligned}
\left(\left.\left(\operatorname{Ind}_{H}^{G} \phi\right)\right|_{G^{i}}, 1_{G^{i}}\right) & =\sum_{G^{i} \backslash G / H}\left(\operatorname{Ind}_{x H x^{-1} \cap G^{i}}^{G^{i}} \phi^{x}, 1_{G^{i}}\right) \\
& =\sum_{G^{i} \backslash G / H}\left(\phi^{x}, 1_{x H x^{-1} \cap G^{i}}\right)=\sum_{G^{i} \backslash G / H}\left(\phi, 1_{H \cap G^{i}}\right)=0,
\end{aligned}
$$

where for each $x \in G, \phi^{x}$ is the character of $x H x^{-1} \cap G^{i}$ given by $g \mapsto \phi\left(x^{-1} g x\right)$. Thus, $\operatorname{Ind}_{H}^{G} \phi$ contains no characters of level less than or equal to $i$, which means that $n\left(G, \operatorname{Ind}_{H}^{G} \phi\right)-\sum_{\chi \in S^{i}}\left(\chi, \operatorname{Ind}_{H}^{G} \phi\right) n(G, \chi)=n\left(G, \operatorname{Ind}_{H}^{G} \phi\right)$ in this case. Now the theorem follows from conditions AHC2 and AHC3.

Corollary 3.14. Let $\phi_{0}$ be a 1-dimensional character of a subgroup $H$ of $G$, and $S_{\phi_{0}}^{i}$ the set of irreducible characters of level $i$ occurring in $\operatorname{Ind}_{H}^{G} \phi_{0}$. Then

$$
\sum_{\chi \in S_{\phi_{0}}^{i}}\left(\chi, \operatorname{Ind}_{H}^{G} \phi_{0}\right) n(G, \chi) \geq 0
$$

Proof. If $\phi_{0}$ is non-trivial on $H \cap G^{i}$, the last paragraph of the proof of Theorem 3.13 gives $\left(\chi, \operatorname{Ind}_{H}^{G} \phi_{0}\right)=0$ for all $\chi \in S_{\phi_{0}}^{i}$, and the corollary follows immediately. Otherwise, $\phi_{0}$ extends uniquely to a character $\phi$ of $H \cdot G^{i}$ which is trivial on $G^{i}$. Then Theorem 3.13 (by replacing $H$ and $i$ by $H \cdot G^{i}$ and $i-1$, respectively) implies that

$$
n\left(G, \operatorname{Ind}_{H \cdot G^{i}}^{G} \phi\right)-\sum_{\chi \in S^{i-1}}\left(\chi, \operatorname{Ind}_{H \cdot G^{i}}^{G} \phi\right) n(G, \chi) \geq 0
$$

By Lemma 3.11, the above difference is equal to
$\sum_{\chi \in S^{i}}\left(\chi, \operatorname{Ind}_{H}^{G} \phi_{0}\right) n(G, \chi)-\sum_{\chi \in S^{i-1}}\left(\chi, \operatorname{Ind}_{H}^{G} \phi_{0}\right) n(G, \chi)=\sum_{\chi \in S_{\phi_{0}}^{i}}\left(\chi, \operatorname{Ind}_{H}^{G} \phi_{0}\right) n(G, \chi)$,
where $S^{j}$ denotes the set of irreducible characters of $G$ of level less than or equal to $j$. Hence, the corollary follows.

We however are not able to show [19, Theorem 4.2]. But we can instead prove the following weaker result conjectured by Murty and Raghuram in [24].

Theorem 3.15. For each $i \geq 1$,

$$
\sum_{\chi \in \operatorname{Irr}(G) \backslash S^{i}} n(G, \chi)^{2} \leq\left(n\left(G, \operatorname{Reg}_{G}\right)-n\left(G, \operatorname{Ind}_{G^{i}}^{G} 1_{G^{i}}\right)\right)^{2}
$$

where $S^{i}$ denotes the set of irreducible characters of $G$ of level at most $i$.
Proof. Again, we define a truncated Heilbronn character with respect to $S^{i}$

$$
\Theta_{G}^{S^{i}}=\sum_{\chi \in \operatorname{Irr}(G) \backslash S^{i}} n(G, \chi) \chi
$$

Taking norms on both sides of the above equation, we get

$$
\frac{1}{|G|} \sum_{g \in G}\left|\Theta_{G}^{S^{i}}(g)\right|^{2}=\sum_{\chi \in \operatorname{Irr}(G) \backslash S^{i}} n(G, \chi)^{2} .
$$

Using the Heilbronn-Stark lemma, one has

$$
\begin{aligned}
\Theta_{G}^{S^{i}}(g) & =\Theta_{G}(g)-\sum_{\chi \in S^{i}} n(G, \chi) \chi(g)=\Theta_{\langle g\rangle}(g)-\sum_{\chi \in S^{i}} n(G, \chi) \chi(g) \\
& =\sum_{\phi \in \operatorname{Irr}(\langle g\rangle)}\left(n(\langle g\rangle, \phi)-\sum_{\chi \in S^{i}} n(G, \chi)\left(\chi, \operatorname{Ind}_{\langle g\rangle}^{G} \phi\right)\right) \phi(g) .
\end{aligned}
$$

