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COUNTING ZEROS OF DEDEKIND ZETA FUNCTIONS

ELCHIN HASANALIZADE, QUANLI SHEN, AND PENG-JIE WONG

ABSTRACT. Given a number field K of degree n_K and with absolute discriminant d_K, we obtain an explicit bound for the number $N_K(T)$ of non-trivial zeros (counted with multiplicity), with height at most T, of the Dedekind zeta function $\zeta_K(s)$ of K. More precisely, we show that for $T \geq 1$,

$$\left|N_K(T) - \frac{T}{\pi}\log\left(d_K\left(\frac{T}{2\pi e}\right)^{n_K}\right)\right| \leq 0.228(\log d_K + n_K \log T) + 23.108 n_K + 4.520,$$

which improves previous results of Kadiri and Ng, and Trudgian. The improvement is based on ideas from the recent work of Bennett et al. on counting zeros of Dirichlet L-functions.

1. Introduction

Given a number field K, the Dedekind zeta function $\zeta_K(s)$ of K is defined by

$$\zeta_K(s) = \sum_{\mathfrak{a} \neq 0} \frac{1}{\mathrm{N}(\mathfrak{a})^s},$$

for $\mathfrak{Re}(s) > 1$, where the sum is over non-zero integral ideals of K. It is known that $\zeta_K(s)$ has an analytic continuation to a meromorphic function on $\mathbb{C}$ with only a simple pole at $s = 1$, and its zeros $\rho = \beta + i\gamma$ encode deep arithmetic information of K. For instance, the generalised Riemann hypothesis, asserting that if $\zeta_K(\rho) = 0$ and $\beta \in (0, 1)$, then $\beta = \frac{1}{2}$, leads to the strongest form of the prime ideal theorem. A related prominent question is to count the zeros of $\zeta_K(s)$ in the critical strip $0 < \mathfrak{Re}(s) < 1$. For $T \geq 0$, we set

$$N_K(T) = \#\{\rho \in \mathbb{C} \mid \zeta_K(\rho) = 0,\ 0 < \beta < 1,\ |\gamma| \leq T\},$$

counted with multiplicity if there are any multiple zeros. The estimate of $N_K(T)$ is crucial for proving effective versions of the Chebotarev density theorem as well as bounding the least prime in the Chebotarev density theorem (see [4, 5]). Moreover, to make these results explicit, it is natural to further require a determination of the implied constants for the estimate of $N_K(T)$.

Adapting the arguments of Backlund [1], McCurley [6], and Rosser [8], in [3], Kadiri and Ng showed that for $T \geq 1$, one has

$$\left|N_K(T) - \frac{T}{\pi}\log\left(d_K\left(\frac{T}{2\pi e}\right)^{n_K}\right)\right| \leq D_1(\log d_K + n_K \log T) + D_2 n_K + D_3, \tag{1.1}$$

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with admissible $(D_1, D_2, D_3) = (0.506, 16.950, 7.663)$, where n_K and d_K are the degree and absolute discriminant of K, respectively; also, D_1 can be taken as small as $(\pi \log 2)^{-1} \approx 0.459$ at expense of larger $D_2 n_K + D_3$. This was improved by Trudgian [9] (not only for Dedekind zeta functions but also for Dirichlet L-functions). In particular, as asserted in [9], the estimate (1.1) is valid with $(D_1, D_2, D_3) = (0.316, 5.872, 3.655)$, and the constant D_1 in (1.1) could be made as small as 0.247 (with larger $D_2 n_K + D_3$). Unfortunately, as pointed out by Bennett, Martin, O'Bryant, and Rechnitzer [2], there is an error in [9] that appears as the ranges of various parameters used in the argument of [9] were not verified properly. In [2], Bennett et al. fixed this problem for Dirichlet L-functions.

The objective of this article is to prove Theorem 1.1.

Theorem 1.1. *Given a number field K of degree n_K and with absolute discriminant d_K and r_1 real places, for any $T \geq 1$, we have*

$$\begin{aligned} &\left| N_K(T) - \frac{T}{\pi} \log \left(d_K \left(\frac{T}{2\pi e} \right)^{n_K} \right) + \frac{r_1}{4} \right| \\ &\qquad \leq 0.22737 \log \left(\frac{d_K (T+2)^{n_K}}{(2\pi)^{n_K}} \right) + 23.02528 n_K + 4.51954. \end{aligned} \tag{1.2}$$

In addition, writing the right of (1.2) as $C_1 \log \left(\frac{d_K (T+2)^{n_K}}{(2\pi)^{n_K}} \right) + C_2 n_K + C_3$, we have further admissible triples (C_1, C_2, C_3) recorded in Table 2 in Section 4. Moreover, recalling that for $T \geq T_0$, $\log(T+2) - \log T \leq \log(1 + \frac{2}{T_0})$, from Theorem 1.1 and the triangle inequality, we derive the following improved bound for $N_K(T)$.

Corollary 1.2. *Given a number field K of degree n_K and with absolute discriminant d_K, for any $T \geq 1$, we have*

$$\left| N_K(T) - \frac{T}{\pi} \log \left(d_K \left(\frac{T}{2\pi e} \right)^{n_K} \right) \right| \leq 0.228 (\log d_K + n_K \log T) + 23.108 n_K + 4.520. \tag{1.3}$$

Furthermore, by Table 2, writing the right of (1.3) as $D_1(\log d_K + n_K \log T) + D_2 n_K + D_3$, we have Table 1 of admissible (D_1, D_2, D_3) that not only repair but also improve all triples given in [9, Table 2]. (Note that, for all number fields K, our D_2 and D_3 yield a smaller vlaue of $D_2 n_K + D_3$ than the one given by Trudgian [9].)

TABLE 1. Admissible (D_1, D_2, D_3) in Corollary 1.2 and in [9]

Trudgian [9]					Our improvement				
	$T \geq 1$		$T \geq 10$			$T \geq 1$		$T \geq 10$	
D_1	D_2	D_3	D_2	D_3	D_1	D_2	D_3	D_2	D_3
0.247	8.851	3.024	8.726	2.081	0.245	6.735	4.213	6.449	3.124
0.265	7.521	3.178	7.396	2.101	0.264	5.276	4.082	4.968	3.051
0.282	6.776	3.335	6.651	2.123	0.281	4.478	4.010	4.149	3.012
0.299	6.262	3.494	6.138	2.146	0.296	3.971	3.969	3.622	2.990

The proof of Theorem 1.1 follows closely the arguments of Bennett, Martin, O'Bryant, and Rechnitzer [2], Kadiri and Ng [3], and Trudgian [9], which are an adaption of the methods of Backlund [1], McCurley [6], and Rosser [8]. We also take advantage of the refined estimates for Gamma factors obtained in [2]. Moreover, following the strategy of Bennett et al. [2], we extend Rademacher's convexity bound for $\zeta_K(s)$ (cf. Propositions 3.8 and 3.9) that, together with "Backlund's trick" (see Section 3.2), plays a central role in improving the leading constants C_1 and D_1. Furthermore, we track all the parameters and related inequalities in a similar manner of Bennett et al. [2] to fix the aforementioned error appearing in [9]. Last but not least, we note that we obtain our results by a direct numerical computation (with help from Maple) and that it may be possible to use the "interval analysis" as in [2] to prove an estimate similar to [2, Theorem 1.1]. Nonetheless, since Corollary 1.2 is already as strong as [2, Corollary 1.2], and it is sufficient for most applications, we shall not devote ourselves to do such an interval analysis here.

