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Counting zeros of the Riemann zeta function

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ABSTRACT

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In this article, we show that

$$\left| N(T) - \frac{T}{2\pi} \log \left(\frac{T}{2\pi e} \right) \right| \leq 0.1038 \log T + 0.2573 \log \log T \\ + 9.3675$$

where $N(T)$ denotes the number of non-trivial zeros ρ , with $0 < \Im(\rho) \leq T$, of the Riemann zeta function. This improves the previous result of Trudgian for sufficiently large T . The improvement comes from the use of various subconvexity bounds and ideas from the work of Bennett et al. on counting zeros of Dirichlet L -functions.

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Table 1Explicit bounds for $N(T)$ in (1.1).

	C_1	C_2	C_3	T_0
Von Mangoldt [19] (1905)	0.4320	1.9167	13.0788	28.5580
Grossmann [6] (1913)	0.2907	1.7862	7.0120	50
Backlund [1] (1918)	0.1370	0.4430	5.2250	200
Rosser [14] (1941)	0.1370	0.4430	2.4630	2
Trudgian [17] (2014)	0.1120	0.2780	3.3850	e
Corollary 1.2	0.1038	0.2573	9.3675	e

1. Introduction

Let $\zeta(s)$ be the Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

for $\Re(s) > 1$, which has an analytic continuation to a meromorphic function on \mathbb{C} with only a simple pole at $s = 1$. The study of zeros of $\zeta(s)$ is an important topic in number theory. In this article, we shall estimate the number of non-trivial zeros $\rho = \beta + i\gamma$, with $0 < \gamma \leq T$, of $\zeta(s)$. For $T \geq 0$, we set

$$N(T) = \#\{\rho \in \mathbb{C} \mid \zeta(\rho) = 0, 0 < \beta < 1, 0 < \gamma \leq T\}.$$

Before stating our results, we shall note that the study of $N(T)$ has a long history. Indeed, for $T \geq T_0$, writing

$$\left| N(T) - \frac{T}{2\pi} \log \left(\frac{T}{2\pi e} \right) \right| \leq C_1 \log T + C_2 \log \log T + C_3, \quad (1.1)$$

we have Table 1 summarising the progress that has been made.

The importance of explicit bounds for $N(T)$ comes from the fact that they are crucial for estimating sums over zeros of $\zeta(s)$, and all the best known bounds for $\pi(x)$ and $\psi(x)$ rely on them (see, e.g., [5]).

In this article, we prove the following general result for $N(T)$ with explicit dependence on the given bounds for $\zeta(s)$ on the both $\frac{1}{2}$ -line and 1-line.

Theorem 1.1. *Let c, r, η be positive real numbers satisfying*

$$-\frac{1}{2} < c-r < 1-c < -\eta < \frac{1}{4} \leq \delta := 2c - \sigma_1 - \frac{1}{2} < \frac{1}{2} < 1+\eta < \sigma_1 := c + \frac{(c-1/2)^2}{r} < c+r$$

and $\theta_{1+\eta} \leq 2.1$, where θ_y is defined in (4.10). Let $c_1, c_2, k_1, k_3 \geq 0$, $k_2 \in [0, \frac{1}{2}]$ and $t_0, t_1 \geq e$ such that for $t \geq t_0$,

$$|\zeta(1+it)| \leq c_1 (\log t)^{c_2}, \quad (1.2)$$

and for $t \geq t_1$,

$$|\zeta(\frac{1}{2} + it)| \leq k_1 t^{k_2} (\log t)^{k_3}. \quad (1.3)$$

Let $T_0 \geq e$ be fixed. Then for any $T \geq T_0$, we have

$$\left| N(T) - \frac{T}{2\pi} \log \left(\frac{T}{2\pi e} \right) + \frac{1}{8} \right| \leq C_1 \log T + C_2 \log \log T + C_3, \quad (1.4)$$

where $C_1 = C_1(c, r, \eta; k_2)$, $C_2 = C_2(c, r, \eta; c_2, k_3)$, $C_3 = C_3(c, r, \eta; c_1, c_2, t_0, k_1, k_2, k_3, t_1; T_0)$ are defined in (5.1), (5.2), (5.3), and (5.4), and some admissible values of C_1 , C_2 , and C_3 are recorded in Table 2 in Section 5.

As a consequence, we obtain an explicit estimate for $N(T)$ as follows.

Corollary 1.2. *For any $T \geq e$, we have*

$$\left| N(T) - \frac{T}{2\pi} \log \left(\frac{T}{2\pi e} \right) \right| \leq 0.1038 \log T + 0.2573 \log \log T + 9.3675. \quad (1.5)$$

Note that $C_1 = 0.1038$ is the smallest value that can be obtained by our argument and computation, and one can make C_2 and C_3 smaller at expense of larger C_1 .

Let $S(T) = \frac{1}{\pi} \Delta_L \arg \zeta(s)$, where L denotes the straight line from 2 to $2 + iT$ and then to $\frac{1}{2} + iT$. We also have the following theorem concerning the argument of $\zeta(s)$ along the critical line.

Theorem 1.3. *In the notation and assumptions of Theorem 1.1, for any $T \geq T_0$, we have*

$$|S(T)| \leq C_1 \log T + C_2 \log \log T + C'_3, \quad (1.6)$$

where $C_1 = C_1(c, r, \eta; k_2)$, $C_2 = C_2(c, r, \eta; c_2, k_3)$, $C'_3 = C'_3(c, r, \eta; c_1, c_2, t_0, k_1, k_2, k_3, t_1; T_0)$ are defined in (5.1), (5.2), (5.3), and (5.7), and some admissible values of C_1 , C_2 , and C'_3 are recorded in Table 2 in Section 5.

A stronger bound for $S(T)$, up to certain given height, can be confirmed using the database of non-trivial zeros of $\zeta(s)$ computed by Platt and made available at [9]. Indeed, nowadays, one has

$$|S(T)| \leq 2.5167 \quad (1.7)$$

for $0 \leq T \leq 30\,610\,046\,000$.¹ Hence, by Theorem 1.3 (with $T_0 = 30\,610\,046\,000$ and Table 2) and (1.7), we derive the following explicit bound for $S(T)$.

¹ Recently, Platt and Trudgian [12] verified the Riemann hypothesis for the height up to $3 \cdot 10^{12}$, which would allow one to further bound $S(T)$ for $0 \leq T \leq 3 \cdot 10^{12}$.

Corollary 1.4. *For any $T \geq e$, we have*

$$\begin{aligned} |S(T)| &\leq \min\{0.1038 \log T + 0.2573 \log \log T + 8.3675, \\ & \quad 0.1095 \log T + 0.2042 \log \log T + 3.0305\}. \end{aligned}$$

We note that this improves the previous best-known explicit bound (for T sufficiently large) due to Platt and Trudgian [11] (see also [15,17]), who showed that for $T \geq e$,

$$|S(T)| \leq 0.110 \log T + 0.290 \log \log T + 2.290.$$

The proofs of Theorems 1.2 and 1.3 are based on the work of [2,7,17,18].² Compared to the considerations of Dirichlet L -functions in [2,18] and Dedekind zeta functions in [7,18], we further use subconvexity bounds for $\zeta(\frac{1}{2} + it)$, together with the Phragmén-Lindelöf principle and the functional equation, to obtain a sharper estimate for $\zeta(s)$ in the strip $0 \leq \Re(s) \leq 1$. Also, based on the idea of [2], we refine the bound for $\zeta(s)$ on $\Re(s) < 0$ used in [17] by bounding $\zeta(s)$ over $\Re(s) < -\frac{1}{2}$. Lastly, we note that most numerical computations were performed in Maple.

2. Main term and bounds for gamma factors

We recall the completed Riemann zeta function $\xi(s)$ is defined by

$$\xi(s) = s(s-1)\gamma(s)\zeta(s), \tag{2.1}$$

where

$$\gamma(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right).$$

It is well-known that $\xi(s)$ can be extended to an entire function of order 1, which satisfies the functional equation

$$\xi(s) = \xi(1-s). \tag{2.2}$$

To follow the argument used in [7], it would be simpler to work with the following “symmetric version” of $N(T)$. We introduce

$$N_{\mathbb{Q}}(T) = \#\{\rho \in \mathbb{C} \mid \zeta(\rho) = 0, 0 < \beta < 1, |\gamma| \leq T\}$$

for $T \geq 0$.³ Note that $N_{\mathbb{Q}}(T) = 2N(T)$.