Applying Theorem 3.13 with $H=\langle g\rangle$, we then obtain

$$
n(\langle g\rangle, \phi)-\sum_{\chi \in S^{i}} n(G, \chi)\left(\chi, \operatorname{Ind}_{\langle g\rangle}^{G} \phi\right) \geq 0 .
$$

Therefore, the triangle inequality and Frobenius reciprocity yield

$$
\begin{aligned}
\left|\Theta_{G}^{S^{i}}(g)\right| & \leq \sum_{\phi \in \operatorname{Irr}(\langle g\rangle)}\left(n(\langle g\rangle, \phi)-\sum_{\chi \in S^{i}} n(G, \chi)\left(\chi, \operatorname{Ind}_{\langle g\rangle}^{G} \phi\right)\right) \\
& =n\left(G, \operatorname{Reg}_{G}\right)-\sum_{\chi \in S^{i}} n(G, \chi)\left(\left.\chi\right|_{\langle g\rangle}, \operatorname{Reg}_{\langle g\rangle}\right) \\
& =n\left(G, \operatorname{Reg}_{G}\right)-n\left(G, \sum_{\chi \in S^{i}} \chi(1) \chi\right) .
\end{aligned}
$$

Using Lemma 3.4 with $H=\left\langle e_{G}\right\rangle$, we have

$$
\operatorname{Reg}_{G}=\operatorname{Ind}_{G^{i}}^{G} 1_{G^{i}}+(*)
$$

where $\left({ }^{*}\right)$ is a sum of monomial characters. Now $\operatorname{Ind}_{G^{i}}^{G} 1_{G^{i}}$ is exactly the sum of characters of $G$ occurring in $\operatorname{Reg}_{G}$ which are "trivial" on $G^{i}$ (or, equivalently, which have level less than or equal to $i$ ). This means that $\operatorname{Ind}_{G^{i}}^{G} 1_{G^{i}}=\sum_{\chi \in S^{i}} \chi(1) \chi$. Therefore, by conditions AHC1 to AHC3, we complete the proof.

By an analogous argument of the proof of Corollary 3.10, one can deduce the following corollary.

Corollary 3.16. Let $G$ be a solvable group. Then $n\left(G, \operatorname{Reg}_{G}\right)-n\left(G, \operatorname{Ind}_{G^{i}}^{G} 1_{G^{i}}\right)$ cannot be 1 .

At the end of this section, we give a simple application of our arithmetic Heilbronn characters to Artin L-functions.

Theorem 3.17. Let $\Theta_{G}$ be an arithmetic Heilbronn character of a group $G$ associated with integers $n(H, \phi)$. Let $\rho$ be a character of $G$. Suppose that for every subgroup $H$ of $G$, and 1-dimensional irreducible character $\phi$ of $H$, we have $n\left(H,\left.\rho\right|_{H} \otimes \phi\right) \geq 0$. Then for every subgroup $H$ of $G$, we have an arithmetic Heilbronn character defined by

$$
\Theta_{H}^{\prime}=\sum_{\phi \in \operatorname{Irr}(H)} n^{\prime}(H, \phi) \phi
$$

where $n^{\prime}(H, \phi):=n\left(H,\left.\rho\right|_{H} \otimes \phi\right)$. In particular, all properties we have shown for arithmetic Heilbronn characters also hold for $\Theta_{H}^{\prime}$.

Proof. The proof is the same as the proof of Theorem 2.7. By linearity of tensor product and the assumption of this theorem, it is easy to see that $n^{\prime}(H, \phi)$ 's satisfy conditions AHC1 and AHC3. Now since tensoring commutes with induction, by condition AHC2, we have

$$
\begin{aligned}
n^{\prime}(H, \phi)=n\left(H,\left.\rho\right|_{H} \otimes \phi\right) & =n\left(G, \operatorname{Ind}_{H}^{G}\left(\left.\rho\right|_{H} \otimes \phi\right)\right) \\
& =n\left(G, \rho \otimes \operatorname{Ind}_{H}^{G} \phi\right)=n^{\prime}\left(G, \operatorname{Ind}_{H}^{G} \phi\right) .
\end{aligned}
$$

Therefore, the theorem follows.
Let $K / k$ be a solvable Galois extension of number fields with Galois group $G$. A deep theorem of Langlands-Tunnell asserts that all 2-dimensional representations of subgroups of $G$ are automorphic. As a consequence, for any 2-dimensional representation $\rho$ of $G$ and any abelian character $\phi$ of a subgroup $H$ of $G$, the Artin L-function $L\left(s,\left.\rho\right|_{H} \otimes \phi, K / K^{H}\right)$ is holomorphic at $s \neq 1$. Fix $s_{0} \neq 1$ and set

$$
n^{\prime}(H, \phi)=\operatorname{ord}_{s=s_{0}} L\left(s,\left.\rho\right|_{H} \otimes \phi, K / K^{H}\right) .
$$

We recall that $n(H, \phi)=\operatorname{ord}_{s=s_{0}} L\left(s, \phi, K / K^{H}\right)$ define the classical Heilbronn character. Hence, the above theorem assures that these $n^{\prime}(H, \phi)$ 's give a new arithmetic Heilbronn character. In particular, we have the following variant of the Uchida-van der Waall theorem and Murty-Raghuram's inequality.