2. The main term and the gamma factor

2.1. **The main term.** Let K be a number field of degree n_K and with absolute discriminant d_K. We let r_1 and r_2 be the numbers of real and complex places, respectively, of K and note that $n_K = r_1 + 2r_2$. We define the completed zeta function $\xi_K(s)$ as

$$\xi_K(s) = s(s-1)d_K^{s/2}\gamma_K(s)\zeta_K(s), \tag{2.1}$$

where

$$\gamma_K(s) = \left(\pi^{-\frac{s+1}{2}}\Gamma\Big(\frac{s+1}{2}\Big)\right)^{r_2}\left(\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big)\right)^{r_1+r_2}.$$

We recall that $\xi_K(s)$ extends to an entire function of order 1 and satisfies the functional equation

$$\xi_K(s) = \xi_K(1-s). \tag{2.2}$$

As in the introduction, we set

$$N_K(T) = \#\{\rho \in \mathbb{C} \mid \zeta_K(\rho) = 0,\ 0 < \beta < 1,\ |\gamma| \le T\}.$$

To estimate $N_K(T)$, we shall apply the argument principle as follows. For any fixed $\sigma_1 > 1$, we consider the rectangle $\mathcal{R}$ with vertices $\sigma_1 - iT$, $\sigma_1 + iT$, $1-\sigma_1+iT$, and $1-\sigma_1-iT$ (that is away from zeros of $\xi_K(s)$).[1] As $\xi_K(s)$ is entire, it follows from the argument principle that

$$N_K(T) = \frac{1}{2\pi}\Delta_{\mathcal{R}} \arg \xi_K(s).$$

[1]Throughout our argument, we will always assume T is away from zeros of $\xi_K(s)$. As shall be seen in Section 4, with this assumption, we will prove (4.2) for T away from zeros of $\xi_K(s)$. Nonetheless, if T is the exact height of a zero, we know that $N_K(T) = N_K(T+\varepsilon)$ for all sufficiently small $\varepsilon > 0$ (in other words, $T+\varepsilon$ is away from zeros). Then, by the triangle inequality, applying (4.2) with $T+\varepsilon$, we see that

$$\begin{aligned}
&\Big|N_K(T) - \frac{T}{\pi}\log\Big(d_K\Big(\frac{T}{2\pi e}\Big)^{n_K}\Big) + \frac{r_1}{4}\Big| \\
&\le \Big|N_K(T+\varepsilon) - \frac{T+\varepsilon}{\pi}\log\Big(d_K\Big(\frac{T+\varepsilon}{2\pi e}\Big)^{n_K}\Big) + \frac{r_1}{4}\Big| \\
&+ \Big|\frac{T+\varepsilon}{\pi}\log\Big(d_K\Big(\frac{T+\varepsilon}{2\pi e}\Big)^{n_K}\Big) - \frac{T}{\pi}\log\Big(d_K\Big(\frac{T}{2\pi e}\Big)^{n_K}\Big)\Big| \\
&\le C_1\log\Big(\frac{d_K(T+\varepsilon+2)^{n_K}}{(2\pi)^{n_K}}\Big) + C_2 n_K + C_3 \\
&+ \Big|\frac{T+\varepsilon}{\pi}\log\Big(d_K\Big(\frac{T+\varepsilon}{2\pi e}\Big)^{n_K}\Big) - \frac{T}{\pi}\log\Big(d_K\Big(\frac{T}{2\pi e}\Big)^{n_K}\Big)\Big|.
\end{aligned}$$

Now, taking $\varepsilon \to 0^+$, we conclude that (4.2) is also valid when T is the exact height of a zero.

Let $\mathcal{C}$ be the part of the contour of $\mathcal{R}$ in $\mathfrak{Re}(s) \geq \frac{1}{2}$ and $\mathcal{C}_0$ be the part of the contour of $\mathcal{R}$ in $\mathfrak{Re}(s) \geq \frac{1}{2}$ and $\mathfrak{Im}(s) \geq 0$. Since $\overline{\xi_K(s)} = \xi_K(\bar{s})$, the functional equation (2.2) then yields

$$\Delta_{\mathcal{R}} \arg \xi_K(s) = 2\Delta_{\mathcal{C}} \arg \xi_K(s) = 4\Delta_{\mathcal{C}_0} \arg \xi_K(s),$$

which implies that

$$N_K(T) = \frac{2}{\pi} \Delta_{\mathcal{C}_0} \arg \xi_K(s). \tag{2.3}$$

Writing $B = d_K/\pi^{n_K}$, by (2.1), we have

$$\begin{aligned} \Delta_{\mathcal{C}_0} \arg \xi_K(s) = \Delta_{\mathcal{C}_0} \arg s + \Delta_{\mathcal{C}_0} \arg B^{s/2} \\ + (r_1 + r_2)\Delta_{\mathcal{C}_0} \arg \Gamma\left(\frac{s}{2}\right) + r_2 \Delta_{\mathcal{C}_0} \arg \Gamma\left(\frac{s+1}{2}\right) \\ + \Delta_{\mathcal{C}_0} \arg\left((s-1)\zeta_K(s)\right). \end{aligned} \tag{2.4}$$

It is clear that

$$\begin{aligned} &\Delta_{\mathcal{C}_0} \arg s = \arctan(2T), \\ &\Delta_{\mathcal{C}_0} \arg B^{s/2} = \frac{T}{2} \log B = \frac{T}{2} \log\left(\frac{d_K}{\pi^{n_K}}\right), \\ &\Delta_{\mathcal{C}_0} \arg \Gamma(s) = \Delta_{\mathcal{C}_0} (\mathfrak{Im} \log \Gamma(s)) = \mathfrak{Im} \log \Gamma\Big(\frac{1}{2} + iT\Big). \end{aligned} \tag{2.5}$$

To control the Gamma factor, we shall appeal for the improved numerical bound established in [2, Sec. 3]. For $a \in \{0, 1\}$, we set

$$g_a(T) = \frac{2}{\pi} \mathfrak{Im} \log \Gamma\Big(\frac{1}{4} + \frac{a}{2} + i\frac{T}{2}\Big) - \frac{T}{\pi} \log\left(\frac{T}{2e}\right) - \frac{2a-1}{4}.$$

It follows from [2, Proposition 3.2] that for $a \in \{0, 1\}$ and $T \geq 5/7$,

$$|g_a(T)| \leq \frac{2-a}{50T}.$$

Hence, setting

$$g_K(T) = (r_1 + r_2) g_0(T) + r_2 g_1(T), \tag{2.6}$$

we then obtain

$$|g_K(T)| \leq \frac{2n_K}{50T} - \frac{r_2}{50T}. \tag{2.7}$$

Now, gathering (2.3), (2.4), (2.5), and (2.6), we obtain

$$N_K(T) = \frac{2}{\pi} \arctan(2T) + g_K(T) + \frac{T}{\pi} \log\left(d_K \left(\frac{T}{2\pi e}\right)^{n_K}\right) - \frac{r_1}{4} + \frac{2}{\pi} \Delta_{\mathcal{C}_0} \arg((s-1)\zeta_K(s)). \tag{2.8}$$

Let $\mathcal{C}_1$ denote the vertical line from σ_1 to $\sigma_1 + iT$ and $\mathcal{C}_2$ denote the horizontal line from $\sigma_1 + iT$ to $\frac{1}{2} + iT$. We require the following two estimates.

Lemma 2.1. *For $s = \sigma + it$ with $\sigma > 1$, one has*

$$\frac{\zeta_K(2\sigma)}{\zeta_K(\sigma)} \leq |\zeta_K(s)| \leq \zeta(\sigma)^{n_K},$$

where, as later, $\zeta(s)$ denotes the Riemann zeta function.

