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On the Chebotarev–Sato–Tate phenomenon

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ABSTRACT

We study the Chebotarev–Sato–Tate phenomenon that concerns the distribution of Artin symbols and Frobenius angles. From the recent work of Barnet-Lamb et al., we derive some unconditional results in the case of Hilbert modular forms. Furthermore, under the Langlands functoriality conjecture and the generalised Riemann hypothesis, we give two effective versions of such distributions, which present a modular variant of the results of Bucur, Kedlaya, and V.K. Murty.

1. Introduction

Over fifty years ago, the following conjecture of Sato and Tate was born.

Conjecture. *Let E be a non-CM elliptic curve defined over $\mathbb{Q}$. The Frobenius angles θ_p at p of E are equidistributed over $[0, \pi]$ with respect to the probability measure μ_{ST}, now called the Sato–Tate measure for E,*

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$$\frac{2}{\pi}\sin^2\theta d\theta.$$

In [34], Serre reformulated Weyl's equidistribution criterion in the language of L-functions associated to compact groups and used the Wiener–Ikehara Tauberian theorem to show that the Sato–Tate conjecture would follow from the certain holomorphy and non-vanishing result of L-functions associated to SU(2), the Sato–Tate group for E. Shortly after Serre's observation, inspired by the Taniyama–Shimura conjecture, which is now the celebrated modularity theorem of Wiles and Breuil–Conrad–Diamond–Taylor, Langlands conjectured that these L-functions associated to E are automorphic. This is, in fact, a part of the Langlands functoriality conjecture and implies the Sato–Tate conjecture. In this direction, several instances have been established by Shimura [37], Gelbart–Jacquet [18], Shahidi [36], and Kim [24].

In a slightly different vein, following Serre, one may consider the Sato–Tate distribution for the Fourier coefficients of the Ramanujan τ-function (or, more generally, non-CM normalised Hecke eigenforms). Indeed, for any non-CM normalised Hecke eigenform f of weight $k \geq 2$, by writing its p-th Fourier coefficient as $a_p(f) = 2p^{(k-1)/2}\cos\theta_p$, it is expected that θ_p's are equidistributed in $[0, \pi]$ with respect to μ_{ST}. It is worth to note that from the above-mentioned modularity theorem, such consideration is stronger than the Sato–Tate conjecture for elliptic curves. Also, it shall be natural to extend Serre's consideration to non-CM Hilbert modular forms or even pairs of non-CM Hilbert modular forms that are not twist-equivalent.

Over the past decade, Barnet-Lamb, Clozel, Gee, Geraghty, Harris, Shepherd-Barron, Taylor, and Thorne in a series of papers [2–4,13–16,20,38–41] showed that the L-functions in the consideration are *potentially automorphic*, and then established the Sato–Tate conjecture via the theory of base change due to Arthur–Clozel [1] and the Brauer induction theorem [8]. (For more historical details, we refer the interested reader to inspiring articles of Clozel [12] and Harris [19]. For the state-of-the-art development, we indicate the reader to the articles [13–15] of Clozel and Thorne.)

Following the grand tradition of analytic number theory, once there is a distribution result, one may ask for the error. Indeed, this has been studied, under the generalised Riemann hypothesis (denoted GRH) and the holomorphy assumption, by V.K. Murty [27] (for non-CM elliptic curves defined over $\mathbb{Q}$) and by Bucur and Kedlaya [10] (for pairs of two non-$\overline{\mathbb{Q}}$-isogenous elliptic curves not of CM-type). Also, assuming the Langlands functoriality conjecture, Thorner [42] gave an error term for the modular variant of the Sato–Tate distribution considered by Serre.

In light of Serre's formalism [34], M.R. Murty and V.K. Murty applied the early potential automorphy result, established in [16,38], to derive a hybrid Chebotarev–Sato–Tate theorem that asserts that the Artin symbols and Frobenius angles (of any given suitable Galois extension and non-CM elliptic curve over a totally real number field, respectively) are equidistributed.

In this note, we will emphasise the role of classical analytic number theory in studying the Chebotarev–Sato–Tate phenomenon for Hilbert modular forms. Throughout this note, by a Hilbert modular form, we mean a Hilbert newform whose weights are at least 2 and have the same parity; we shall further assume every Hilbert modular form over a totally real number field k in the consideration is not of CM, that is, not obtained by automorphic induction from any algebraic Hecke character of any totally imaginary quadratic extension of k.

First of all, by invoking the result of Barnet-Lamb et al. [2,4] on the potential automorphy of symmetric powers of Hilbert modular forms, we generalise the work of M.R. Murty and V.K. Murty in the context of Hilbert modular forms as follows.