Theorem 3.18. Let $K / k$ be a solvable Galois extension of number fields with Galois group $G$, and let $\rho$ be a 2-dimensional representation of $G$. Then for any subgroup $H$ of $G$, the quotient

$$
\frac{L\left(s,\left.\operatorname{Ind}_{H}^{G} \rho\right|_{H}, K / k\right)}{L(s, \rho, K / k)}
$$

is holomorphic at $s \neq 1$. Moreover, for every 1-dimensional character $\chi_{0}$ of $G$, one has

$$
\sum_{\chi \in \operatorname{Irr}(G) \backslash\left\{\chi_{0}\right\}}\left(\operatorname{ord}_{s=s_{0}} L(s, \rho \otimes \chi)\right)^{2} \leq\left(\operatorname{ord}_{s=s_{0}}\left(\frac{\zeta_{K}^{2}(s)}{L\left(s, \rho \otimes \chi_{0}, K / k\right)}\right)\right)^{2}
$$

Proof. By Theorem 3.17, this theorem follows immediately from Theorems 3.6 and 3.7 and the identity

$$
\rho \otimes \operatorname{Reg}_{G}=\rho \otimes \operatorname{Ind}_{\left\langle e_{G}\right\rangle}^{G} 1_{\left\langle e_{G}\right\rangle}=\left.\operatorname{Ind}_{\left\langle e_{G}\right\rangle}^{G} \rho\right|_{\left\langle e_{G}\right\rangle}=2 \operatorname{Ind}_{\left\langle e_{G}\right\rangle}^{G} 1_{\left\langle e_{G}\right\rangle}=2 \operatorname{Reg}_{G} .
$$

## 4. Application to Artin-Hecke L-Functions and L-Functions of CM-Elliptic Curves

To avoid the situation that this note becomes a loyal servant of Nicolas Bourbaki, we shall apply our theory of arithmetic Heilbronn characters to study Artin-Hecke L-functions and L-functions of CM-elliptic curves. The crucial idea is due to Murty and Murty in [22] by setting $n(G, \chi)$ being equal to the orders of certain Artin-Hecke L-functions to establish an elliptic analogues of the Uchida-van der Waall theorem. As we will see, this brilliant idea will allow us to obtain several analytic properties of Artin-Hecke L-functions and L-functions of CM-elliptic curves. In particular, we obtain the non-existence of simple zeros for the quotients of suitable L-functions of CM-elliptic curves.

We now recall the concept of Artin-Hecke L-functions developed by Weil [30].
Definition 4.1. Let $K / k$ be a Galois extension of number fields with Galois group $G$. Let $\psi$ be a Hecke character of $k$ and $\rho$ be a complex representation of $G$ with underlying vector space $V$. The Artin-Hecke L-function attached to $\psi$ and $\rho$ is defined by

$$
L(s, \psi \otimes \rho, K / k)=\prod_{\mathfrak{p}} \operatorname{det}\left(1-\left.\psi(\mathfrak{p}) \rho\right|^{V^{I_{\mathfrak{F}}}}\left(\sigma_{\mathfrak{P}}\right) N \mathfrak{p}^{-s}\right)^{-1}
$$

where the product runs over prime ideals in $\mathcal{O}_{k}, \mathfrak{P}$ denotes a prime ideal above $\mathfrak{p}, I_{\mathfrak{P}}$ is the inertia subgroup at $\mathfrak{P}$, and $V^{I_{\mathfrak{P}}}=\left\{v \in V \mid \rho(g) v=v\right.$ for all $\left.g \in I_{\mathfrak{P}}\right\}$. Usually we write $L(s, \psi \otimes \chi, K / k)$ for $L(s, \psi \otimes \rho, K / k)$ where $\chi=\operatorname{tr} \rho$.

We remark that for every 1-dimensional character $\chi$ of $G$, the Artin-Hecke Lfunction $L(s, \psi \otimes \chi, K / k)$ extends to a meromorphic function over $\mathbb{C}$ with only a possible pole at $s=1$ since the corresponding L-function is a Hecke L-function. Moreover, Weil proved each of these L-functions $L(s, \psi \otimes \rho, K / k)$ extends to a meromorphic function on $\mathbb{C}$ by showing the following lemma and applying the Brauer induction theorem [5].

Lemma 4.2 ([30]). For any characters, $\chi_{1}$ and $\chi_{2}$ of $G$ and every character $\phi$ of $H$, we have
(1) $L\left(s, \psi \otimes\left(\chi_{1}+\chi_{2}\right), K / k\right)=L\left(s, \psi \otimes \chi_{1}, K / k\right) L\left(s, \psi \otimes \chi_{2}, K / k\right)$,
(2) $L\left(s, \psi \otimes \operatorname{Ind}_{H}^{G} \phi, K / k\right)=L\left(s, \psi \circ N_{K^{H} / k} \otimes \phi, K / K^{H}\right)$ where $K^{H}$ is a subfield of $K$ fixed by $H$ and $N_{K^{H} / k}$ is the usual norm of $K^{H} / k$.