Lemma 2.2. *For* $\sigma_1 > 1$,

$$|\Delta_{\mathcal{C}_1} \arg(s-1)\zeta_K(s)| \le \frac{\pi}{2} + n_K \log\zeta(\sigma_1).$$

Proof. Note that

$$\begin{aligned}\Delta_{\mathcal{C}_1} \arg(s-1)\zeta_K(s) &= \Delta_{\mathcal{C}_1} \arg(s-1) + \Delta_{\mathcal{C}_1} \arg\zeta_K(s)\\ &= \arctan\left(\frac{T}{\sigma_1 - 1}\right) + \Delta_{\mathcal{C}_1} \arg\zeta_K(s).\end{aligned}$$

Now, the lemma follows from the estimate

$$|\Delta_{\mathcal{C}_1} \arg\zeta_K(s)| = |\arg\zeta_K(\sigma_1+iT)| \le |\log\zeta_K(\sigma_1+iT)| \le \log\zeta_K(\sigma_1) \le n_K \log\zeta(\sigma_1),$$

where the last inequality is due to Lemma 2.1. □

Thus, by Lemma 2.2 and (2.8), we arrive at

$$\begin{aligned}&\left|N_K(T) - \frac{T}{\pi}\log\left(d_K\left(\frac{T}{2\pi e}\right)^{n_K}\right) + \frac{r_1}{4}\right|\\ &\qquad \le 2 + |g_K(T)| + \frac{2n_K}{\pi}\log\zeta(\sigma_1) + \frac{2}{\pi}|\Delta_{\mathcal{C}_2} \arg((s-1)\zeta_K(s))|.\end{aligned} \tag{2.9}$$

2.2. **Bounding the Gamma factor.** For $a \in \{0,1\}$, $0 \le d < 9/2$ and $T \ge 5/7$, we set

$$\mathcal{E}_a(T,d) = \left|\mathfrak{Im}\log\Gamma\left(\frac{\sigma+a+iT}{2}\right)\Big|_{\sigma=\frac12}^{\frac12+d} + \mathfrak{Im}\log\Gamma\left(\frac{\sigma+a+iT}{2}\right)\Big|_{\sigma=\frac12}^{\frac12-d}\right|,$$

and we define

$$\mathcal{E}_K(T,d) = (r_1+r_2)\mathcal{E}_0(T,d) + r_2\mathcal{E}_1(T,d). \tag{2.10}$$

Following [2, p. 1463], we let

$$\begin{aligned}E_a(T,d) = {}& \frac{2T/3}{(2a+2d+17)^2+4T^2} + \frac{2T/3}{(2a-2d+17)^2+4T^2} - \frac{4T/3}{(2a+17)^2+4T^2}\\ &+ \frac{T}{2}\log\left(1+\frac{(2a+17)^2}{4T^2}\right) - \frac{T}{4}\log\left(1+\frac{(2a+2d+17)^2}{4T^2}\right)\\ &- \frac{T}{4}\log\left(1+\frac{(2a-2d+17)^2}{4T^2}\right) + \frac{(8+6\pi)/45}{((2a+2d+17)^2+4T^2)^{3/2}}\\ &+ \frac{(8+6\pi)/45}{((2a-2d+17)^2+4T^2)^{3/2}} + \frac{2(8+6\pi)/45}{((2a+17)^2+4T^2)^{3/2}}\\ &+ \sum_{k=0}^{3}\Big(2\arctan\frac{2a+1+4k}{2T}\\ &\qquad - \arctan\frac{2a+2d+1+4k}{2T} - \arctan\frac{2a-2d+1+4k}{2T}\Big)\\ &+ \frac{2a+2d+15}{4}\arctan\frac{2a+2d+17}{2T} + \frac{2a-2d+15}{4}\arctan\frac{2a-2d+17}{2T}\\ &- \frac{2a+15}{2}\arctan\frac{2a+17}{2T}.\end{aligned}$$

We shall further set

$$E_K(T,d) = (r_1+r_2)E_0(T,d) + r_2E_1(T,d). \tag{2.11}$$

As shown in [2, p. 1462], $\mathcal{E}_a(T,d) \le E_a(T,d)$ for $0 \le d < 9/2$ and $T \ge 5/7$, and thus

$$\mathcal{E}_K(T,d) \le E_K(T,d) \tag{2.12}$$

for $0 \le d < 9/2$ and $T \ge 5/7$. In addition, from [2, Lemma 3.4] and our definition of $E_K(T,d)$, we have Lemma 2.3.

Lemma 2.3. *For $0 \le \delta_1 \le d < 9/2$ and $T \ge 5/7$,*

$$0 < E_K(T,\delta_1) \le E_K(T,d).$$

Furthermore, for $d \in [\frac{1}{4}, \frac{5}{8}]$ and $T \ge 5/7$,

$$\frac{E_K(T,d)}{\pi} \le (r_1+r_2)\frac{640d-112}{1536(3T-1)} + r_2\frac{(640+216)d-112-39}{1536(3T+3-1)} + \frac{n_K}{2^{10}}.$$

3. Backlund's trick and the Jensen integral

3.1. **Introducing the auxiliary function f_N.** For the sake of convenience, we shall set $\mathcal{Z}(w) = (w-1)\zeta_K(w)$. In order to analyse the variation of the argument of $\mathcal{Z}(w)$ on $\mathcal{C}_2$, we shall introduce an auxiliary function

$$f_N(s) = \frac{1}{2}\Big(\mathcal{Z}(s+iT)^N + \mathcal{Z}(s-iT)^N\Big)$$

for $N \in \mathbb{N}$. For $\sigma \in \mathbb{R}$, it is clear that

$$\begin{aligned} f_N(\sigma) &= \frac{1}{2}\Big(\mathcal{Z}(\sigma+iT)^N + \mathcal{Z}(\sigma-iT)^N\Big) = \frac{1}{2}\Big(\mathcal{Z}(\sigma+iT)^N + \overline{\mathcal{Z}(\sigma+iT)^N}\Big) \\ &= \mathfrak{Re}(\mathcal{Z}(\sigma+iT)^N). \end{aligned}$$

We need Definition 3.1 that measures the variation of the argument of $\mathcal{Z}(w)^N$ on $\mathcal{C}_2$.

Definition 3.1. Let b_N denote the non-negative integer, depending on N, such that

$$b_N \le \frac{1}{\pi}\Big|\Delta_{\mathcal{C}_2} \arg \mathcal{Z}(w)^N\Big| < b_N + 1.$$

From this definition and the fact that $\arg \mathcal{Z}(w)^N = N \arg \mathcal{Z}(w)$, we immediately obtain

$$\frac{b_N}{N} \le \frac{1}{\pi}\Big|\Delta_{\mathcal{C}_2} \arg \mathcal{Z}(w)\Big| < \frac{b_N+1}{N}. \tag{3.1}$$

In addition, we have Lemma 3.2 concerning the zeros of $f_N(\sigma)$.

Lemma 3.2. *In the notation of Definition 3.1, the function $f_N(\sigma)$ has at least b_N zeros in $[\frac{1}{2}, \sigma_1]$.*

Proof. By Definition 3.1, there are at least b_N different values of σ such that $\frac{1}{2} + \frac{1}{\pi}\arg \mathcal{Z}(\sigma+iT)^N \in \mathbb{Z}$. Thus, for such values of σ, $\mathcal{Z}(\sigma+iT)^N$ is purely imaginary, which means that

$$f_N(\sigma) = \mathfrak{Re}(\mathcal{Z}(\sigma+iT)^N) = 0$$

for at least b_N different values σ. □

We shall also require Lemma 3.3 regarding the limiting behaviour of f_N.

Lemma 3.3. *For any $c > 1$, there is an infinite sequence of natural numbers $(N_m)_{m=1}^{\infty}$ such that $f_{N_m}(c) \neq 0$. Moreover, we have*

$$\limsup_{m\to\infty} \Big(-\frac{1}{N_m} \log |f_{N_m}(c)| \Big) \leq \log \Big(\frac{1}{\sqrt{(c-1)^2+T^2}} \frac{\zeta_K(c)}{\zeta_K(2c)} \Big).$$

Proof. Write $\mathcal{Z}(c+iT) = Re^{i\phi}$ for some $R, \phi \in \mathbb{R}$. It is clear that $\mathcal{Z}(c-iT) = Re^{-i\phi}$. Also, as $\mathcal{Z}(c+iT) \neq 0$ for any $c > 1$, we know that $R > 0$. Thus, we have

$$\frac{f_N(c)}{\mathcal{Z}(c+iT)^N} = \frac{1}{2}\Big(1 + \frac{\mathcal{Z}(c-iT)^N}{\mathcal{Z}(c+iT)^N}\Big) = \frac{1}{2}(1+e^{-2N\phi i})$$

for any $N \in \mathbb{N}$.