Theorem 1.1. *Let π_1 and π_2 be non-CM Hilbert modular forms of k that are not twist-equivalent. Let K/k be a Galois extension of totally real number fields with Galois group $G_1 \times G_2$, and C a conjugacy class of G. Suppose that K^{G_1}/k is totally real and that G_1 is abelian. Let $\theta_{1,v}$ and $\theta_{2,v}$ be Frobenius angles at v of π_1 and π_2, respectively. Then the density of (good) primes v of k for which the Artin symbol $\sigma_v = C$ and $\theta_{1,v} \in [\alpha, \beta]$, with $0 \le \alpha \le \beta \le \pi$, is*

$$\frac{2|C|}{\pi|G|} \int_{\alpha}^{\beta} \sin^2 \theta d\theta.$$

Also, the density of primes v of k for which the Artin symbol $\sigma_v = C$ and the Frobenius angles $\theta_{i,v} \in [\alpha_i, \beta_i]$, with $0 \le \alpha_i \le \beta_i \le \pi$, is

$$\frac{4|C|}{\pi^2|G|} \int_{\alpha_1}^{\beta_1} \sin^2 \theta d\theta \int_{\alpha_2}^{\beta_2} \sin^2 \theta d\theta.$$

In Section 4, we shall study the error of such Chebotarev–Sato–Tate distributions, which gives a modular analogue of the works of Bucur–Kedlaya and V.K. Murty.

Theorem 1.2. *Let K/k be a Galois extension of totally real number fields with Galois group G, and let π be a non-CM Hilbert modular form of k with conductor $q = q(\pi)$. Suppose that all the irreducible characters χ of G and all the symmetric powers $\mathrm{Sym}^m \pi$ of π are cuspidal over k, and that each $L(s, \mathrm{Sym}^m \pi \otimes \chi)$ satisfies GRH. Assuming Serre's bound (2.3) for π, then*

$$\sum_C \frac{1}{|C|} \left| \pi_{C,I}(x) - \frac{|C|}{|G|} \mu_{ST}(I) \operatorname{Li}(x) \right|^2 \ll \left(x^{\frac{3}{4}} n_k^{\frac{3}{2}} (\log(x\mathcal{M}(K/k)q))^{\frac{1}{2}} \right)^2,$$

where $\pi_{C,I}(x)$ denotes number of primes v for which $Nv \le x$, the Artin symbol $\sigma_v = C$, and the Frobenius angle $\theta_v \in I = [\alpha, \beta]$, $n_k = [k : \mathbb{Q}]$ is the degree of $k/\mathbb{Q}$, $\mathcal{M}(K/k)$ is a computable constant, defined in (2.7), depending only on K/k, and the implied constant depends only on π. In particular, we have

$$\pi_{C,I}(x) - \frac{|C|}{|G|}\mu_{ST}(I)\,\mathrm{Li}(x) \ll x^{\frac{3}{4}}|C|^{\frac{1}{2}}n_k^{\frac{3}{2}}(\log(x\mathcal{M}(K/k)q))^{\frac{1}{2}}.$$

Also, we have the following variant for the pairs of Hilbert modular forms.

Theorem 1.3. *Let K/k be a Galois extension of totally real number fields with Galois group G. Let π_1 and π_2 be non-twist-equivalent Hilbert modular forms of k not of CM type. Suppose that all the irreducible characters χ of G and all the symmetric powers $\mathrm{Sym}^{m_i}\pi_i$ are cuspidal over k, and that each $L(s, \mathrm{Sym}^{m_1}\pi_1 \otimes \mathrm{Sym}^{m_2}\pi_2 \otimes \chi)$ satisfies GRH and the analytic properties predicted by the functoriality of tensor product. If Serre's bound holds for π_1 and π_2, then*

$$\sum_C \frac{1}{|C|}\left|\pi_{C,I_1,I_2}(x) - \frac{|C|}{|G|}\mu_{ST}(I_1)\mu_{ST}(I_2)\,\mathrm{Li}(x)\right|^2 \ll \left(x^{\frac{5}{6}}n_k^{\frac{4}{3}}(\log(x\mathcal{M}(K/k)q_1q_2))^{\frac{1}{3}}\right)^2,$$

where the sum is over conjugacy classes of G, $\pi_{C,I_1,I_2}(x)$ denotes number of primes v for which $Nv \le x$, $\sigma_v = C$, and $\theta_{i,v} \in I_i = [\alpha_i, \beta_i]$, $q_i = q(\pi_i)$ is the conductor of π_i, n_k and $\mathcal{M}(K/k)$ are defined as before, and the implied constant depends only on π_1 and π_2. In particular, we have

$$\pi_{C,I_1,I_2}(x) - \frac{|C|}{|G|}\mu_{ST}(I_1)\mu_{ST}(I_2)\,\mathrm{Li}(x) \ll x^{\frac{5}{6}}|C|^{\frac{1}{2}}n_k^{\frac{4}{3}}(\log(x\mathcal{M}(K/k)q_1q_2))^{\frac{1}{3}}.$$