Now, we consider a (non-trivial) Hecke character $\psi$ of infinite type of $k$, and fix a point $s_{0} \in \mathbb{C}$. We may set $n^{\psi}(H, \phi)=\operatorname{ord}_{s=s_{0}} L\left(s, \psi \circ N_{K^{H} / k} \otimes \phi, K / K^{H}\right)$ for
every character $\phi$ of any subgroup $H$ of $G$. Using Lemma 4.2, it is easy to see that such $n^{\psi}(H, \phi)$ 's define an arithmetic Heilbronn character. Moreover, by "linearity" of tensor product, for any Hecke characters $\psi_{1}$ and $\psi_{2}$ of infinite type of $k$, the integers $n^{\psi_{1}, \psi_{2}}(H, \phi):=n^{\psi_{1}}(H, \phi)+n^{\psi_{2}}(H, \phi)$ also give an arithmetic Heilbronn character.

We now recall two necessary facts from the theory of elliptic curves that allow us to apply our theory of arithmetic Heilbronn characters to L-functions of elliptic curves.

Lemma $4.3([\mathbf{7}-\mathbf{1 0}])$. Let $E$ be an elliptic curve defined over $k$. Suppose that $E$ has $C M$ by an order in an imaginary quadratic field $F$. If $F \subseteq k$, then the $L$ function $L(s, E, k)$ of $E$ is the product of two Hecke L-functions of $k$. If $F \nsubseteq k$, then $L(s, E, k)$ is equal to a Hecke L-function of $k F$ which is a quadratic extension of $k$.

Using this lemma, Murty and Murty in [22] showed the following lemma, which was proved earlier by Shimura for CM-elliptic curves over $\mathbb{Q}$ by using Weil's converse theorem.

Lemma 4.4 ([22, Lemma 2]). The generalized Taniyama conjecture is true for CM-elliptic curves. In other words, every L-function of a CM-elliptic curve can be written in terms of Hecke L-functions.

Now fix $s_{0} \in \mathbb{C}$ and suppose that $K / k$ is a Galois extension of number fields with Galois group $G$. Let $L\left(s, E, K^{H}\right)$ be the L-function of $E / K^{H}$. By the above theorems, this L-function is either a single Hecke L-function or a product of two Hecke L-functions of $K^{H}$. Following the proof of [22, Theorem 1], for each subgroup $H$ of $G$ and complex character $\phi$ of $H$, let $n(H, \phi)$ be the order of the L-function $L\left(s, \phi, E, K^{H}\right)$ at $s=s_{0}$, where $L\left(s, \phi, E, K^{H}\right.$ ) is the "twist" (by $\phi$ ) of $L\left(s, E, K^{H}\right.$ ) (in particular, it is either a single Artin-Hecke L-function or a product of two Artin-Hecke L-functions). According to the conclusion of our previous discussion of Artin-Hecke L-functions, such integers $n(H, \phi)$ define an arithmetic Heilbronn character, and we hence are able to use the theory developed in the previous sections to these integers.

We do not intend to state all theorems and corollaries we can get, but just mention two results. First of all, we have the following theorem that generalizes Murty and Murty's elliptic analogue of the Uchida-van der Waall theorem. Also, this theorem gives an elliptic analogue of Murty-Raghuram's inequality.

Theorem 4.5. Suppose $K / k$ is a solvable Galois extension with Galois group $G$, and let $H$ be a subgroup of $G$. Let $\chi$ and $\phi$ be 1-dimensional characters of $G$ and $H$, respectively. Then

$$
\frac{L\left(s, \operatorname{Ind}_{H}^{G} \phi, E, k\right)}{L(s, \chi, E, k)^{(\chi \mid H, \phi)}}
$$

is entire. In addition, for every 1-dimensional character $\chi_{0}$ of $G$, one has

$$
\sum_{\chi \in \operatorname{Irr}(G) \backslash\left\{\chi_{0}\right\}}\left(\operatorname{ord}_{s=s_{0}} L(s, \chi, E, k)\right)^{2} \leq\left(\operatorname{ord}_{s=s_{0}}\left(\frac{L(s, E, K)}{L\left(s, \chi_{0}, E, k\right)}\right)\right)^{2} .
$$

Moreover, we have below an interesting result for L-functions of CM-elliptic curves by just applying Corollary 3.16.

Corollary 4.6. Suppose $K / k$ is a solvable Galois extension with Galois group $G$. Then for all $i \geq 1$,

$$
\frac{L(s, E, K)}{L\left(s, E, K^{G^{i}}\right)}
$$

cannot have any simple zero, where $G^{0}=G, G^{i}=\left[G^{i-1}, G^{i-1}\right]$ for $i \geq 1, K^{G^{i}}$ is the fixed field of $G^{i}$, and $L\left(s, E, K^{G^{i}}\right)$ is the L-function of $E / K^{G^{i}}$.