Now, applying Dirichlet's approximation theorem, for any ϕ, there is an infinite sequence of natural numbers $(N_m)_{m=1}^{\infty}$ such that as $m \to \infty$, $-2N_m\phi \to 0$ modulo 2π and $N_m \to \infty$. Thus, $\frac{f_{N_m}(c)}{\mathcal{Z}(c+iT)^{N_m}} \to 1$ as $m \to \infty$, and hence

$$\begin{aligned}&\lim_{m\to\infty} \Big(-\frac{1}{N_m}(\log|f_{N_m}(c)| - N_m \log|\mathcal{Z}(c+iT)|)\Big) \\ &\qquad = \Big(\lim_{m\to\infty} \frac{-1}{N_m}\Big)\Big(\lim_{m\to\infty} \log\Big|\frac{f_{N_m}(c)}{\mathcal{Z}(c+iT)^{N_m}}\Big|\Big) = 0.\end{aligned}$$

Moreover, by the left inequality of Lemma 2.1, we have

$$|\mathcal{Z}(c+iT)| \geq \sqrt{(c-1)^2+T^2}\frac{\zeta_K(2c)}{\zeta_K(c)},$$

which, combined with the above identity, gives

$$\begin{aligned}0 &\geq \limsup_{m\to\infty} \Big(-\frac{1}{N_m}\log|f_{N_m}(c)| + \log\Big(\sqrt{(c-1)^2+T^2}\frac{\zeta_K(2c)}{\zeta_K(c)}\Big)\Big) \\ &= \limsup_{m\to\infty} \Big(-\frac{1}{N_m}\log|f_{N_m}(c)|\Big) + \log\Big(\sqrt{(c-1)^2+T^2}\frac{\zeta_K(2c)}{\zeta_K(c)}\Big).\end{aligned}$$

Herein, we complete the proof. $\square$

Let $D(c,r)$ be the open disk centred at c with radius r. Let $(N_m)_{m=1}^{\infty}$ be given as in Lemma 3.3. For any $N \in (N_m)_{m=1}^{\infty}$, we set

$$S_N(c,r) = \frac{1}{N} \sum_{z \in \mathcal{S}_N(D(c,r))} \log \frac{r}{|z-c|},$$

where $\mathcal{S}_N(D(c,r))$ denotes the set of zeros of $f_N(s)$ in $D(c,r)$. As in [2, Theorem 5.1], we have the following version of Jensen's formula.

Theorem 3.4 (Jensen's formula)**.** *For $c \in \mathbb{C}$ and $r > 0$, if $f_N(c) \neq 0$, then*

$$S_N(c,r) = -\frac{1}{N}\log|f_N(c)| + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{N}\log|f_N(c+re^{i\theta})|d\theta.$$

Applying Jensen's formula and Lemma 3.3, we obtain the following upper bound for $S_N(c,r)$.

Proposition 3.5. *Let c, r, and σ_1 be real numbers such that*

$$c - r < \frac{1}{2} < 1 < c < \sigma_1 < c + r.$$

Let $F_{c,r} : [-\pi, \pi] \to \mathbb{R}$ *be an even function such that* $F_{c,r}(\theta) \geq \frac{1}{N_m} \log |f_{N_m}(c + re^{i\theta})|$. *Then we have*

$$\limsup_{m\to\infty} S_{N_m}(c,r) \leq \log\left(\frac{1}{\sqrt{(c-1)^2+T^2}}\frac{\zeta_K(c)}{\zeta_K(2c)}\right) + \frac{1}{\pi}\int_0^\pi F_{c,r}(\theta)d\theta.$$

3.2. **Backlund's trick.** We start with the following technical estimate.

Lemma 3.6. *Let* $0 \leq d < 1/2$ *and* $T \geq 5/7$. *Then we have*

$$\left|\arg\Big((\sigma-1+iT)\zeta_K(\sigma+iT)\Big)^N\Big|_{\sigma=\frac12}^{\frac12+d}\right| \leq \left|\arg\Big((\sigma-1+iT)\zeta_K(\sigma+iT)\Big)^N\Big|_{\sigma=\frac12}^{\frac12-d}\right| + N\mathcal{E}_K(T,d) + N\frac{\pi}{2},$$

where $\mathcal{E}_K(T,d)$ *is defined as in* (2.10).

Proof. By the functional equation (2.2) and the fact that $\xi_K(s) = \overline{\xi_K(\bar{s})}$, we have

$$\arg\xi_K(\sigma+iT)\Big|_{\sigma=\frac12}^{\frac12+d} = -\arg\xi_K(\sigma+iT)\Big|_{\sigma=\frac12}^{\frac12-d}. \tag{3.2}$$

Since

$$\arg(\sigma+iT) + \arg B^{(\sigma+iT)/2} = \arctan\frac{T}{\sigma} + \frac{T}{2}\log B,$$

by (2.1), we have

$$\begin{aligned}\arg\xi_K(\sigma+iT) &= \arctan\frac{T}{\sigma} + \frac{T}{2}\log B + (r_1+r_2)\mathfrak{Im}\log\Gamma\Big(\frac{\sigma+iT}{2}\Big)\\ &\quad + r_2\mathfrak{Im}\log\Gamma\Big(\frac{\sigma+iT+1}{2}\Big)\\ &\quad + \arg\Big((\sigma+iT-1)\zeta_K(\sigma+iT)\Big).\end{aligned} \tag{3.3}$$

As we know that for $\pm xy < 1$,

$$\arctan x \pm \arctan y = \arctan\frac{x\pm y}{1\mp xy},$$

for $0 \leq d < 1/2$, we have

$$\begin{aligned}&\left|\arctan\frac{T}{\frac12+d} - \arctan\frac{T}{\frac12} + \arctan\frac{T}{\frac12-d} - \arctan\frac{T}{\frac12}\right|\\ &= \left|\arctan\frac{\frac{T}{\frac12+d}-\frac{T}{\frac12}}{1+\frac{T}{\frac12+d}\frac{T}{\frac12}} + \arctan\frac{\frac{T}{\frac12-d}-\frac{T}{\frac12}}{1+\frac{T}{\frac12-d}\frac{T}{\frac12}}\right|\\ &\leq \frac{\pi}{2}.\end{aligned} \tag{3.4}$$

Now, applying the triangle inequality, by (3.2), (3.3), and (3.4), we obtain

$$\begin{aligned}&\left|\arg\Big((\sigma-1+iT)\zeta_K(\sigma+iT)\Big)\Big|_{\sigma=\frac12}^{\frac12+d}\right|\\ &\leq \left|\arg\Big((\sigma-1+iT)\zeta_K(\sigma+iT)\Big)\Big|_{\sigma=\frac12}^{\frac12-d}\right| + \mathcal{E}_K(T,d) + \frac{\pi}{2}.\end{aligned}$$

Recalling that

$$\arg\Big((\sigma-1+iT)\zeta_K(\sigma+iT)\Big)^N\Big|_{\sigma=\frac12}^{\frac12\pm d} = N\arg\Big((\sigma-1+iT)\zeta_K(\sigma+iT)\Big)\Big|_{\sigma=\frac12}^{\frac12\pm d},$$

we conclude the proof. $\square$

As argued in [2] and [9], we require the following version of "Backlund's trick".

Proposition 3.7 (Backlund's trick)**.** *Let c and r be real numbers. Set*

$$\sigma_1 = c + \frac{(c-1/2)^2}{r} \quad \text{and} \quad \delta = 2c - \sigma_1 - \frac{1}{2}.$$

If $1 < c < r$ and $0 < \delta < \frac{1}{2}$, then

$$\left| \arg\Big((\sigma + iT - 1)\zeta_K(\sigma+iT)\Big)\Big|_{\sigma=\sigma_1}^{1/2} \right| \le \frac{\pi S_N(c,r)}{2\log(r/(c-1/2))} + \frac{E_K(T,\delta)}{2} + \frac{\pi}{N} + \frac{\pi}{2N} + \frac{\pi}{4}.$$

Proof. By the conditions on c and r and the definitions of σ_1 and δ, we know that

$$c - r < \frac{1}{2} - \delta \le \frac{1}{2} \le \frac{1}{2} + \delta = 2c - \sigma_1 \le c \le \sigma_1 < c + r.$$

As $\log \frac{r}{|z-c|} > 0$ for $z \in D(c,r)$, we see that

$$S_N(c,r) = \frac{1}{N} \sum_{z \in \mathcal{S}_N(D(c,r))} \log \frac{r}{|z-c|} \ge \frac{1}{N} \sum_{z \in \mathcal{S}_N((c-r,\sigma_1])} \log \frac{r}{|z-c|}.$$