² Regrettably, as pointed out in [2], there is an error appearing in [17,18] (and [11], where the erroneous result of [17] was used) since the ranges of parameters involved in the final formulae were not verified properly; [2,7] fix this issue for [18]. In a certain degree, the objective of the presented paper is to fix the error occurring in [17].

³ This notation agrees with [7], where we defined $N_K(T)$, the zero-counting function for the Dedekind zeta function of a number field K .

Let $\sigma_1 > 1$, and let \mathcal{R} be the rectangle with vertices $\sigma_1 - iT$, $\sigma_1 + iT$, $1 - \sigma_1 + iT$, and $1 - \sigma_1 - iT$ (that is away from zeros of $\xi(s)$). Since $\xi(s)$ is entire, applying the argument principle, we know that

$$N_{\mathbb{Q}}(T) = \frac{1}{2\pi} \Delta_{\mathcal{R}} \arg \xi(s).$$

We let \mathcal{C} be the part of the contour of \mathcal{R} in $\Re(s) \geq \frac{1}{2}$ and \mathcal{C}_0 be the part of the contour of \mathcal{R} in $\Re(s) \geq \frac{1}{2}$ and $\Im(s) \geq 0$. From the functional equation (2.2) and the fact that $\overline{\xi(s)} = \xi(\bar{s})$, it follows that

$$\Delta_{\mathcal{R}} \arg \xi(s) = 2\Delta_{\mathcal{C}} \arg \xi(s) = 4\Delta_{\mathcal{C}_0} \arg \xi(s),$$

and thus

$$N_{\mathbb{Q}}(T) = \frac{2}{\pi} \Delta_{\mathcal{C}_0} \arg \xi(s). \quad (2.3)$$

Now, by (2.1), we arrive at

$$\Delta_{\mathcal{C}_0} \arg \xi(s) = \Delta_{\mathcal{C}_0} \arg s + \Delta_{\mathcal{C}_0} \arg \pi^{-s/2} + \Delta_{\mathcal{C}_0} \arg \Gamma\left(\frac{s}{2}\right) + \Delta_{\mathcal{C}_0} \arg ((s-1)\zeta(s)). \quad (2.4)$$

In addition, by a straightforward calculation, we have

$$\begin{aligned} \Delta_{\mathcal{C}_0} \arg s &= \arctan(2T), \\ \Delta_{\mathcal{C}_0} \arg \pi^{-s/2} &= \frac{T}{2} \log\left(\frac{1}{\pi}\right), \\ \Delta_{\mathcal{C}_0} \arg \Gamma(s) &= \Delta_{\mathcal{C}_0} (\Im \log \Gamma(s)) = \Im \log \Gamma\left(\frac{1}{2} + iT\right). \end{aligned} \quad (2.5)$$

In order to control the contribution of the gamma factor in (2.4), we set

$$g(T) = \frac{2}{\pi} \Im \log \Gamma\left(\frac{1}{4} + i\frac{T}{2}\right) - \frac{T}{\pi} \log\left(\frac{T}{2e}\right) + \frac{1}{4} \quad (2.6)$$

and recall that by [2, Proposition 3.2], one has

$$|g(T)| \leq \frac{1}{25T} \quad (2.7)$$

for $T \geq 5/7$. Now, gathering (2.3), (2.4), (2.5), and (2.6), we establish

$$N_{\mathbb{Q}}(T) = \frac{2}{\pi} \arctan(2T) + g(T) + \frac{T}{\pi} \log\left(\frac{T}{2\pi e}\right) - \frac{1}{4} + \frac{2}{\pi} \Delta_{\mathcal{C}_0} \arg((s-1)\zeta(s)). \quad (2.8)$$

To control $\Delta_{C_0} \arg((s-1)\zeta(s))$, we let C_1 be the vertical line from σ_1 to $\sigma_1 + iT$ and C_2 be the horizontal line from $\sigma_1 + iT$ to $\frac{1}{2} + iT$. As

$$\Delta_{C_1} \arg(s-1)\zeta(s) = \Delta_{C_1} \arg(s-1) + \Delta_{C_1} \arg \zeta(s) = \arctan\left(\frac{T}{\sigma_1 - 1}\right) + \Delta_{C_1} \arg \zeta(s)$$

and for $\sigma_1 > 1$,

$$|\Delta_{C_1} \arg \zeta(s)| = |\arg \zeta(\sigma_1 + iT)| \leq |\log \zeta(\sigma_1 + iT)| \leq \log \zeta(\sigma_1),$$

we obtain

$$|\Delta_{C_1} \arg(s-1)\zeta(s)| \leq \frac{\pi}{2} + \log \zeta(\sigma_1). \quad (2.9)$$

Hence, from (2.7), (2.8), and (2.9), it follows that

$$\left| N_{\mathbb{Q}}(T) - \frac{T}{\pi} \log \left(\frac{T}{2\pi e} \right) + \frac{1}{4} \right| \leq 2 + \frac{1}{25T} + \frac{2}{\pi} \log \zeta(\sigma_1) + \frac{2}{\pi} |\Delta_{C_2} \arg((s-1)\zeta(s))|. \quad (2.10)$$

To end this section, we shall borrow some estimates for the gamma function from [2] as follows. For $0 \leq d < 9/2$ and $T \geq 5/7$, we define

$$\mathcal{E}(T, d) = \left| \Im \log \Gamma \left(\frac{\sigma + iT}{2} \right) \Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}+d} + \Im \log \Gamma \left(\frac{\sigma + iT}{2} \right) \Big|_{\sigma=\frac{1}{2}}^{\frac{1}{2}-d} \right|.$$

As in [2, p. 1463], we set

$$\begin{aligned} E(T, d) &= \frac{2T/3}{(2d+17)^2 + 4T^2} + \frac{2T/3}{(-2d+17)^2 + 4T^2} - \frac{4T/3}{17^2 + 4T^2} \\ &\quad + \frac{T}{2} \log \left(1 + \frac{17^2}{4T^2} \right) - \frac{T}{4} \log \left(1 + \frac{(2d+17)^2}{4T^2} \right) - \frac{T}{4} \log \left(1 + \frac{(-2d+17)^2}{4T^2} \right) \\ &\quad + \frac{(8+6\pi)/45}{((2d+17)^2 + 4T^2)^{3/2}} + \frac{(8+6\pi)/45}{((-2d+17)^2 + 4T^2)^{3/2}} + \frac{2(8+6\pi)/45}{(17^2 + 4T^2)^{3/2}} \\ &\quad + \sum_{k=0}^3 \left(2 \arctan \frac{1+4k}{2T} - \arctan \frac{2d+1+4k}{2T} - \arctan \frac{-2d+1+4k}{2T} \right) \\ &\quad + \frac{2d+15}{4} \arctan \frac{2d+17}{2T} + \frac{-2d+15}{4} \arctan \frac{-2d+17}{2T} - \frac{15}{2} \arctan \frac{17}{2T}. \end{aligned}$$

It has been shown in [2, p. 1462] that $\mathcal{E}(T, d) \leq E(T, d)$ for $0 \leq d < 9/2$ and $T \geq 5/7$. Moreover, one has the following lemma established in [2, Lemma 3.4].

Lemma 2.1. *For $0 \leq \delta_1 \leq d < 9/2$ and $T \geq 5/7$, one has*

$$0 < E(T, \delta_1) \leq E(T, d).$$

Also, for $d \in [\frac{1}{4}, \frac{5}{8}]$ and $T \geq 5/7$, one has

$$\frac{E(T, d)}{\pi} \leq \frac{640d - 112}{1536(3T - 1)} + \frac{1}{2^{10}}.$$

We remark that the number $5/7$ above is not random but is borrowed directly from [2]. In [2], it is chosen to obtain a proper numerical bound (in order to count the zeros of Dirichlet L -functions). One may replace $5/7$ with an appropriate larger T_0 to get a better result.

3. Backlund's trick

In order to estimate $\Delta_{C_2} \arg((s-1)\zeta(s))$, we shall borrow some results from [7]. Define

$$f_N(s) = \frac{1}{2} \left(((s+iT-1)\zeta(s+iT))^N + ((s-iT-1)\zeta(s-iT))^N \right) \quad (3.1)$$

for $N \in \mathbb{N}$. Let $D(c, r)$ be the open disk centred at c with radius r . For any $N \in \mathbb{N}$, we define

$$S_N(c, r) = \frac{1}{N} \sum_{z \in \mathcal{S}_N(D(c, r))} \log \frac{r}{|z - c|},$$

where $\mathcal{S}_N(D(c, r))$ denotes the set of zeros of $f_N(s)$ in $D(c, r)$. In [7, Proposition 3.5], the authors prove the following upper bound for $S_N(c, r)$.