## 5. Application to Automorphic L-Functions and L-Functions of Elliptic Curves

In this section, we will follow the path enlightened by [22] to demonstrate how arithmetic Heilbronn characters play a role in studying automorphic L-functions.

We will begin by collecting some important results from the theory of automorphic L-functions.

Theorem 5.1 ([1, Arthur and Clozel]). Let $K / k$ be a cyclic Galois extension of number fields of prime degree, and $\pi$ and $\Pi$ denote automorphic representations induced from cuspidal of $G L_{n}\left(\mathbb{A}_{k}\right)$ and $G L_{n}\left(\mathbb{A}_{K}\right)$ respectively (or, in particular, cuspidal automorphic representations of $G L_{n}\left(\mathbb{A}_{k}\right)$ and $G L_{n}\left(\mathbb{A}_{K}\right)$ respectively). Then the base change $B(\pi)$ of $\pi$ and the automorphic induction $I(\Pi)$ of $\Pi$ exist. Moreover, $I(\Pi)$ is induced from cuspidal.

We remark that Arthur-Clozel's theorem in fact enables one to use the base change and the automorphic induction if $K / k$ is a solvable Galois extension of number fields. Moreover, their theorem yields that all nilpotent groups are of automorphic type, which is predicted by Langlands' reciprocity conjecture. We refer the reader to [1] for the complete details and to [20] for a nice introduction. We also recall the theory of Rankin-Selberg L-functions due to Jacquet, Piatetski-Shapiro and Shalika, and its connection with Arthur-Clozel's theory.

Theorem 5.2 ([17, Jacquet, Piatetski-Shapiro and Shalika]). Let $\pi$ and $\sigma$ be two cuspidal unitary automorphic representations of $G L_{n}\left(\mathbb{A}_{k}\right)$ and $G L_{m}\left(\mathbb{A}_{k}\right)$, respectively. Then the Rankin-Selberg L-function $L(s, \pi \otimes \sigma)$ of $\pi$ and $\sigma$ extends to a meromorphic function of $s$.

Theorem 5.3 ([16, Jacquet]). Let $K / k$ be a cyclic Galois extension of number fields of prime degree, and $\pi$ and $\sigma$ be two cuspidal unitary automorphic representations of $G L_{n}\left(\mathbb{A}_{k}\right)$ and $G L_{m}\left(\mathbb{A}_{K}\right)$, respectively. Then the Rankin-Selberg L-functions
satisfy the following formal identity:

$$
L(s, B(\pi) \otimes \sigma)=L(s, \pi \otimes I(\sigma))
$$

In light of [22, Proof of Theorem 2], we prove the following lemma that allows us to construct arithmetic Heilbronn characters later.

Lemma 5.4. Let $K / k$ be a Galois extension of number fields with Galois group $G, \rho$ a representation of $G$, and $n \geq 2$. Suppose that $\pi$ is a cuspidal automorphic representation of $G L_{n}\left(\mathbb{A}_{k}\right)$ such that for every intermediate field $M$ of $K / k$ with $K / M$ solvable, $\left.\pi\right|_{M}$ is automorphic (over $M$ ). Then the Rankin-Selberg L-function $L(s, \pi \otimes \rho)$ extends to a meromorphic function of $s$.

Proof. By the Brauer induction theorem, one can write

$$
\operatorname{tr} \rho=\sum_{i} m_{i} \operatorname{Ind}_{H_{i}}^{G} \chi_{i},
$$

where $m_{i} \in \mathbb{Z}, \chi_{i}$ is an abelian character of an elementary subgroup $H_{i}$ of $G$, which is nilpotent. By Artin reciprocity, for each $i, \chi_{i}$ corresponds to a cuspidal automorphic representation of $G L_{1}\left(\mathbb{A}_{K^{H_{i}}}\right)$. Since each $H_{i}$ is nilpotent, $H_{i}$ is solvable, and so $\left.\pi\right|_{K^{H_{i}}}$ is automorphic. Now the Rankin-Selberg theory ensures that every $L\left(s,\left.\pi\right|_{K^{H_{i}}} \otimes \chi_{i}\right)$ extends to an entire function. Thus, $L(s, \pi \otimes \rho)$ extends to a meromorphic function over $\mathbb{C}$.

We first note that if Langlands' reciprocity conjecture holds for $K / k$, then the automorphy assumption on $\left.\pi\right|_{M}$ can be easily removed by just applying the theory of Rankin-Selberg L-functions. On the other hand, if one knows how to associate Galois representations to $\pi$ and its "descents", then one can apply Arthur-Clozel's theory of base change to derive the desired automorphy result. In particular, if $K / k$ is a totally real solvable extension and $\pi$ is a "RAESDC" (regular algebraic essentially self-dual cuspidal) automorphic representation, then by the work of Taylor and his school, the extra automorphy assumption in the above lemma can be dropped (for more details and references, see the next section, especially, Theorems 6.2 and 6.3).