Recall that by Lemma 3.2, there are at least b_N values of σ satisfying $\sigma \in [1/2, \sigma_1]$ and $f_N(\sigma) = 0$, where b_N is defined as in Definition 3.1. For $1 \le k \le b_N$, we then set δ_k as the smallest non-negative real number such that

(3.5)
$$f_N(1/2 + \delta_k) = 0 \quad \text{and} \quad k - 1 \le \frac{1}{\pi}\left| \arg\Big((\sigma + iT - 1)\zeta_K(\sigma + iT)\Big)^N \Big|_{\sigma=1/2}^{1/2+\delta_k} \right|.$$

Writing $z_k = \frac{1}{2} + \delta_k$, we let x_1 denote the number of z_k with $z_k \in [1/2, 1/2 + \delta) = [1/2, 2c - \sigma_1)$ and let x_2 denote the number of z_k with $z_k \in [2c - \sigma_1, \sigma_1]$. We note that $x_2 = b_N - x_1$ and that

$$0 \le \delta_1 < \delta_2 < \cdots < \delta_{x_1} < \delta \le \delta_{x_1+1} < \cdots < \delta_{b_N} \le \sigma_1 - 1/2.$$

From (2.12), (3.5), and Lemma 3.6, it follows that

$$k - 1 \le \frac{1}{\pi}\left| \arg\Big((\sigma - 1 + iT)\zeta_K(\sigma + iT)\Big)^N \Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}-\delta_k} \right| + \frac{1}{\pi} N E_K(T, \delta_k) + \frac{N}{2}$$

whenever $1 \le k \le x_1$ (which implies that $\delta_k < \delta < \frac{1}{2}$).

For each $j \ge 1$, if there exists a k (chosen to be minimal) such that

$$k - 1 - \frac{1}{\pi} N E_K(T, \delta_k) - \frac{N}{2} \ge j,$$

then f_N has at least j zeros in $[1/2 - \delta_k, 1/2)$ since

$$\frac{1}{\pi}\left| \arg\Big((\sigma + iT - 1)\zeta_K(\sigma + iT)\Big)^N \Big|_{\sigma=1/2}^{1/2-\delta_k} \right| \ge k - 1 - \frac{1}{\pi} N E_K(T, \delta_k) - \frac{N}{2} \ge j.$$

For such an instance, we define δ_{-k} as the smallest values of these zeros (to avoid possible repetition), and we shall say that the zero $z_k = 1/2 + \delta_k$ has a pair $z_{-k} = 1/2 - \delta_{-k}$. We note that $\delta_{-k} \le \delta_k$ by the construction.

By the same argument as in [2, pp. 1467-1468], we have

$$S_N(c,r) \ge \frac{2b_N - \frac{N E_K(T,\delta) + \frac{N\pi}{2} + \pi}{\pi}}{N} \log\left(\frac{r}{c - 1/2}\right),$$

and thus

$$\frac{b_N}{N} \leq \frac{S_N(c,r)}{2\log(r/(c-1/2))} + \frac{E_K(T,\delta)}{2\pi} + \frac{1}{4} + \frac{1}{2N},$$

which combined with (3.1) completes the proof. □

3.3. **Constructing and bounding $F_{c,r}$.** We first recall the convexity bound for $\zeta_K(s)$ established by Rademacher [7, Theorem 4].

Proposition 3.8. *Let $\eta \in (0, \frac{1}{2}]$ and $s = \sigma + it$. If $-\eta \leq \sigma \leq 1+\eta$, then one has*

$$|\zeta_K(s)| \leq 3\Big|\frac{1+s}{1-s}\Big|\Big(d_K\Big(\frac{|1+s|}{2\pi}\Big)^{n_K}\Big)^{\frac{1+\eta-\sigma}{2}}\zeta(1+\eta)^{n_K}.$$

Also, for $\sigma \in [-\frac{1}{2}, 0)$, one has

$$|\zeta_K(s)| \leq 3\Big|\frac{1+s}{1-s}\Big|\Big(d_K\Big(\frac{|1+s|}{2\pi}\Big)^{n_K}\Big)^{\frac{1}{2}-\sigma}\zeta(1-\sigma)^{n_K}. \tag{3.6}$$

We note that the second inequality follows from the first bound by taking $\eta = -\sigma$. Moreover, Rademacher's argument [7] can be used to extend (3.6) for $\sigma < 0$ as follows (cf. [2, Theorem 5.7]). For $x \in \mathbb{R}$, let $[x]$ be the integer closest to x; when there are two integers equally close to x, we shall choose the one closer to 0.

Proposition 3.9. *Let $s = \sigma + it$ with $\sigma < 0$. Then we have*

$$|\zeta_K(s)| \leq \Big(\frac{d_K}{(2\pi)^{n_K}}\Big)^{\frac{1}{2}-\sigma}|1+s-[\sigma]|^{n_K(\frac{1}{2}+[\sigma]-\sigma)}\prod_{j=1}^{-[\sigma]}|s+j-1|^{n_K}\zeta(1-\sigma)^{n_K}.$$

Proof. From the functional equation (2.2) we have

$$\begin{aligned}|\zeta_K(s)| &\leq d_K^{1/2-\sigma}\Big|\frac{\gamma_K(1-s)}{\gamma_K(s)}\Big||\zeta_K(1-s)|\\ &= d_K^{1/2-\sigma}\pi^{(\sigma-\frac{1}{2})n_K}\Big|\frac{\Gamma(\frac{1}{2}+\frac{1-s}{2})}{\Gamma(\frac{1}{2}+\frac{s}{2})}\Big|^{r_2}\Big|\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\Big|^{r_1+r_2}|\zeta_K(1-s)|.\end{aligned}$$

As $\sigma < 0$, by Lemma 2.1, we have $|\zeta_K(1-s)| \leq \zeta(1-\sigma)^{n_K}$. It remains to estimate the ratios of gamma functions. It was obtained in the proof of [2, Theorem 5.7] that for $a, b \in \{0,1\}$ and $k \in \mathbb{Z}$,

$$\frac{\Gamma(\frac{a}{2}+\frac{1-s}{2})}{\Gamma(\frac{a}{2}+\frac{s}{2})} = \frac{\Gamma(\frac{b}{2}+\frac{1-(s+k)}{2})}{\Gamma(\frac{b}{2}+\frac{s+k}{2})}2^{-k}\Big(\prod_{j=1}^{k}(s+j-1)\Big)\frac{\sin(\frac{\pi}{2}(s+k+1-b))}{\sin(\frac{\pi}{2}(s+1-a))}.$$

Setting $a = 0$ and $a = 1$ and taking $b \equiv k \pmod 2$ and $b \equiv k+1 \pmod 2$, respectively, we can make sine factors ± 1. Thus, upon choosing $k = -[\sigma]$ and applying [7, Lemmata 1 and 2] to $\frac{\Gamma(\frac{b}{2}+\frac{1-(s+k)}{2})}{\Gamma(\frac{b}{2}+\frac{s+k}{2})}$, we conclude that

$$\Big|\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\Big|^{r_1+r_2} \leq \Big(\frac{1}{2}|1+s-[\sigma]|\Big)^{(\frac{1}{2}+[\sigma]-\sigma)(r_1+r_2)}2^{[\sigma](r_1+r_2)}\Big(\prod_{j=1}^{-[\sigma]}|s+j-1|\Big)^{r_1+r_2}$$

and

$$\Big|\frac{\Gamma(\frac{1}{2}+\frac{1-s}{2})}{\Gamma(\frac{1}{2}+\frac{s}{2})}\Big|^{r_2} \leq \Big(\frac{1}{2}|1+s-[\sigma]|\Big)^{(\frac{1}{2}+[\sigma]-\sigma)r_2}2^{[\sigma]r_2}\Big(\prod_{j=1}^{-[\sigma]}|s+j-1|\Big)^{r_2}.$$

Collecting above estimates and recalling the fact that $n_K = r_1 + 2r_2$, we obtain the desired result. $\square$