Proposition 3.1. *Let c , r , and σ_1 be real numbers such that*

$$c - r < \frac{1}{2} < 1 < c < \sigma_1 < c + r.$$

Let $F_{c,r} : [-\pi, \pi] \rightarrow \mathbb{R}$ be an even function such that $F_{c,r}(\theta) \geq \frac{1}{N_m} \log |f_{N_m}(c + re^{i\theta})|$. Then there is an infinite sequence of natural numbers $(N_m)_{m=1}^{\infty}$ such that

$$\limsup_{m \rightarrow \infty} S_{N_m}(c, r) \leq \log \left(\frac{1}{\sqrt{(c-1)^2 + T^2}} \frac{\zeta(c)}{\zeta(2c)} \right) + \frac{1}{\pi} \int_0^\pi F_{c,r}(\theta) d\theta.$$

To end this section, we recall the following version of Backlund's trick established in [7, Proposition 3.7] (cf. [2,17,18]). As explained in [18, Sec. 3], using Backlund's trick, one can track contribution from zeros of $f_N(\sigma)$ in $[1 - \sigma_1, \frac{1}{2}]$ whence there are zeros in $[\frac{1}{2}, \sigma_1]$.

Proposition 3.2 (*Backlund's trick*). Let c and r be real numbers. Set

$$\sigma_1 = c + \frac{(c - 1/2)^2}{r} \quad \text{and} \quad \delta = 2c - \sigma_1 - \frac{1}{2}.$$

If $1 < c < r$ and $0 < \delta < \frac{1}{2}$, then

$$\left| \arg((\sigma + iT - 1)\zeta(\sigma + iT)) \right|_{\sigma=\sigma_1}^{1/2} \leq \frac{\pi S_N(c, r)}{2 \log(r/(c - 1/2))} + \frac{E(T, \delta)}{2} + \frac{\pi}{N} + \frac{\pi}{2N} + \frac{\pi}{4}.$$

4. Convexity and subconvexity bounds and $F_{c,r}(\theta)$

4.1. Convexity and subconvexity bounds

In light of Propositions 3.1 and 3.2, to estimate (2.10), we shall construct an appropriate $F_{c,r}(\theta)$. We first recall the following version of the Phragmén-Lindelöf principle established by Trudgian [17, Lemma 3].

Proposition 4.1 (*Phragmén-Lindelöf principle*). Let a, b, Q be real numbers such that $b > a$ and $Q + a > 1$. Let $f(s)$ be a holomorphic function on the strip $a \leq \Re(s) \leq b$ such that

$$|f(s)| < C \exp(e^{k|t|})$$

for some $C > 0$ and $0 < k < \frac{\pi}{b-a}$. Suppose, further, that there are $A, B, \alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ such that $\alpha_1 \geq \beta_1$ and

$$|f(s)| \leq \begin{cases} A|Q + s|^{\alpha_1} (\log |Q + s|)^{\alpha_2} & \text{for } \Re(s) = a; \\ B|Q + s|^{\beta_1} (\log |Q + s|)^{\beta_2} & \text{for } \Re(s) = b. \end{cases}$$

Then for $a \leq \Re(s) \leq b$, one has

$$|f(s)| \leq \{A|Q + s|^{\alpha_1} (\log |Q + s|)^{\alpha_2}\}^{\frac{b-\Re(s)}{b-a}} \{B|Q + s|^{\beta_1} (\log |Q + s|)^{\beta_2}\}^{\frac{\Re(s)-a}{b-a}}.$$

We shall assume that there are $c_1, c_2, k_1, k_3 \geq 0$, $k_2 \in [0, \frac{1}{2}]$ and $t_0, t_1 \geq e$ such that

$$|\zeta(1 + it)| \leq c_1(\log t)^{c_2} \tag{4.1}$$

for $t \geq t_0$, and

$$|\zeta(\frac{1}{2} + it)| \leq k_1 t^{k_2} (\log t)^{k_3} \tag{4.2}$$

for $t \geq t_1$. Recall that $0 < \eta \leq \frac{1}{2}$. Now, we split our consideration into the following six cases.

(1) Assume $\sigma \geq 1 + \eta$. The trivial bound for the zeta function immediately gives

$$|\zeta(s)| \leq \zeta(\sigma). \quad (4.3)$$

(2) Assume $1 \leq \sigma \leq 1 + \eta$. From (4.1), it follows that there is $Q_0 > 0$ such that

$$|(1+it-1)\zeta(1+it)| \leq c_1|Q_0 + (1+it)|(\log|Q_0 + (1+it)|)^{c_2} \quad (4.4)$$

for all t . (We note that Q_0 depends on c_1, c_2, t_0 , and it can be computed explicitly.) Also, by (4.3), it is clear that

$$|(1+\eta+it-1)\zeta(1+\eta+it)| \leq \zeta(1+\eta)|Q_0 + (1+\eta+it)|.$$

Thus, by Proposition 4.1, for $1 \leq \sigma \leq 1 + \eta$,

$$|(s-1)\zeta(s)| \leq (c_1|Q_0 + s|(\log|Q_0 + s|)^{c_2})^{\frac{1+\eta-\sigma}{\eta}} (\zeta(1+\eta)|Q_0 + s|)^{\frac{\sigma-1}{\eta}},$$

which implies

$$|\zeta(s)| \leq \frac{1}{|s-1|} (c_1|Q_0 + s|(\log|Q_0 + s|)^{c_2})^{\frac{1+\eta-\sigma}{\eta}} (\zeta(1+\eta)|Q_0 + s|)^{\frac{\sigma-1}{\eta}}.$$

(3) Let $\frac{1}{2} \leq \sigma \leq 1$. Using (4.2), we deduce

$$|(\frac{1}{2}+it-1)\zeta(\frac{1}{2}+it)| \leq k_1|Q_1 + (\frac{1}{2}+it)|^{k_2+1}(\log|Q_1 + (\frac{1}{2}+it)|)^{k_3}, \quad (4.5)$$

for all t , where $Q_1 > 0$ is a constant depending only on k_1, k_2, k_3, t_1 (which can be computed directly). Now, by (4.4), (4.5), and Proposition 4.1, for $\frac{1}{2} \leq \sigma \leq 1$, we have

$$|\zeta(s)| \leq \frac{1}{|s-1|} (k_1|Q_2 + s|^{k_2+1}(\log|Q_2 + s|)^{k_3})^{2-2\sigma} (c_1|Q_2 + s|(\log|Q_2 + s|)^{c_2})^{2\sigma-1},$$

where $Q_2 = \max\{Q_0, Q_1\}$.

(4) Assume $0 \leq \sigma \leq \frac{1}{2}$. On the one hand, by (4.2), there is $Q_3 > 0$ such that

$$|\zeta(\frac{1}{2}+it)| \leq k_1|Q_3 + (\frac{1}{2}+it)|^{k_2}(\log|Q_3 + (\frac{1}{2}+it)|)^{k_3} \quad (4.6)$$

for all t . On the other hand, as (2.2) gives

$$\zeta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}-\frac{s}{2})}{\Gamma(\frac{s}{2})} \zeta(1-s), \quad (4.7)$$

it follows from (4.1) and the estimate

$$\left| \frac{\Gamma(\frac{1}{2}-\frac{s}{2})}{\Gamma(\frac{s}{2})} \right| \leq \left(\frac{|1+s|}{2} \right)^{\frac{1}{2}-\sigma}, \quad (4.8)$$

for $-\frac{1}{2} \leq \sigma \leq \frac{1}{2}$, that

$$|\zeta(it)| \leq c_1 \left(\frac{|1+it|}{2\pi} \right)^{\frac{1}{2}} (\log t)^{c_2}$$

for $t \geq t_0$. Hence, checking small values of t , we can find $Q_4 \geq 1$ such that

$$|\zeta(0+it)| \leq \frac{c_1}{\sqrt{2\pi}} |Q_4 + it|^{\frac{1}{2}} (\log |Q_4 + it|)^{c_2} \quad (4.9)$$

for all t . Now, together with (4.6), by Proposition 4.1, we then obtain

$$|\zeta(s)| \leq \left(\frac{c_1}{\sqrt{2\pi}} |Q_5 + s|^{\frac{1}{2}} (\log |Q_5 + s|)^{c_2} \right)^{1-2\sigma} (k_1 |Q_5 + s|^{k_2} (\log |Q_5 + s|)^{k_3})^{2\sigma},$$

where $Q_5 = \max\{Q_3, Q_4\}$.