Under the above assumption and notation, we now further assume that $K / k$ is totally real and solvable. We let $H$ be a subgroup of $G$ and $\phi$ a character of $H$, and fix $s_{0} \in \mathbb{C}$. We define $n(H, \phi)$ to be the order of the Rankin-Selberg L-function $L\left(s,\left.\pi\right|_{K^{H}} \otimes \phi\right)$ at $s=s_{0}$. Since $K / K^{H}$ is still a solvable Galois extension, by Theorem 5.3 and Lemma 5.4, we know that $n(H, \phi)$ 's define an arithmetic Heilbronn character. Again, we do not intend to restate all results established in the previous section but just mention two of them.

First of all, applying Theorem 3.5, we obtain the following theorem that can be seen as an analogue of Murty and Raghuram's variant of the Uchida-van der Waall theorem.

Theorem 5.5. Under the assumption and notation as above. Let $\chi$ and $\phi$ be 1dimensional characters of $G$ and $H$, respectively. Then the quotient

$$
\frac{L\left(s,\left.\pi\right|_{H} \otimes \phi\right)}{L(s, \pi \otimes \chi)^{\left(\left.\chi\right|_{H}, \phi\right)}}
$$

is entire. Moreover, for every 1-dimensional character $\chi_{0}$ of $G$, one has

$$
\sum_{\chi \in \operatorname{Irr}(G) \backslash\left\{\chi_{0}\right\}}\left(\operatorname{ord}_{s=s_{0}} L(s, \pi \otimes \chi)\right)^{2} \leq\left(\operatorname{ord}_{s=s_{0}}\left(\frac{L(s, B(\pi))}{L\left(s, \pi \otimes \chi_{0}\right)}\right)\right)^{2}
$$

where $B(\pi)$ is the base change of $\pi$ to $K$.
In fact, this also generalizes [22, Theorem 4] that asserts that $\frac{L\left(s,\left.\pi\right|_{H}\right)}{L(s, \pi)}$ is entire. On the other hand, one can use Theorem 3.16 to get the following.

Theorem 5.6. Under the assumption and notation as above. Then for all $i \geq 1$,

$$
\frac{L(s, B(\pi))}{L\left(s, B^{i}(\pi)\right)}
$$

cannot have any simple zero where $G^{0}=G, G^{i}$ denotes $\left[G^{i-1}, G^{i-1}\right]$ for all $i \geq 1$, $K^{G^{i}}$ is the fixed field of $G^{i}, B(\pi)$ is the base change of $\pi$ to $K$, and $B^{i}(\pi)$ is the base change of $\pi$ to $K^{G^{i}}$, the fixed field of $G^{i}$.

We note that the existence of $B^{i}(\pi)$ in the above theorem is due to the ArthurClozel theorem and the fact that each $G^{i}$ is normal in $G$. We remark that these theorems also have other arithmetic applications. For instance, as mentioned in [22], the zeta function of any CM abelian variety over an arbitrary number field is given in terms of Hecke L-functions, and the Jacobian of a modular curve has the zeta function that is equal to a product of L-functions attached to modular forms by a theorem of Shimura. In both instances, one may obtain appropriate generalization by setting integers equal to the orders of suitable L-functions (at $s=s_{0} \in \mathbb{C}$ ) to define an arithmetic Heilbronn character.

At the end of this section, we shall apply the previous results to symmetric power L-functions. Suppose that $M / k$ is an extension of number fields contained in a totally real solvable Galois extension $K / k$ with $G=\operatorname{Gal}(K / k)$. We denote $H_{M}$ to be a subgroup of $G$ such that $K^{H_{M}}=M$. Let $E$ be a non-CM elliptic curve defined over $k$. For every intermediate field $F$ of $K / k$, let $\rho_{F}=\rho_{E, F}$ denote a compatible system of $\ell$-adic representations attached to $E$ over $F$, i.e. for each prime $\ell$,

$$
\rho_{F}:=\rho_{\ell, F}: \operatorname{Gal}(\bar{k} / F) \rightarrow \operatorname{Aut}\left(T_{\ell}(E, F)\right),
$$

where $T_{\ell}(E, F)$ denotes ( $\ell$-adic) Tate module of $E / F$ (for more details, see [22] and [23]). Now assuming the $m$ th symmetric power of $\rho_{k}$ is automorphic, Lemma 5.4 implies that for every character $\chi$ of $G$, the Rankin-Selberg L-function

$$
L\left(s, \operatorname{Sym}^{m} \rho_{k} \otimes \chi\right)
$$

extends to a meromorphic function over $\mathbb{C}$. On the other hand, since for every intermediate field $F$ of $K / k, T_{\ell}(E, F)=T_{\ell}(E, k)$ as $\operatorname{Gal}(\bar{k} / F)$-modules, it follows that

$$
\left.\left(\operatorname{Sym}^{m} \rho_{k}\right)\right|_{\operatorname{Gal}(\bar{k} / F)}=\operatorname{Sym}^{m} \rho_{F} .
$$