Lemma 3.10. *Let $\eta \in (0, \frac{1}{2}]$, $s = \sigma + it$, and $T > 0$. If $\sigma \geq 1 + \eta$, then we have*

$$\frac{1}{N} \log |f_N(s)| \leq \frac{1}{2} \log((\sigma - 1)^2 + (|t| + T)^2) + n_K \log \zeta(\sigma).$$

If $-\eta \leq \sigma \leq 1 + \eta$, then we have

$$\begin{aligned}\frac{1}{N} \log |f_N(s)| \leq \log 3 + \frac{n_K(1 + \eta - \sigma) + 2}{4} \log((\sigma + 1)^2 + (|t| + T)^2) \\ + \frac{1 + \eta - \sigma}{2} \log\left(\frac{d_K}{(2\pi)^{n_K}}\right) + n_K \log \zeta(1 + \eta).\end{aligned}$$

If $\sigma \leq -\eta$, then we have

$$\begin{aligned}\frac{1}{N} \log |f_N(s)| \leq n_K \log \zeta(1 - \sigma) + \frac{1}{2} \log((\sigma - 1)^2 + (|t| + T)^2) \\ + \frac{1 - 2\sigma}{2} \log\left(\frac{d_K}{(2\pi)^{n_K}}\right) \\ + \frac{(1 - 2\sigma + 2[\sigma])n_K}{4} \log((1 + \sigma - [\sigma])^2 + (|t| + T)^2) \\ + \frac{n_K}{2} \sum_{j=1}^{-[\sigma]} \log((\sigma + j - 1)^2 + (|t| + T)^2).\end{aligned}$$

Proof. Since $\sigma \geq 1 + \eta > 1$, by Lemma 2.1, we derive

$$\begin{aligned}|f_N(s)| &\leq \frac{1}{2}\left(|s + iT - 1|^N |\zeta_K(s + iT)|^N + |s - iT - 1|^N |\zeta_K(s - iT)|^N\right) \\ &\leq \left((\sigma - 1)^2 + (|t| + T)^2\right)^{\frac{N}{2}} \zeta(\sigma)^{n_K N}.\end{aligned}$$

Now, the first estimate follows from taking logarithms and dividing both sides by N.

Secondly, if $-\eta \leq \sigma \leq 1 + \eta$, then by Proposition 3.8, we see that $|f_N(s)|$ is at most

$$\begin{aligned}&\frac{1}{2}\left(3^N |s + iT + 1|^N + 3^N |s - iT + 1|^N\right) \\ &\times \left(d_K \left(\frac{\sqrt{(\sigma + 1)^2 + (|t| + T)^2}}{2\pi}\right)^{n_K}\right)^{\frac{(1+\eta-\sigma)N}{2}} \zeta(1 + \eta)^{n_K N} \\ &\leq 3^N \left((\sigma + 1)^2 + (|t| + T)^2\right)^{\frac{N}{2}} \\ &\times \left(d_K \left(\frac{\sqrt{(\sigma + 1)^2 + (|t| + T)^2}}{2\pi}\right)^{n_K}\right)^{\frac{(1+\eta-\sigma)N}{2}} \zeta(1 + \eta)^{n_K N}.\end{aligned}$$

Again, taking logarithms yields the second bound.

Lastly, for $\sigma \le -\eta$, it follows from Proposition 3.9 that

$$
\begin{aligned}
|f_N(s)| \le &\Big((\sigma-1)^2+(|t|+T)^2\Big)^{\frac{N}{2}}\\
&\times\Big(\frac{d_K}{(2\pi)^{n_K}}\Big)^{N(\frac12-\sigma)}|(1+\sigma-[\sigma])^2+(|t|+T)^2|^{\frac{(1-2\sigma+2[\sigma])Nn_K}{4}}\\
&\times\Big(\prod_{j=1}^{-[\sigma]}((\sigma+j-1)^2+(|t|+T)^2)\Big)^{\frac{n_K N}{2}}\zeta(1-\sigma)^{n_K N}.
\end{aligned}
$$

We then conclude the proof by taking logarithms. □

Following [2], to proceed further, we introduce some notation and auxiliary functions. We first set

$$L_j(\theta)=\log\frac{(j+c+r\cos\theta)^2+(|r\sin\theta|+T)^2}{(T+2)^2},$$

and note that $L_j(\theta)$ is an even function of θ. Moreover, if $\theta\in[0,\pi]$ and $T\ge 5/7$, by the inequality $\log x\le x-1$, one has $L_j(\theta)\le\frac{L_j^\star(\theta)}{T+2}$, where

$$L_j^\star(\theta)=2r\sin\theta-4+\frac{7}{19}((j+c+r\cos\theta)^2+(r\sin\theta-2)^2).$$

In light of the choice of $F_{c,r}(\theta)$ (for Dirichlet L-functions) in [2, Definition 5.10], we shall use the following $F_{c,r}(\theta)$ for $\zeta_K(s)$.

Definition 3.11. For $\theta\in[-\pi,\pi]$, we let $\sigma=c+r\cos\theta$, with $c-r>-\frac12$, and $t=r\sin\theta$. For $\sigma\ge 1+\eta$, we define

$$F_{c,r}(\theta)=n_K\log\zeta(\sigma)+\frac12 L_{-1}(\theta)+\log(T+2).$$

For $-\eta\le\sigma\le 1+\eta$, we define

$$
\begin{aligned}
F_{c,r}(\theta)=&n_K\log\zeta(1+\eta)+\frac{n_K(1+\eta-\sigma)+2}{4}L_1(\theta)+\frac{n_K(1+\eta-\sigma)+2}{2}\log(T+2)\\
&+\frac{1+\eta-\sigma}{2}\Big(\log\frac{d_K}{(2\pi)^{n_K}}\Big)+\log 3.
\end{aligned}
$$

For $\sigma<-\eta$, we define

$$
\begin{aligned}
F_{c,r}(\theta)=&n_K\log\zeta(1-\sigma)+\frac12 L_{-1}(\theta)+\log(T+2)+\frac{1-2\sigma}{2}\log\Big(\frac{d_K(T+2)^{n_K}}{(2\pi)^{n_K}}\Big)\\
&+\frac{(1-2\sigma+2[\sigma])n_K}{4}L_{1-[\sigma]}(\theta)+\frac{n_K}{2}\sum_{j=1}^{-[\sigma]}L_{j-1}(\theta).
\end{aligned}
$$

We note that $F_{c,r}(\theta)$ is an even function of θ satisfying $F_{c,r}(\theta)\ge\frac1N\log|f_N(c+re^{i\theta})|$. In order to bound $F_{c,r}(\theta)$, following [2], for $c\in\mathbb{R}$ and $r>0$, we define

$$
\theta_y=\begin{cases}0 & \text{if } c+r\le y;\\ \arccos\frac{y-c}{r} & \text{if } c-r\le y\le c+r;\\ \pi & \text{if } y\le c-r.\end{cases}
$$

For the sake of convenience, we define

$$\kappa_1 = \int_{\theta_{1+\eta}}^{\theta_{-\eta}} \frac{1+\eta-\sigma}{2} d\theta + \int_{\theta_{-\eta}}^{\pi} \frac{1-2\sigma}{2} d\theta.$$

For $J_1, J_2 \in \mathbb{N}$, we shall set

$$\kappa_2(J_1) = \frac{\pi}{4J_1}\Big(\log\zeta(c+r) + 2\sum_{j=1}^{J_1-1}\log\zeta\Big(c+r\cos\frac{\pi j}{2J_1}\Big)\Big),$$

and

$$\kappa_3(J_2) = \frac{\pi-\theta_{1-c}}{2J_2}\Big(\log\zeta(1-c+r) + 2\sum_{j=1}^{J_2-1}\log\zeta\Big(1-c-r\cos\Big(\frac{\pi j}{J_2}+\Big(1-\frac{j}{J_2}\Big)\theta_{1-c}\Big)\Big)\Big).$$

In addition, we define

$$\kappa_4 = \frac{1}{4}\int_{\theta_{1+\eta}}^{\theta_{-\eta}} (1+\eta-\sigma)L_1^{\star}(\theta)d\theta,$$

$$\kappa_5 = \frac{1}{4}\int_{\theta_{-\eta}}^{\theta_{-1/2}} (1-2\sigma)L_1^{\star}(\theta)d\theta.$$

Similar to [2, Proposition 5.13], we have Proposition 3.12 regarding the upper bound of $\int_0^{\pi} F_{c,r}(\theta)d\theta$.