(5) For $-\eta \leq \sigma \leq 0$, using (4.3), the functional equation (4.7), and the Gamma bound (4.8), as $Q_4 \geq 1$, we obtain

$$|\zeta(-\eta+it)| \leq \left(\frac{|1-\eta+it|}{2\pi} \right)^{\frac{1}{2}+\eta} \zeta(1+\eta) \leq \left(\frac{|Q_4-\eta+it|}{2\pi} \right)^{\frac{1}{2}+\eta} \zeta(1+\eta)$$

for all t . This estimate, combined with (4.9) and Proposition 4.1, then yields

$$|\zeta(s)| \leq \left(\frac{1}{(2\pi)^{\frac{1}{2}+\eta}} \zeta(1+\eta) |Q_4 + s|^{\frac{1}{2}+\eta} \right)^{\frac{-\sigma}{\eta}} \left(\frac{c_1}{\sqrt{2\pi}} |Q_4 + s|^{\frac{1}{2}} (\log |Q_4 + s|)^{c_2} \right)^{\frac{\sigma+\eta}{\eta}}.$$

(6) Finally, for $\sigma \leq -\eta$, we use (4.7) to deduce

$$|\zeta(s)| \leq \pi^{\sigma-\frac{1}{2}} \left| \frac{\Gamma(\frac{1}{2}-\frac{s}{2})}{\Gamma(\frac{s}{2})} \zeta(1-s) \right| \leq \pi^{\sigma-\frac{1}{2}} \left| \frac{\Gamma(\frac{1}{2}-\frac{s}{2})}{\Gamma(\frac{s}{2})} \right| \zeta(1-\sigma).$$

From the proof of [2, Theorem 5.7], it follows that for $a, b \in \{0, 1\}$ and $k \in \mathbb{N}$,

$$\frac{\Gamma(\frac{a}{2} + \frac{1-s}{2})}{\Gamma(\frac{a}{2} + \frac{s}{2})} = \frac{\Gamma(\frac{b}{2} + \frac{1-(s+k)}{2})}{\Gamma(\frac{b}{2} + \frac{s+k}{2})} 2^{-k} \left(\prod_{j=1}^k (s+j-1) \right) \frac{\sin(\frac{\pi}{2}(s+k+1-b))}{\sin(\frac{\pi}{2}(s+1-a))}.$$

Now, for $x \in \mathbb{R}$, we let $[x]$ be the integer closest to x (if there are two integers equally close to x , we then choose the one closer to 0). Note that for $a = 0$ and $b \equiv k \pmod{2}$, the sine factors above are ± 1 . Thus, upon taking $k = -[\sigma]$ and applying [13, Lemmata 1 and 2] to bound the ratio $\Gamma(\frac{b}{2} + \frac{1-(s+k)}{2})/\Gamma(\frac{b}{2} + \frac{s+k}{2})$, we arrive at

$$\left| \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \right| \leq \left(\frac{1}{2} |1+s-[s]| \right)^{\frac{1}{2}+[\sigma]-\sigma} 2^{[\sigma]} \left(\prod_{j=1}^{-[\sigma]} |s+j-1| \right),$$

which gives

$$|\zeta(s)| \leq \zeta(1-\sigma) \left(\frac{1}{2\pi}\right)^{\frac{1}{2}-\sigma} (|1+s-[s]|)^{\frac{1}{2}+[\sigma]-\sigma} \left(\prod_{j=1}^{-[s]} |s+j-1|\right).$$

4.2. Constructing and estimating $F_{c,r}(\theta)$

4.2.1. Bounding $\frac{1}{N} \log |f_N(s)|$

With the above convexity and subconvexity bounds in hand, we are in a position to bound $\frac{1}{N} \log |f_N(s)|$, where $f_N(s)$ is defined in (3.1). For $\sigma \geq 1 + \eta > 1$, we have

$$\begin{aligned} |f_N(s)| &\leq \frac{1}{2} (|s+iT-1|^N |\zeta(s+iT)|^N + |s-iT-1|^N |\zeta(s-iT)|^N) \\ &\leq ((\sigma-1)^2 + (|t|+T)^2)^{\frac{N}{2}} \zeta(\sigma)^N. \end{aligned}$$

Taking logarithms and dividing both sides by N gives

$$\frac{1}{N} \log |f_N(s)| \leq \frac{1}{2} \log((\sigma-1)^2 + (|t|+T)^2) + \log \zeta(\sigma).$$

For $1 \leq \sigma \leq 1 + \eta$, we have

$$\begin{aligned} |f_N(s)| &\leq \left(c_1 \sqrt{(Q_0+\sigma)^2 + (|t|+T)^2} \left(\log \sqrt{(Q_0+\sigma)^2 + (|t|+T)^2} \right)^{c_2} \right)^{\frac{N(1+\eta-\sigma)}{\eta}} \\ &\times \left(\zeta(1+\eta) \sqrt{(Q_0+\sigma)^2 + (|t|+T)^2} \right)^{\frac{N(\sigma-1)}{\eta}}. \end{aligned}$$

Taking logarithms of both sides and dividing by N , we obtain

$$\begin{aligned} \frac{1}{N} \log |f_N(s)| &\leq \frac{1+\eta-\sigma}{\eta} \log \frac{c_1}{2^{c_2}} + \frac{\sigma-1}{\eta} \log \zeta(1+\eta) + \frac{1}{2} \log ((Q_0+\sigma)^2 + (|t|+T)^2) \\ &+ \frac{c_2(1+\eta-\sigma)}{\eta} \log \log ((Q_0+\sigma)^2 + (|t|+T)^2). \end{aligned}$$

For $\frac{1}{2} \leq \sigma \leq 1$, from

$$\begin{aligned} |f_N(s)| &\leq \left(k_1 ((Q_2+\sigma)^2 + (|t|+T)^2)^{\frac{k_2+1}{2}} \left(\log \sqrt{(Q_2+\sigma)^2 + (|t|+T)^2} \right)^{k_3} \right)^{(2-2\sigma)N} \\ &\times \left(c_1 ((Q_2+\sigma)^2 + (|t|+T)^2)^{\frac{1}{2}} \left(\log \sqrt{(Q_2+\sigma)^2 + (|t|+T)^2} \right)^{c_2} \right)^{(2\sigma-1)N}, \end{aligned}$$

it follows that

$$\begin{aligned} \frac{1}{N} \log |f_N(s)| &\leq (2 - 2\sigma) \log k_1 + (2\sigma - 1) \log c_1 - (k_3(2 - 2\sigma) + c_2(2\sigma - 1)) \log 2 \\ &\quad + \frac{(2 - 2\sigma)(k_2 + 1) + 2\sigma - 1}{2} \log((Q_2 + \sigma)^2 + (|t| + T)^2) \\ &\quad + (k_3(2 - 2\sigma) + c_2(2\sigma - 1)) \log \log((Q_2 + \sigma)^2 + (|t| + T)^2). \end{aligned}$$

Assume $0 \leq \sigma \leq \frac{1}{2}$. From

$$\begin{aligned} |f_N(s)| &\leq ((\sigma - 1)^2 + (|t| + T)^2)^{\frac{N}{2}} \\ &\times \left(\frac{c_1}{\sqrt{2\pi}} ((Q_5 + \sigma)^2 + (|t| + T)^2)^{\frac{1}{4}} \left(\log \sqrt{(Q_5 + \sigma)^2 + (|t| + T)^2} \right)^{c_2} \right)^{(1-2\sigma)N} \\ &\times \left(k_1 ((Q_5 + \sigma)^2 + (|t| + T)^2)^{\frac{k_2}{2}} \left(\log \sqrt{(Q_5 + \sigma)^2 + (|t| + T)^2} \right)^{k_3} \right)^{2\sigma N}, \end{aligned}$$

we derive

$$\begin{aligned} \frac{1}{N} \log |f_N(s)| &\leq (1 - 2\sigma) \log \left(\frac{c_1}{2^{c_2+\frac{1}{2}} \sqrt{\pi}} \right) + 2\sigma \log \frac{k_1}{2^{k_3}} + \frac{1}{2} \log ((\sigma - 1)^2 + (|t| + T)^2) \\ &\quad + \frac{1 - 2\sigma + 4k_2\sigma}{4} \log ((Q_5 + \sigma)^2 + (|t| + T)^2) \\ &\quad + (c_2(1 - 2\sigma) + 2k_3\sigma) \log \log ((Q_5 + \sigma)^2 + (|t| + T)^2). \end{aligned}$$