But

$$
\operatorname{Ind}_{\operatorname{Gal}(\bar{k} / F)}^{\operatorname{Gal}(\bar{k} / k)}\left(\left.\left(\operatorname{Sym}^{m} \rho_{k}\right)\right|_{\operatorname{Gal}(\bar{k} / F)}\right)=\operatorname{Sym}^{m} \rho_{k} \otimes \operatorname{Ind}_{\operatorname{Gal}(\bar{k} / F)}^{\operatorname{Gal}(\bar{k} / k)} 1 .
$$

Putting everything together, we finally obtain

$$
\begin{align*}
L\left(s, \operatorname{Sym}^{m} \rho_{F}\right) & =L\left(s,\left.\left(\operatorname{Sym}^{m} \rho_{k}\right)\right|_{\operatorname{Gal}(\bar{k} / F)}\right) \\
& =L\left(s, \operatorname{Sym}^{m} \rho_{k} \otimes \operatorname{Ind}_{H_{F}}^{G} 1\right) \tag{5.1}
\end{align*}
$$

where $H_{F}$ is a subgroup of $G$ such that $K^{H_{F}}=F$.
Now fix $s=s_{0} \in \mathbb{C}$, and for every character $\phi$, define $n(H, \phi)$ to be the order of the L-function

$$
L\left(s,\left.\left(\operatorname{Sym}^{m} \rho_{k}\right)\right|_{K^{H}} \otimes \phi\right)
$$

at $s=s_{0}$, where $\left.\left(\operatorname{Sym}^{m} \rho_{k}\right)\right|_{K^{H}}$ is obtained in the same manner as in the proof of Theorem 6.1 (we note that Arthur-Clozel's theory of base change asserts that $\left.\left(\operatorname{Sym}^{m} \rho_{k}\right)\right|_{K}$ is automorphic). Therefore, $n(H, \phi)$ 's define an arithmetic Heilbronn character. As a consequence, we have the following elliptic analogue of the Uchidavan der Waall theorem that generalizes [22, Theorem 2].

Theorem 5.7. Under the assumption and notation as above. Let $\chi$ and $\phi$ be 1dimensional representations of $G$ and $H$, respectively. Then

$$
\frac{L\left(s, \operatorname{Sym}^{m} \rho_{K^{H}} \otimes \phi\right)}{L\left(s, \operatorname{Sym}^{m} \rho_{k} \otimes \chi\right)^{\left(\left.\chi\right|_{H}, \phi\right)}}
$$

is entire. Moreover, by Eq. (5.1), for every intermediate field $F$ of $K / k$,

$$
\frac{L\left(s, \operatorname{Sym}^{m} \rho_{F}\right)}{L\left(s, \operatorname{Sym}^{m} \rho_{k}\right)}
$$

is entire.
On the other hand, Theorem 5.6 and Eq. (5.1) give below a surprising result.
Theorem 5.8. Under the assumption and notation as above. Then for all $i \geq 1$,

$$
\frac{L\left(s, \operatorname{Sym}^{m} \rho_{K}\right)}{L\left(s, \operatorname{Sym}^{m} \rho_{K^{G^{i}}}\right)}
$$

cannot admit any simple zero. In particular,

$$
\frac{L(s, E, K)}{L\left(s, E, K^{G^{i}}\right)}
$$

has no simple zeros, where for any intermediate field $F$ of $K / k, L(s, E, F)$ denotes the L-function of $E / F$.

Also, we have an elliptic analogue of Murty-Raghuram's inequality.

Theorem 5.9. Under the assumption and notation as above. Suppose $K / k$ is a totally real solvable Galois extension with Galois group G. Then for every 1dimensional character $\chi_{0}$ of $G$, one has

$$
\sum_{\chi \in \operatorname{Irr}(G) \backslash\left\{\chi_{0}\right\}}\left(\operatorname{ord}_{s=s_{0}} L\left(s, \operatorname{Sym}^{m} \rho_{k} \otimes \chi\right)\right)^{2} \leq\left(\operatorname{ord}_{s=s_{0}}\left(\frac{L\left(s, \operatorname{Sym}^{m} \rho_{K}\right)}{L\left(s, \operatorname{Sym}^{m} \rho_{k} \otimes \chi_{0}\right)}\right)\right)^{2} .
$$

## 6. Application of Weak Heilbronn Characters

As one can see, arithmetic Heilbronn characters indeed play a role which helps us to obtain analytic properties of L-functions. Meanwhile, one may wonder if we really need the notion of weak arithmetic Heilbronn characters, which seems impractical and unnecessary. Thanks to the recent groundbreaking work of Taylor and his school, this wonder may not really be an issue.

In his paper [26], Taylor proved the potential automorphy for certain symmetric power L-functions of non-CM elliptic curves, and then deduced the Sato-Tate conjecture (over totally real fields). He was building on his earlier work [6] and [14] with Clozel, Harris, and Shepherd-Barron. More recently, Barnet-Lamb, Geraghty, Harris, and Taylor [3] proved the potential automorphy for symmetric power L-functions in a more general setting.

As we will demonstrate, it is possible to utilize the above-mentioned results of potential automorphy and our weak arithmetic Heilbronn characters to study Lfunctions. However, for the sake of conceptual clarity, we shall only use Taylor's potential automorphy result here.