Proposition 3.12. *Let c, r, and η be positive real numbers satisfying*

$$-\frac{1}{2} < c-r < -\eta < 1+\eta < c \tag{3.7}$$

and $0<\eta\le\frac{1}{2}$. Then for $T\ge\frac{5}{7}$, we have

$$\begin{aligned}\int_0^{\pi} F_{c,r}(\theta)d\theta \le{}& n_K\int_0^{\theta_{1+\eta}}\log\zeta(\sigma)d\theta + \frac{1}{2(T+2)}\int_0^{\theta_{1+\eta}} L_{-1}^{\star}(\theta)d\theta + \theta_{1+\eta}\log(T+2)\\
&+ n_K(\log\zeta(1+\eta))(\theta_{-\eta}-\theta_{1+\eta}) + \Big(\log\frac{d_K(T+2)^{n_K}}{(2\pi)^{n_K}}\Big)\kappa_1\\
&+ \frac{n_K}{T+2}\kappa_4 + \frac{1}{2(T+2)}\int_{\theta_{1+\eta}}^{\theta_{-\eta}} L_1^{\star}(\theta)d\theta + (\theta_{-\eta}-\theta_{1+\eta})\log(3(T+2))\\
&+ n_K\int_{\theta_{-\eta}}^{\pi}\log\zeta(1-\sigma)d\theta + \frac{1}{2(T+2)}\int_{\theta_{-\eta}}^{\pi} L_{-1}^{\star}(\theta)d\theta\\
&+ (\pi-\theta_{-\eta})\log(T+2) + \frac{n_K}{T+2}\kappa_5.\end{aligned}$$

Proof. We first write

$$\int_0^{\pi} F_{c,r}(\theta)d\theta = \int_0^{\theta_{1+\eta}} F_{c,r}(\theta)d\theta + \int_{\theta_{1+\eta}}^{\theta_{-\eta}} F_{c,r}(\theta)d\theta + \int_{\theta_{-\eta}}^{\pi} F_{c,r}(\theta)d\theta.$$

By the definition of $F_{c,r}(\theta)$, we have

$$\begin{aligned}&\int_0^{\theta_{1+\eta}} F_{c,r}(\theta)d\theta\\
&\quad= n_K\int_0^{\theta_{1+\eta}}\log\zeta(\sigma)d\theta + \frac{1}{2}\int_0^{\theta_{1+\eta}} L_{-1}(\theta)d\theta + \int_0^{\theta_{1+\eta}}\log(T+2)d\theta\\
&\quad\le n_K\int_0^{\theta_{1+\eta}}\log\zeta(\sigma)d\theta + \frac{1}{2(T+2)}\int_0^{\theta_{1+\eta}} L_{-1}^{\star}(\theta)d\theta + \theta_{1+\eta}\log(T+2).\end{aligned}$$

Secondly, we compute

(3.8)

$$\begin{aligned}
&\int_{\theta_{1+\eta}}^{\theta_{-\eta}} F_{c,r}(\theta)d\theta \\
&\qquad = n_K \int_{\theta_{1+\eta}}^{\theta_{-\eta}} \log\zeta(1+\eta)d\theta + \Big(\log\frac{d_K}{(2\pi)^{n_K}}\Big)\int_{\theta_{1+\eta}}^{\theta_{-\eta}} \frac{1+\eta-\sigma}{2}d\theta + \log 3\int_{\theta_{1+\eta}}^{\theta_{-\eta}} 1d\theta \\
&\qquad + \int_{\theta_{1+\eta}}^{\theta_{-\eta}} \frac{n_K(1+\eta-\sigma)+2}{4}L_1(\theta)d\theta + \log(T+2)\int_{\theta_{1+\eta}}^{\theta_{-\eta}} \frac{n_K(1+\eta-\sigma)+2}{2}d\theta.
\end{aligned}$$

The first three integrals on the right of (3.8) are

$$\begin{aligned}
&n_K(\log\zeta(1+\eta))(\theta_{-\eta}-\theta_{1+\eta}) \\
&\qquad + \Big(\log\frac{d_K}{(2\pi)^{n_K}}\Big)\int_{\theta_{1+\eta}}^{\theta_{-\eta}} \frac{1+\eta-\sigma}{2}d\theta + (\log 3)(\theta_{-\eta}-\theta_{1+\eta}).
\end{aligned}$$

As $1+\eta-\sigma \geq 0$ for $\theta \in [\theta_{1+\eta}, \theta_{-\eta}]$, it follows that the last two integrals on the right of (3.8) are

$$\begin{aligned}
&\frac{n_K}{4}\int_{\theta_{1+\eta}}^{\theta_{-\eta}} (1+\eta-\sigma)L_1(\theta)d\theta + \frac{1}{2}\int_{\theta_{1+\eta}}^{\theta_{-\eta}} L_1(\theta)d\theta + n_K\log(T+2)\int_{\theta_{1+\eta}}^{\theta_{-\eta}} \frac{1+\eta-\sigma}{2}d\theta \\
&+(\theta_{-\eta}-\theta_{1+\eta})\log(T+2) \\
&\leq \frac{n_K}{(T+2)}\kappa_4 + \frac{1}{2(T+2)}\int_{\theta_{1+\eta}}^{\theta_{-\eta}} L_1^{\star}(\theta)d\theta + n_K\log(T+2)\int_{\theta_{1+\eta}}^{\theta_{-\eta}} \frac{1+\eta-\sigma}{2}d\theta \\
&+(\theta_{-\eta}-\theta_{1+\eta})\log(T+2).
\end{aligned}$$

Lastly, we have

(3.9)

$$\begin{aligned}
\int_{\theta_{-\eta}}^{\pi} F_{c,r}(\theta)d\theta &= n_K\int_{\theta_{-\eta}}^{\pi}\log\zeta(1-\sigma)d\theta + \frac{1}{2}\int_{\theta_{-\eta}}^{\pi} L_{-1}(\theta)d\theta + \int_{\theta_{-\eta}}^{\pi}\log(T+2)d\theta \\
&+ \Big(\log\frac{d_K(T+2)^{n_K}}{(2\pi)^{n_K}}\Big)\int_{\theta_{-\eta}}^{\pi}\frac{1-2\sigma}{2}d\theta + n_K\int_{\theta_{-\eta}}^{\theta_{-\frac{1}{2}}}\frac{1-2\sigma}{4}L_1(\theta)d\theta \\
&+ n_K\sum_{j=1}^{\infty}\int_{\theta_{-j+\frac{1}{2}}}^{\theta_{-j-\frac{1}{2}}}\Big(\frac{1-2\sigma-2j}{4}L_{j+1}(\theta) + \frac{1}{2}\sum_{k=1}^{j}L_{k-1}(\theta)\Big)d\theta.
\end{aligned}$$

The first four integrals on the right of (3.9) are

$$\begin{aligned}
&\leq n_K\int_{\theta_{-\eta}}^{\pi}\log\zeta(1-\sigma)d\theta + \frac{1}{2(T+2)}\int_{\theta_{-\eta}}^{\pi} L_{-1}^{\star}(\theta)d\theta + \log(T+2)(\pi-\theta_{-\eta}) \\
&+ \Big(\log\frac{d_K(T+2)^{n_K}}{(2\pi)^{n_K}}\Big)\int_{\theta_{-\eta}}^{\pi}\frac{1-2\sigma}{2}d\theta.
\end{aligned}$$

Note that as $-\frac{1}{2} < c-r$, we have $\theta_{-j+\frac{1}{2}} = \theta_{-j-\frac{1}{2}} = \pi$ for $j \geq 1$. Thus, the remaining integral and sum on the the right of (3.9) is

$$n_K\int_{\theta_{-\eta}}^{\theta_{-\frac{1}{2}}}\frac{1-2\sigma}{4}L_1(\theta)d\theta \leq \frac{n_K}{T+2}\int_{\theta_{-\eta}}^{\theta_{-\frac{1}{2}}}\frac{1-2\sigma}{4}L_1^{\star}(\theta)d\theta = \frac{n_K}{T+2}\kappa_5.$$

Putting all the estimates together, we complete the proof. □

To control "zeta integrals" in the above proposition, we shall borrow two estimates from [2, Lemmata 5.14 and 5.15] as follows.