For $-\eta \leq \sigma \leq 0$, we have

$$\begin{aligned} |f_N(s)| &\leq \left(\sqrt{(\sigma - 1)^2 + (|t| + T)^2} \right)^N \\ &\times \left(\frac{1}{(2\pi)^{\frac{1}{2}+\eta}} \zeta(1 + \eta) \left(\sqrt{(Q_4 + \sigma)^2 + (|t| + T)^2} \right)^{\frac{1}{2}+\eta} \right)^{-\frac{\sigma}{\eta}N} \\ &\times \left(\frac{c_1}{\sqrt{2\pi}} \left(\sqrt{(Q_4 + \sigma)^2 + (|t| + T)^2} \right)^{\frac{1}{2}} \left(\log \sqrt{(Q_4 + \sigma)^2 + (|t| + T)^2} \right)^{c_2} \right)^{\frac{\sigma+\eta}{\eta}N}. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{1}{N} \log |f_N(s)| &\leq -\frac{\sigma}{\eta} \log \frac{1}{(2\pi)^{\frac{1}{2}+\eta}} - \frac{\sigma}{\eta} \log(1 + \eta) + \frac{\sigma + \eta}{\eta} \log \frac{c_1}{\sqrt{2\pi}} - \frac{\sigma + \eta}{\eta} c_2 \log 2 \\ &\quad + \frac{1}{2} \log((\sigma - 1)^2 + (|t| + T)^2) \\ &\quad + \left(-\frac{\sigma(1 + 2\eta)}{4\eta} + \frac{\sigma + \eta}{4\eta} \right) \log((Q_4 + \sigma)^2 + (|t| + T)^2) \\ &\quad + \frac{\sigma + \eta}{\eta} c_2 \log \log((Q_4 + \sigma)^2 + (|t| + T)^2). \end{aligned}$$

Lastly, for $\sigma \leq -\eta$, as

$$\begin{aligned} |f_N(s)| &\leq ((\sigma - 1)^2 + (|t| + T)^2)^{\frac{N}{2}} \left(\frac{1}{2\pi} \right)^{N(\frac{1}{2} - \sigma)} ((1 + \sigma - [\sigma])^2 + (|t| + T)^2)^{\frac{(1-2\sigma+2[\sigma])N}{4}} \\ &\times \left(\prod_{j=1}^{-[\sigma]} ((\sigma + j - 1)^2 + (|t| + T)^2) \right)^{\frac{N}{2}} \zeta(1 - \sigma)^N, \end{aligned}$$

we obtain

$$\begin{aligned} \frac{1}{N} \log |f_N(s)| &\leq \log \zeta(1 - \sigma) + \frac{1}{2} \log((\sigma - 1)^2 + (|t| + T)^2) \\ &+ \frac{2\sigma - 1}{2} \log 2\pi + \frac{(1 - 2\sigma + 2[\sigma])}{4} \log((1 + \sigma - [\sigma])^2 + (|t| + T)^2) \\ &+ \frac{1}{2} \sum_{j=1}^{-[\sigma]} \log((\sigma + j - 1)^2 + (|t| + T)^2). \end{aligned}$$

4.2.2. Constructing and bounding $F_{c,r}(\theta)$

With the above bounds of $\frac{1}{N} \log |f_N(s)|$ in mind, similar to the construction of $F_{c,r}(\theta)$ for Dirichlet L -functions in [2, Definition 5.10] and Dedekind zeta functions in [7], we shall construct $F_{c,r}(\theta)$ for $\zeta(s)$ as follows. (We note that the main difference between our construction and the ones in [2,7] lies in the range $0 \leq \Re(s) \leq 1 + \eta$ as we have sharper bounds for $\zeta(s)$ in this range.)

Similar to [2,7], we first introduce some auxiliary functions and notation. For $\theta \in [-\pi, \pi]$, we let $\sigma = c + r \cos \theta$, with $c - r > -\frac{1}{2}$, and $t = r \sin \theta$. We define

$$\begin{aligned} L_j(\theta) &= \log \frac{(j + c + r \cos \theta)^2 + (|r \sin \theta| + T)^2}{T^2}, \\ M_j(\theta) &= \log \log((j + c + r \cos \theta)^2 + (|r \sin \theta| + T)^2) - \log \log(T^2). \end{aligned}$$

Now, we give upper bounds for $L_j(\theta)$ and $M_j(\theta)$. From the inequality $\log x \leq x - 1$, it follows that

$$\begin{aligned} L_j(\theta) &\leq \frac{(j + c + r \cos \theta)^2 + (|r \sin \theta| + T)^2}{T^2} - 1 \\ &= \frac{(j + c + r \cos \theta)^2 + (r \sin \theta)^2}{T^2} + \frac{2r \sin \theta}{T} \end{aligned}$$

for $\theta \in [0, \pi]$. Fix $T_0 \geq 1$. For $\theta \in [0, \pi]$, we let

$$L_j^*(\theta) = \frac{1}{T_0} (j + c + r \cos \theta)^2 + \frac{1}{T_0} (r \sin \theta)^2 + 2r \sin \theta.$$

It is clear that for $T \geq T_0$ and $\theta \in [0, \pi]$,

$$L_j(\theta) \leq \frac{L_j^*(\theta)}{T}.$$

Similarly, for $T \geq T_0$ and $\theta \in [0, \pi]$, we have

$$\begin{aligned} M_j(\theta) &\leq \frac{\log((j + c + r \cos \theta)^2 + (|r \sin \theta| + T)^2)}{\log(T^2)} - 1 \\ &= \frac{1}{\log(T^2)} \left(\log \frac{(j + c + r \cos \theta)^2 + (|r \sin \theta| + T)^2}{T^2} + \log(T^2) \right) - 1 \\ &= \frac{1}{\log(T^2)} L_j(\theta) \\ &\leq \frac{L_j^*(\theta)}{2T \log T}. \end{aligned}$$

We are now in a position to construct $F_{c,r}(\theta)$.

Definition 4.2. For $\theta \in [-\pi, \pi]$, we let $\sigma = c + r \cos \theta$, with $c - r > -\frac{1}{2}$, and $t = r \sin \theta$. For $\sigma > 1 + \eta$, we define

$$F_{c,r}(\theta) = \frac{1}{2} L_{-1}(\theta) + \log T + \log \zeta(\sigma).$$

For $1 \leq \sigma \leq 1 + \eta$, we define

$$\begin{aligned} F_{c,r}(\theta) &= \frac{1 + \eta - \sigma}{\eta} \log c_1 + \frac{\sigma - 1}{\eta} \log \zeta(1 + \eta) + \frac{1}{2} L_{Q_0}(\theta) + \log T \\ &\quad + \frac{c_2(1 + \eta - \sigma)}{\eta} M_{Q_0}(\theta) + \frac{c_2(1 + \eta - \sigma)}{\eta} \log \log T. \end{aligned}$$

For $\frac{1}{2} \leq \sigma \leq 1$, we define

$$\begin{aligned} F_{c,r}(\theta) &= (2 - 2\sigma) \log k_1 + (2\sigma - 1) \log c_1 \\ &\quad + ((2 - 2\sigma)(k_2 + 1) + 2\sigma - 1) \left(\frac{L_{Q_2}(\theta)}{2} + \log T \right) \\ &\quad + (k_3(2 - 2\sigma) + c_2(2\sigma - 1))(M_{Q_2}(\theta) + \log \log T). \end{aligned}$$

For $0 \leq \sigma \leq \frac{1}{2}$, we define

$$\begin{aligned} F_{c,r}(\theta) &= (1 - 2\sigma) \log \frac{c_1}{\sqrt{2\pi}} + 2\sigma \log k_1 + \frac{1}{2} L_{-1}(\theta) + \log T \\ &\quad + \frac{1 - 2\sigma + 4k_2\sigma}{2} \left(\frac{L_{Q_5}(\theta)}{2} + \log T \right) \\ &\quad + (c_2(1 - 2\sigma) + 2k_3\sigma)(M_{Q_5}(\theta) + \log \log T). \end{aligned}$$

For $-\eta \leq \sigma \leq 0$, we define

$$\begin{aligned} F_{c,r}(\theta) = & -\frac{\sigma}{\eta} \log \frac{1+\eta}{c_1(2\pi)^\eta} + \log \frac{c_1}{\sqrt{2\pi}} + \frac{1}{2} L_{-1}(\theta) + \log T \\ & + \left(-\frac{\sigma(1+2\eta)}{2\eta} + \frac{\sigma+\eta}{2\eta} \right) \left(\frac{L_{Q_4}(\theta)}{2} + \log T \right) + \frac{\sigma+\eta}{\eta} c_2(M_{Q_4}(\theta) + \log \log T). \end{aligned}$$

For $\sigma \leq -\eta$, we define

$$\begin{aligned} F_{c,r}(\theta) = & \log \zeta(1-\sigma) + \frac{1}{2} L_{-1}(\theta) + \left(1 + \frac{1-2\sigma}{2} \right) \log T - \frac{1-2\sigma}{2} \log 2\pi \\ & + \frac{(1-2\sigma+2[\sigma])}{4} L_{1-[\sigma]}(\theta) + \frac{1}{2} \sum_{j=1}^{-[\sigma]} L_{j-1}(\theta). \end{aligned}$$

It is clear that $F_{c,r}(\theta)$ is an even function of θ such that $F_{c,r}(\theta) \geq \frac{1}{N} \log |f_N(c+re^{i\theta})|$.