We recall that Taylor's main theorem is: let $k$ be a totally real field and $E / k$ a non-CM elliptic curve with at least one prime of multiplicative reduction. Then for any finite set $S$ of (odd) natural numbers, there is a (finite) totally real Galois extension $L / k$ such that for every $m \in S, \operatorname{Sym}^{m} \rho_{k}$ is automorphic over $L$, i.e. $\left.\left(\operatorname{Sym}^{m} \rho_{k}\right)\right|_{L}$ is automorphic.

From now on, we fix a finite set $S$ of natural numbers and let $L$ be a (finite) totally real Galois extension $L / k$ such that for every $m \in S, \operatorname{Sym}^{m} \rho_{k}$ is automorphic over $L$, which is given by Taylor's theorem. We now recall two of key steps of the proof of the Sato-Tate conjecture.

Theorem 6.1 ([14]). For any intermediate field $F$ of $L / k$ with $L / F$ solvable,

$$
\left.\left(\operatorname{Sym}^{m} \rho_{k}\right)\right|_{F}
$$

is automorphic.
This is proved in [14] by Harris, Shepherd-Barron and Taylor. The proof essentially applies the above-mentioned Arthur-Clozel theorem of base change and the fact that $\left.\left(\operatorname{Sym}^{m} \rho_{k}\right)\right|_{L}$ is Galois-invariant. Since every irreducible character $\phi$ of a cyclic subgroup $H$ of $G=\operatorname{Gal}(L / k)$ can be identified as an automorphic representation of $G L_{1}\left(\mathbb{A}_{L^{H}}\right)$ via Artin reciprocity, the above theorem and the Rankin-Selberg
theory yield

$$
L\left(s,\left.\left(\operatorname{Sym}^{m} \rho_{k}\right)\right|_{L^{H}} \otimes \phi\right)
$$

is entire.
On the other hand, applying Theorem 6.1, Artin reciprocity, and the Brauer induction theorem, Taylor and his school showed the following.

Theorem 6.2. $L\left(s, \operatorname{Sym}^{m} \rho_{k}\right)$ extends to a meromorphic function over $\mathbb{C}$.
In light of their method, one can show the following.
Theorem 6.3. For every character $\chi$ of $G=\operatorname{Gal}(L / k), L\left(s,\left(\operatorname{Sym}^{m} \rho_{k}\right) \otimes \chi\right)$ extends to a meromorphic function over $\mathbb{C}$.

Proof. As usual, the Brauer induction theorem asserts

$$
\chi=\sum_{i} n_{i} \operatorname{Ind}_{H_{i}}^{G} \phi_{i},
$$

where for each $i, n_{i}$ is an integer, and $\phi_{i}$ is a 1-dimensional character of a nilpotent subgroup $H_{i}$ of $G$. According to Artin reciprocity, $\phi_{i}$ can be seen as a Hecke character over $L^{H_{i}}$. Putting everything together, one has

$$
L\left(s,\left(\operatorname{Sym}^{m} \rho_{k}\right) \otimes \chi\right)=\prod_{i} L\left(s,\left.\left(\operatorname{Sym}^{m} \rho_{k}\right)\right|_{L^{H_{i}}} \otimes \phi_{i}\right)^{n_{i}},
$$

where $\phi_{i} \in \mathfrak{A}\left(G L_{1}\left(\mathbb{A}_{L^{H_{i}}}\right)\right)$. By Theorem 6.1, $\left.\left(\operatorname{Sym}^{m} \rho_{k}\right)\right|_{L^{H_{i}}}$ is automorphic over $L^{H_{i}}$. Now the Rankin-Selberg theory tells us that each

$$
L\left(s,\left.\left(\operatorname{Sym}^{m} \rho_{k}\right)\right|_{L^{H_{i}}} \otimes \phi_{i}\right)
$$

is entire, which completes the proof.
Therefore, for $H$ cyclic or $H=G$, fixing $s_{0} \in \mathbb{C}$ and setting

$$
n(H, \phi)=\operatorname{ord}_{s=s_{0}} L\left(s,\left.\left(\operatorname{Sym}^{m} \rho_{k}\right)\right|_{L^{H}} \otimes \phi\right),
$$

the above discussion yields that $n(H, \phi)$ 's define a weak arithmetic Heilbronn character. In particular, by Theorem 2.4, we then deduce:

## Theorem 6.4.

$$
\sum_{\chi \in \operatorname{Irr}(G)} n(G, \chi)^{2} \leq\left(\operatorname{ord}_{s=s_{0}} L\left(s, \operatorname{Sym}^{m} \rho_{L}\right)\right)^{2}
$$

In particular, (if we choose $S$ containing 1 in the very beginning)

$$
\left|\operatorname{ord}_{s=s_{0}} L\left(s, \rho_{k}\right)\right| \leq \operatorname{ord}_{s=s_{0}} L\left(s, \rho_{L}\right) .
$$

We remark that the last inequality of analytic ranks is a consequence predicted by the Birch-Swinnerton-Dyer conjecture for $s_{0}=1$.

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