Lemma 3.13. *Let c, r and η be positive real numbers, satisfying* (3.7)*, and J_1 and J_2 be positive integers. If $\theta_{1+\eta} \leq 2.1$, then for $\sigma = c + r\cos\theta$, one has*

$$\int_0^{\theta_{1+\eta}} \log\zeta(\sigma)d\theta \leq \frac{\log\zeta(1+\eta)+\log\zeta(c)}{2}\left(\theta_{1+\eta}-\frac{\pi}{2}\right)+\frac{\pi}{4J_1}\log\zeta(c)+\kappa_2(J_1).$$

In addition, assuming further $r > 2c-1$, one has

$$\begin{aligned}\int_{\theta_{-\eta}}^{\pi} \log\zeta(1-\sigma)d\theta \leq{}& \frac{\log\zeta(1+\eta)+\log\zeta(c)}{2}(\theta_{1-c}-\theta_{-\eta})\\ &+\frac{\pi-\theta_{1-c}}{2J_2}\log\zeta(c)+\kappa_3(J_2).\end{aligned}$$

4. Completing the proof

Gathering (2.9) and Propositions 3.5 and 3.7, for

$$\begin{aligned}-\frac{1}{2} < c-r < 1-c < -\eta < 0 < \frac{1}{4} \leq \delta = 2c-\sigma_1-\frac{1}{2} < \frac{1}{2} < 1 < 1+\eta < c < \sigma_1\\ = c+\frac{(c-1/2)^2}{r} < c+r,\end{aligned}$$

satisfying $\theta_{1+\eta} \leq 2.1$, we have

$$\begin{aligned}&\left|N_K(T)-\frac{T}{\pi}\log\left(d_K\left(\frac{T}{2\pi e}\right)^{n_K}\right)+\frac{r_1}{4}\right|\\ &\leq \frac{5}{2}+|g_K(T)|+\frac{2n_K}{\pi}\log\zeta(\sigma_1)+\frac{\log\left(\frac{1}{\sqrt{(c-1)^2+T^2}}\frac{\zeta_K(c)}{\zeta_K(2c)}\right)}{\log\frac{r}{c-\frac{1}{2}}}\\ &\quad+\frac{1}{\pi\log\frac{r}{c-\frac{1}{2}}}\int_0^{\pi}F_{c,r}(\theta)d\theta+\frac{E_K(T,\delta)}{\pi},\end{aligned}\tag{4.1}$$

where $g_K(T)$ and $E_K(T,\delta)$ are defined as in (2.6) and (2.11), respectively, and

$$\log\frac{\zeta_K(c)}{\zeta_K(2c)} = \int_c^{2c}-\frac{\zeta_K'}{\zeta_K}(\sigma)d\sigma \leq n_K\int_c^{2c}-\frac{\zeta'}{\zeta}(\sigma)d\sigma \leq n_K\log\frac{\zeta(c)}{\zeta(2c)}.$$

Finally, using (2.7), Lemma 2.3, Proposition 3.12, and Lemma 3.13 to bound (4.1) and recalling that $r_1+2r_2=n_K$, for any $T_0 \geq \frac{5}{7}$, we obtain

$$\left|N_K(T)-\frac{T}{\pi}\log\left(d_K\left(\frac{T}{2\pi e}\right)^{n_K}\right)+\frac{r_1}{4}\right| \leq C_1\log\left(\frac{d_K(T+2)^{n_K}}{(2\pi)^{n_K}}\right)+C_2n_K+C_3\tag{4.2}$$

whenever $T \geq T_0$, where

$$C_1 = \kappa_1 \Big(\pi \log \frac{r}{c-\frac{1}{2}}\Big)^{-1},$$

$$\begin{aligned}
C_2 &= \frac{1}{25T_0} + \frac{2}{\pi}\log\zeta(\sigma_1) + \frac{640\delta - 112}{1536(3T_0-1)} \\
&+ \max\Big\{0, \frac{856\delta-151}{1536(3T_0+2)} - \frac{640\delta-112}{1536(3T_0-1)}\Big\} + \frac{1}{2^{10}} \\
&+ \Big(\pi\log\frac{r}{c-\frac{1}{2}}\Big)^{-1}\Big(\frac{\log\zeta(1+\eta)+\log\zeta(c)}{2}\Big(\theta_{1+\eta}-\frac{\pi}{2}\Big) + \frac{\pi}{4J_1}\log\zeta(c) + \kappa_2(J_1)\Big) \\
&+ \Big(\pi\log\frac{r}{c-\frac{1}{2}}\Big)^{-1}\Big(\frac{\log\zeta(1+\eta)+\log\zeta(c)}{2}(\theta_{1-c}-\theta_{-\eta}) + \frac{\pi-\theta_{1-c}}{2J_2}\log\zeta(c) + \kappa_3(J_2)\Big) \\
&+ \Big(\pi\log\frac{r}{c-\frac{1}{2}}\Big)^{-1}\Big((\log\zeta(1+\eta))(\theta_{-\eta}-\theta_{1+\eta}) + \max\Big\{0, \frac{\kappa_4+\kappa_5}{T_0+2}\Big\} + \pi\log\frac{\zeta(c)}{\zeta(2c)}\Big),
\end{aligned}$$

$$\begin{aligned}
C_3 = \frac{5}{2} &+ \Big(\pi\log\frac{r}{c-\frac{1}{2}}\Big)^{-1}\Big(\pi\log\Big(1+\frac{2}{T_0}\Big) + (\theta_{-\eta}-\theta_{1+\eta})\log 3\Big) \\
&+ \max\Big\{0, \Big(\pi\log\frac{r}{c-\frac{1}{2}}\Big)^{-1}\Big(\frac{1}{2(T_0+2)}\Big(\int_0^{\theta_{1+\eta}} L_{-1}^{\star}(\theta)d\theta \\
&+ \int_{\theta_{1+\eta}}^{\theta_{-\eta}} L_1^{\star}(\theta)d\theta + \int_{\theta_{-\eta}}^{\pi} L_{-1}^{\star}(\theta)d\theta\Big)\Big)\Big\}.
\end{aligned}$$

For $T_0 = 1$ and $T_0 = 10$, choosing $J_1 = 64$ and $J_2 = 39$, via a Maple numerical computation, we have Table 2 of admissible (C_1, C_2, C_3).

TABLE 2. Choices of parameters (c, r, η) and resulting admissible (C_1, C_2, C_3)

				$T \geq 1$		$T \geq 10$	
c	r	η	C_1	C_2	C_3	C_2	C_3
1.000011314	1.064340602	$4.2826451 \cdot 10^{-6}$	0.22737	23.02528	4.51954	22.97204	3.30668
1.042877508	1.259860485	0.01737451737	0.24493	6.66558	4.21201	6.60397	3.12362
1.079779637	1.410370323	0.03441682600	0.26304	5.22032	4.08149	5.15251	3.05074
1.114294066	1.538391756	0.05247813411	0.28032	4.43521	4.00936	4.36214	3.01124
1.145720440	1.645584376	0.07107039918	0.29590	3.93889	3.96852	3.86136	2.98903

One may find functioning Maple code at `https://arxiv.org/abs/2102.04663`

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Department of Mathematics and Computer Science, University of Lethbridge, 4401 University Drive, Lethbridge, Alberta T1K 3M4, Canada
Email address: e.hasanalizade@uleth.ca

Department of Mathematics and Computer Science, University of Lethbridge, 4401 University Drive, Lethbridge, Alberta T1K 3M4, Canada
Email address: quanli.shen@uleth.ca

Department of Mathematics and Computer Science, University of Lethbridge, 4401 University Drive, Lethbridge, Alberta T1K 3M4, Canada
Email address: pengjie.wong@uleth.ca