Now, we shall provide an upper bound for $\int_0^\pi F_{c,r}(\theta) d\theta$. In light of work of [2] and [7], for $c \in \mathbb{R}$ and $r > 0$, we define

$$\theta_y = \begin{cases} 0 & \text{if } c+r \leq y; \\ \arccos \frac{y-c}{r} & \text{if } c-r \leq y \leq c+r; \\ \pi & \text{if } y \leq c-r. \end{cases} \quad (4.10)$$

Now, we let c, r , and η be positive real numbers satisfying

$$-\frac{1}{2} < c-r < 1-c < -\eta < 1+\eta < c \quad (4.11)$$

and $0 < \eta \leq \frac{1}{2}$. To bound $\int_0^\pi F_{c,r}(\theta) d\theta$, we consider the splitting

$$\int_0^\pi = \int_0^{\theta_{1+\eta}} + \int_{\theta_{1+\eta}}^{\theta_1} + \int_{\theta_1}^{\theta_{\frac{1}{2}}} + \int_{\theta_{\frac{1}{2}}}^{\theta_0} + \int_{\theta_0}^{\theta_{-\eta}} + \int_{\theta_{-\eta}}^\pi.$$

Firstly, we have

$$\int_0^{\theta_{1+\eta}} F_{c,r}(\theta) d\theta \leq \log T \int_0^{\theta_{1+\eta}} 1 d\theta + \int_0^{\theta_{1+\eta}} \log \zeta(\sigma) d\theta + \frac{1}{2T} \int_0^{\theta_{1+\eta}} L_{-1}^*(\theta) d\theta.$$

Secondly, $\int_{\theta_{1+\eta}}^{\theta_1} F_{c,r}(\theta) d\theta$ is bounded above by

$$\log T \int_{\theta_{1+\eta}}^{\theta_1} 1 d\theta + \frac{c_2}{\eta} \log \log T \int_{\theta_{1+\eta}}^{\theta_1} (1+\eta-\sigma) d\theta + \frac{\log c_1}{\eta} \int_{\theta_{1+\eta}}^{\theta_1} (1+\eta-\sigma) d\theta$$

$$+ \frac{\log \zeta(1+\eta)}{\eta} \int_{\theta_{1+\eta}}^{\theta_1} (\sigma - 1)d\theta + \frac{1}{2T} \int_{\theta_{1+\eta}}^{\theta_1} L_{Q_0}^*(\theta)d\theta + \frac{c_2}{2\eta T \log T} \int_{\theta_{1+\eta}}^{\theta_1} (1 + \eta - \sigma)L_{Q_0}^*(\theta)d\theta.$$

A direction calculation shows that

$$\begin{aligned} \int_{\theta_1}^{\theta_{\frac{1}{2}}} F_{c,r}(\theta)d\theta &\leq \log T \int_{\theta_1}^{\theta_{\frac{1}{2}}} (2 - 2\sigma)(k_2 + 1) + 2\sigma - 1 d\theta \\ &+ \log \log T \int_{\theta_1}^{\theta_{\frac{1}{2}}} k_3(2 - 2\sigma) + c_2(2\sigma - 1) d\theta \\ &+ \int_{\theta_1}^{\theta_{\frac{1}{2}}} (2 - 2\sigma) \log k_1 + (2\sigma - 1) \log c_1 d\theta \\ &+ \frac{1}{T} \int_{\theta_1}^{\theta_{\frac{1}{2}}} \frac{(2 - 2\sigma)(k_2 + 1) + 2\sigma - 1}{2} L_{Q_2}^*(\theta)d\theta \\ &+ \frac{1}{2T \log T} \int_{\theta_1}^{\theta_{\frac{1}{2}}} (k_3(2 - 2\sigma) + c_2(2\sigma - 1)) L_{Q_2}^*(\theta)d\theta. \end{aligned}$$

Also, we can bound $\int_{\theta_{\frac{1}{2}}}^{\theta_0} F_{c,r}(\theta)d\theta$ above by

$$\begin{aligned} &\log T \int_{\theta_{\frac{1}{2}}}^{\theta_0} 1 d\theta + \frac{\log T}{2} \int_{\theta_{\frac{1}{2}}}^{\theta_0} 1 - 2\sigma + 4k_2\sigma d\theta + \log \log T \int_{\theta_{\frac{1}{2}}}^{\theta_0} c_2(1 - 2\sigma) + 2k_3\sigma d\theta \\ &+ \left(\log \frac{c_1}{\sqrt{2\pi}} \right) \int_{\theta_{\frac{1}{2}}}^{\theta_0} 1 - 2\sigma d\theta + 2 \log k_1 \int_{\theta_{\frac{1}{2}}}^{\theta_0} \sigma d\theta + \frac{1}{2T} \int_{\theta_{\frac{1}{2}}}^{\theta_0} L_{-1}^*(\theta)d\theta \\ &+ \frac{1}{4T} \int_{\theta_{\frac{1}{2}}}^{\theta_0} (1 - 2\sigma + 4k_2\sigma) L_{Q_5}^*(\theta)d\theta + \frac{1}{2T \log T} \int_{\theta_{\frac{1}{2}}}^{\theta_0} (c_2(1 - 2\sigma) + 2k_3\sigma) L_{Q_5}^*(\theta)d\theta. \end{aligned}$$

Moreover, $\int_{\theta_0}^{\theta_{-\eta}} F_{c,r}(\theta)d\theta$ is bounded above by

$$\log T \int_{\theta_0}^{\theta_{-\eta}} 1 - \frac{\sigma(1 + 2\eta)}{2\eta} + \frac{\sigma + \eta}{2\eta} d\theta + \log \log T \int_{\theta_0}^{\theta_{-\eta}} \frac{\sigma + \eta}{\eta} c_2 d\theta$$

$$\begin{aligned}
& + \int_{\theta_0}^{\theta_{-\eta}} -\frac{\sigma}{\eta} \log \frac{1+\eta}{c_1(2\pi)^\eta} + \log \frac{c_1}{\sqrt{2\pi}} d\theta + \frac{1}{2T} \int_{\theta_0}^{\theta_{-\eta}} L_{-1}^*(\theta) d\theta \\
& + \frac{1}{T} \int_{\theta_0}^{\theta_{-\eta}} \left(-\frac{\sigma(1+2\eta)}{4\eta} + \frac{\sigma+\eta}{4\eta} \right) L_{Q_4}^*(\theta) d\theta + \frac{1}{2T \log T} \int_{\theta_0}^{\theta_{-\eta}} \frac{\sigma+\eta}{\eta} c_2 L_{Q_4}^*(\theta) d\theta.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\int_{\theta_{-\eta}}^{\pi} F_{c,r}(\theta) d\theta & \leq \log T \int_{\theta_{-\eta}}^{\pi} 1 + \frac{1-2\sigma}{2} d\theta + \int_{\theta_{-\eta}}^{\pi} \log \zeta(1-\sigma) d\theta - \log 2\pi \int_{\theta_{-\eta}}^{\pi} \frac{1-2\sigma}{2} d\theta \\
& + \frac{1}{2T} \int_{\theta_{-\eta}}^{\pi} L_{-1}^*(\theta) d\theta + \frac{1}{T} \int_{\theta_{-\eta}}^{\theta_{-\frac{1}{2}}} \frac{1-2\sigma}{4} L_1^*(\theta) d\theta \\
& + \sum_{j=1}^{\infty} \int_{\theta_{-j+\frac{1}{2}}}^{\theta_{-j-\frac{1}{2}}} \left(\frac{1-2\sigma-2j}{4} L_{j+1}(\theta) + \frac{1}{2} \sum_{k=1}^j L_{k-1}(\theta) \right) d\theta.
\end{aligned}$$

(We note that from the assumption $-\frac{1}{2} < c - r$ it follows that $\theta_{-j+\frac{1}{2}} = \theta_{-j-\frac{1}{2}} = \pi$ for $j \geq 1$, and thus the last term in the above inequality is equal to zero.)

To end this section, we require the following two estimates from [2, Lemmata 5.14 and 5.15] to control the zeta integrals appearing in the above estimates.

Lemma 4.3. *Let c, r and η be positive real numbers, satisfying (4.11), and J_1 and J_2 be positive integers. If $\theta_{1+\eta} \leq 2.1$, then for $\sigma = c + r \cos \theta$, one has*

$$\int_0^{\theta_{1+\eta}} \log \zeta(\sigma) d\theta \leq \frac{\log \zeta(1+\eta) + \log \zeta(c)}{2} \left(\theta_{1+\eta} - \frac{\pi}{2} \right) + \frac{\pi}{4J_1} \log \zeta(c) + \kappa_1(J_1),$$

where

$$\kappa_1(J_1) = \frac{\pi}{4J_1} \left(\log \zeta(c+r) + 2 \sum_{j=1}^{J_1-1} \log \zeta \left(c + r \cos \frac{\pi j}{2J_1} \right) \right).$$

In addition, assuming further $r > 2c - 1$, one has

$$\int_{\theta_{-\eta}}^{\pi} \log \zeta(1-\sigma) d\theta \leq \frac{\log \zeta(1+\eta) + \log \zeta(c)}{2} (\theta_{1-c} - \theta_{-\eta}) + \frac{\pi - \theta_{1-c}}{2J_2} \log \zeta(c) + \kappa_2(J_2),$$

where

$$\begin{aligned} \kappa_2(J_2) = & \frac{\pi - \theta_{1-c}}{2J_2} \left(\log \zeta(1 - c + r) \right. \\ & \left. + 2 \sum_{j=1}^{J_2-1} \log \zeta \left(1 - c - r \cos \left(\frac{\pi j}{J_2} + \left(1 - \frac{j}{J_2} \right) \theta_{1-c} \right) \right) \right). \end{aligned}$$

5. Final formulae

In this section, we shall first prove Theorem 1.1.

Proof of Theorem 1.1. Using (2.10) and Propositions 3.1 and 3.2, for $c, r, \eta > 0$ such that

$$-\frac{1}{2} < c - r < 1 - c < -\eta < 0 < \frac{1}{4} \leq \delta = 2c - \sigma_1 - \frac{1}{2} < \frac{1}{2}$$

and

$$1 < 1 + \eta < c < \sigma_1 = c + \frac{(c - 1/2)^2}{r} < c + r,$$

satisfying $\theta_{1+\eta} \leq 2.1$, we have

$$\begin{aligned} & \left| N_{\mathbb{Q}}(T) - \frac{T}{\pi} \log \left(\frac{T}{2\pi e} \right) + \frac{1}{4} \right| \\ & \leq \frac{5}{2} + \frac{1}{25T} + \frac{2}{\pi} \log \zeta(\sigma_1) + \frac{1}{\log(r/(c - 1/2))} \log \frac{\zeta(c)}{\zeta(2c)} - \frac{1}{\log(r/(c - 1/2))} \log T \\ & + \frac{1}{\pi \log(r/(c - 1/2))} \int_0^\pi F_{c,r}(\theta) d\theta + \frac{E(T, \delta)}{\pi}. \end{aligned}$$

Thus, recalling that $N_{\mathbb{Q}}(T) = 2N(T)$ and applying Lemma 2.1 (to bound $E(T, \delta)$) and the estimates from Section 4.2.2 (to bound $\int_0^\pi F_{c,r}(\theta) d\theta$), for $T \geq T_0$, we have

$$\left| N(T) - \frac{T}{2\pi} \log \left(\frac{T}{2\pi e} \right) + \frac{1}{8} \right| \leq C_1 \log T + C_2 \log \log T + C_3,$$

where for $j = 1, 2, 3$,

$$C_j = \frac{\tilde{C}_j}{2\pi \log(r/(c - 1/2))}, \quad (5.1)$$

$$\tilde{C}_1 = \int_{\theta_1}^{\theta_{\frac{1}{2}}} (2 - 2\sigma)(k_2 + 1) + 2\sigma - 2d\theta + \frac{1}{2} \int_{\theta_{\frac{1}{2}}}^{\theta_0} 1 - 2\sigma + 4k_2\sigma d\theta \\ + \int_{\theta_0}^{\theta_{-\eta}} -\frac{\sigma(1 + 2\eta)}{2\eta} + \frac{\sigma + \eta}{2\eta} d\theta + \int_{\theta_{-\eta}}^{\pi} \frac{1 - 2\sigma}{2} d\theta, \quad (5.2)$$

$$\tilde{C}_2 = \frac{c_2}{\eta} \int_{\theta_{1+\eta}}^{\theta_1} (1 + \eta - \sigma) d\theta + \int_{\theta_1}^{\theta_{\frac{1}{2}}} k_3(2 - 2\sigma) + c_2(2\sigma - 1) d\theta \\ + \int_{\theta_{\frac{1}{2}}}^{\theta_0} c_2(1 - 2\sigma) + 2k_3\sigma d\theta + \int_{\theta_0}^{\theta_{-\eta}} \frac{\sigma + \eta}{\eta} c_2 d\theta, \quad (5.3)$$

$$\tilde{C}_3 = \pi \log(r/(c - 1/2)) \left(\frac{640\delta - 112}{1536(3T_0 - 1)} + \frac{1}{2^{10}} + \frac{5}{2} + \frac{1}{25T_0} + \frac{2}{\pi} \log \zeta(\sigma_1) \right) + \pi \log \frac{\zeta(c)}{\zeta(2c)} \\ + \frac{\log c_1}{\eta} \int_{\theta_{1+\eta}}^{\theta_1} (1 + \eta - \sigma) d\theta + \frac{\log \zeta(1 + \eta)}{\eta} \int_{\theta_{1+\eta}}^{\theta_1} (\sigma - 1) d\theta \\ + \int_{\theta_1}^{\theta_{\frac{1}{2}}} (2 - 2\sigma) \log k_1 + (2\sigma - 1) \log c_1 d\theta + \left(\log \frac{c_1}{\sqrt{2\pi}} \right) \int_{\theta_{\frac{1}{2}}}^{\theta_0} 1 - 2\sigma d\theta + 2 \log k_1 \int_{\theta_{\frac{1}{2}}}^{\theta_0} \sigma d\theta \\ + \int_{\theta_0}^{\theta_{-\eta}} -\frac{\sigma}{\eta} \log \frac{1 + \eta}{c_1(2\pi)^{\eta}} + \log \frac{c_1}{\sqrt{2\pi}} d\theta - (\log 2\pi) \int_{\theta_{-\eta}}^{\pi} \frac{1 - 2\sigma}{2} d\theta \\ + \frac{\log \zeta(1 + \eta) + \log \zeta(c)}{2} \left(\theta_{1+\eta} - \frac{\pi}{2} \right) + \frac{\pi}{4J_1} \log \zeta(c) + \kappa_1(J_1) \\ + \frac{\log \zeta(1 + \eta) + \log \zeta(c)}{2} (\theta_{1-c} - \theta_{-\eta}) + \frac{\pi - \theta_{1-c}}{2J_2} \log \zeta(c) + \kappa_2(J_2) + \kappa_3(T_0), \quad (5.4)$$

and $\kappa_3(T_0)$ is equal to

$$\frac{1}{2T_0} \max \left\{ 0, \int_0^{\theta_{1+\eta}} L_{-1}^*(\theta) d\theta + \int_{\theta_{1+\eta}}^{\theta_1} L_{Q_0}^*(\theta) d\theta + \int_{\theta_1}^{\theta_{\frac{1}{2}}} ((2 - 2\sigma)(k_2 + 1) + 2\sigma - 1) L_{Q_2}^*(\theta) d\theta \right. \\ \left. + \int_{\theta_{\frac{1}{2}}}^{\theta_0} L_{-1}^*(\theta) d\theta + \frac{1}{2} \int_{\theta_{\frac{1}{2}}}^{\theta_0} (1 - 2\sigma + 4k_2\sigma) L_{Q_5}^*(\theta) d\theta + \int_{\theta_0}^{\theta_{-\eta}} L_{-1}^*(\theta) d\theta \right)$$

Table 2Choices of parameters (c, r, η) and resulting admissible (C_1, C_2, C_3, C'_3) .

c	r	η	C_1	C_2	C_3	C'_3
1.000011314	1.064340602	$4.2826451 \cdot 10^{-6}$	0.103787	0.257297	9.367419	8.367419
1.025253504	1.182375395	0.009944751381	0.109410	0.204142	4.030486	3.030486
1.035766557	1.229059659	0.014325507360	0.111973	0.189768	3.746756	2.746756

$$\begin{aligned}
& + \int_{\theta_0}^{\theta_{-\eta}} \left(-\frac{\sigma(1+2\eta)}{2\eta} + \frac{\sigma+\eta}{2\eta} \right) L_{Q_4}^*(\theta) d\theta + \int_{\theta_{-\eta}}^{\pi} L_{-1}^*(\theta) d\theta + \int_{\theta_{-\eta}}^{\theta_{-\frac{1}{2}}} \frac{1-2\sigma}{2} L_1^*(\theta) d\theta \Big\} \\
& + \frac{1}{2T_0 \log T_0} \max \left\{ 0, \int_{\theta_{1+\eta}}^{\theta_1} \frac{c_2}{\eta} (1+\eta-\sigma) L_{Q_0}^*(\theta) d\theta \right. \\
& + \int_{\theta_1}^{\theta_{\frac{1}{2}}} (k_3(2-2\sigma) + c_2(2\sigma-1)) L_{Q_2}^*(\theta) d\theta \\
& \left. + \int_{\theta_0}^{\theta_{-\eta}} (c_2(1-2\sigma) + 2k_3\sigma) L_{Q_5}^*(\theta) d\theta + \int_{\theta_0}^{\theta_{-\eta}} \frac{\sigma+\eta}{\eta} c_2 L_{Q_4}^*(\theta) d\theta \right\}.
\end{aligned}$$

Recall that Patel [10] showed that $|\zeta(1+it)| \leq \log t$ for $t \geq 3$ (i.e., in (1.2), $(c_1, c_2, t_0) = (1, 1, 3)$ is admissible) and that by the work of Hiary [8], $(k_1, k_2, k_3, t_1) = (0.77, \frac{1}{6}, 1, 3)$ is admissible for (1.3).⁴ In addition, for $(c_1, c_2, t_0) = (1, 1, 3)$ and $(k_1, k_2, k_3, t_1) = (0.77, \frac{1}{6}, 1, 3)$, we may take

$$(Q_0, Q_1, Q_2, Q_3, Q_4, Q_5) = (1, 1.18, 1.18, 3.9, 2.3, 3.9).$$

Finally, for $T_0 = 30\,610\,046\,000$, choosing $J_1 = 64$ and $J_2 = 39$, we calculate admissible (C_1, C_2, C_3) and record them in Table 2. \square

Now, we are in a position to prove Corollary 1.2.

Proof of Corollary 1.2. By Theorem 1.1 (with $T_0 = 30\,610\,046\,000$ and Table 2), it is sufficient to verify the corollary for $e \leq T \leq 30\,610\,046\,000$. We note that by (2.8),

⁴ In fact, Hiary [8] showed that $(k_1, k_2, k_3) = (0.63, \frac{1}{6}, 1)$ was admissible. However, as pointed out by [10], due to an error in [8], one can only take $(k_1, k_2, k_3) = (0.77, \frac{1}{6}, 1)$.

We also note that there are two bounds used in [17, Sec. 5] and [11] that may be no longer valid. On one hand, Trudgian in [17, Sec. 5] used a result from [16] that $|\zeta(1+it)| \leq \frac{3}{4} \log t$ for $t \geq 3$. On the other hand, [11, Theorem 1] states that $(k_1, k_2, k_3) = (0.732, \frac{1}{6}, 1)$ is admissible. However, as pointed out by [10], both bounds made use of an incorrect result obtained by Cheng-Graham in [4]. We refer the reader to [10] for a detailed discussion.

$$S(T) = \frac{1}{\pi} \Delta_{C_0} \arg \zeta(s) = \frac{1}{2} \left(N_{\mathbb{Q}}(T) - \frac{T}{\pi} \log \left(\frac{T}{2\pi e} \right) + \frac{1}{4} - g(T) - 2 \right), \quad (5.5)$$

where $g(T)$ is defined as in (2.6). Thus, by (1.7) and (2.7), we obtain

$$\left| N(T) - \frac{T}{2\pi} \log \left(\frac{T}{2\pi e} \right) + \frac{1}{8} \right| \leq |S(T)| + \frac{1}{2}|g(T)| + 1 \leq 2.5167 + \frac{1}{50e} + 1 \quad (5.6)$$

for $e \leq T \leq 30\,610\,046\,000$. (We remark that one may apply [3, Lemma 2] to improve (5.6) for $T \geq 2\pi$.) As the quantity on the right of (5.6) clearly is less than $0.1038 \log T + 0.2573 \log \log T + 9.3675$ for any $T \geq e$. We then conclude the proof by using the triangle inequality. \square

To end this section, we shall prove Theorem 1.3.

Proof of Theorem 1.3. Note that

$$S(T) = \frac{1}{\pi} \Delta_{C_0} \arg \zeta(s) = \frac{1}{\pi} \Delta_{C_1} \arg \zeta(s) + \frac{1}{\pi} \Delta_{C_2} \arg(s-1) \zeta(s) - \frac{1}{\pi} \Delta_{C_2} \arg(s-1).$$

We know that

$$|\Delta_{C_1} \arg \zeta(s)| \leq \log \zeta(\sigma_1)$$

and

$$|\Delta_{C_2} \arg(s-1)| = \arctan \left(\frac{\sigma_1 - 1}{T} \right) + \arctan \left(\frac{1}{2T} \right) \leq \arctan \left(\frac{\sigma_1 - 1}{T_0} \right) + \arctan \left(\frac{1}{2T_0} \right)$$

for $T \geq T_0$. From Propositions 3.1 and 3.2, it follows that for $c, r, \eta > 0$ such that

$$-\frac{1}{2} < c - r < 1 - c < -\eta < 0 < \frac{1}{4} \leq \delta = 2c - \sigma_1 - \frac{1}{2} < \frac{1}{2}$$

and

$$1 < 1 + \eta < c < \sigma_1 = c + \frac{(c - 1/2)^2}{r} < c + r,$$

satisfying $\theta_{1+\eta} \leq 2.1$,

$$\begin{aligned} |S(T)| &\leq \frac{1}{4} + \frac{1}{\pi} \log \zeta(\sigma_1) + \frac{1}{2 \log(r/(c-1/2))} \log \frac{\zeta(c)}{\zeta(2c)} - \frac{1}{2 \log(r/(c-1/2))} \log T \\ &\quad + \frac{1}{2\pi \log(r/(c-1/2))} \int_0^\pi F_{c,r}(\theta) d\theta + \frac{E(T, \delta)}{2\pi} + \frac{1}{\pi} \arctan \left(\frac{\sigma_1 - 1}{T_0} \right) \\ &\quad + \frac{1}{\pi} \arctan \left(\frac{1}{2T_0} \right). \end{aligned}$$

Thus, applying Lemma 2.1 (to bound $E(T, \delta)$) and the estimates from Section 4.2.2 (to bound $\int_0^\pi F_{c,r}(\theta) d\theta$), for $T \geq T_0$, we have

$$|S(T)| \leq C_1 \log T + C_2 \log \log T + C'_3,$$

where

$$C'_3 = C_3 - 1 + \frac{1}{\pi} \arctan \left(\frac{\sigma_1 - 1}{T_0} \right) + \frac{1}{\pi} \arctan \left(\frac{1}{2T_0} \right), \quad (5.7)$$

$C_1 = C_1(c, r, \eta; k_2)$, $C_2 = C_2(c, r, \eta; c_2, k_3)$, $C_3 = C_3(c, r, \eta; c_1, c_2, t_0, k_1, k_2, k_3, t_1; T_0)$ are given in (5.1), (5.2), (5.3), and (5.4). In particular, for $T_0 = 30610046000$, we have admissible (C_1, C_2, C'_3) recorded in Table 2. Finally, applying (1.7), we conclude the proof. \square

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