2. Preliminaries on L-functions

2.1. Standard automorphic L-functions

In order to study Chebotarev–Sato–Tate distributions, we shall first recall some definitions and analytic properties of standard automorphic L-functions. To do this, we closely follow the exposition in [9]. For an (irreducible unitary) cuspidal representation $\pi = \otimes_v \pi_v$ of $\mathrm{GL}_n(\mathbb{A}_k)$, where as later, $\mathbb{A}_k$ stands for the adèle ring of k, the (incomplete) L-function of π is given by

$$L(s,\pi) = \prod_{v<\infty} L(s,\pi_v),$$

for $\mathfrak{Re}(s) > 1$, where for unramified v of π, the local L-function $L(s,\pi_v)$ is defined by

$$L(s,\pi_v) = \prod_{i=1}^{n}(1-\alpha_\pi(v,i)Nv^{-s})^{-1}, \tag{2.1}$$

and $\alpha_\pi(v,i)$ are Satake parameters of π_v. For ramified v of π, one may use the Langlands parameters of π_v to define $L(s,\pi_v)$, which is the reciprocal of a polynomial, in Nv^{-s}, of degree at most n, and can be written in the form of (2.1). For each infinite place v, the local L-function $L(s,\pi_v)$ is defined by

$$L(s, \pi_v) = \prod_{i=1}^{n} \Gamma_{k_v}(s + \mu_\pi(v, i)),$$

where $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$, and $\mu_\pi(v, i)$ are Langlands parameters of π_v. Luo–Rudnick–Sarnak [25] showed that for each finite v,

$$|\alpha_\pi(v, i)| \leq Nv^{1/2-(n^2+1)^{-1}},$$

and that for each infinite place v,

$$|\mathfrak{Re}(\mu_\pi(v, i))| \leq \frac{1}{2} - \frac{1}{n^2+1}.$$

We shall note that these bounds may be improved as predicted by the Ramanujan–Petersson conjecture (denoted RPC). We will address this for Hilbert modular forms in Section 2.3, and indicate the interested reader to the beautiful articles [6] and [7] of Blomer and Brumley for a comprehensive discussion and the contemporary development. Also, the bound above implies that $L(s, \pi)$ converges on $\mathfrak{Re}(s) > 3/2 - (n^2+1)^{-1}$. The convergence of $L(s, \pi)$ on $\mathfrak{Re}(s) > 1$ is established by the Rankin–Selberg theory.

Attached to π is the contragredient $\check{\pi}$, which is also a cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_k)$, such that for every place v of k, $\check{\pi}_v$ is equivalent to the complex conjugation $\overline{\pi}_v$. In particular, one has

$$\{\alpha_{\check{\pi}}(v, i)\} = \{\overline{\alpha_\pi(v, i)}\}, \text{ and } \{\mu_{\check{\pi}}(v, i)\} = \{\overline{\mu_\pi(v, i)}\}.$$

The complete L-function of π is defined as

$$\Lambda(s, \pi) = L(s, \pi) \prod_{v|\infty} L(s, \pi_v),$$

which extends to an entire function of order 1 unless π is trivial so that $L(s, \pi)$ is the Dedekind zeta function of k; $\Lambda(s, \pi)$ satisfies the functional equation

$$\Lambda(s, \pi) = W(\pi)q(\pi)^{1/2-s}\Lambda(1 - s, \check{\pi}),$$

where $W(\pi)$ is a complex number of absolute value 1, called the root number of π, and $q(\pi)$ is the (arithmetic) conductor of π. The analytic conductor of π is defined by

$$\mathfrak{q}(t, \pi) = q(\pi) \prod_{v|\infty} \prod_{j=1}^{n} (1 + |it + \mu_\pi(v, j)|).$$

It shall be convenient to denote $\mathfrak{q}(\pi) = \mathfrak{q}(0, \pi)$.

2.2. Rankin–Selberg L-functions

Let $\pi = \otimes_v \pi_v$ and $\pi' = \otimes_v \pi'_v$ be cuspidal representations of $\mathrm{GL}_n(\mathbb{A}_k)$ and $\mathrm{GL}_{n'}(\mathbb{A}_k)$, respectively. The (incomplete) Rankin–Selberg L-function attached to π and π' is defined to be

$$L(s, \pi \times \pi') = \prod_{v<\infty} L(s, \pi_v \times \pi'_v),$$

for $\mathfrak{Re}(s) > 1$, as established in [23], where

$$L(s, \pi_v \times \pi'_v) = \prod_{i=1}^{n} \prod_{j=1}^{n'} (1 - \alpha_{\pi\times\pi'}(v, i, j) N v^{-s})^{-1},$$

with

$$|\alpha_{\pi\times\pi'}(v, i, j)| \leq N v^{1-(n^2+1)^{-1}-(n'^2+1)^{-1}}.$$

In addition, for finite unramified v of either π or π',

$$\{\alpha_{\pi\times\pi'}(v, i, j)\} = \{\alpha_\pi(v, i)\alpha_{\pi'}(v, j)\}.$$

For each infinite place v, the local L-function $L(s, \pi_v \times \pi'_v)$ is defined by

$$L(s, \pi_v \times \pi'_v) = \prod_{i=1}^{n} \prod_{j=1}^{n'} \Gamma_{k_v}(s + \mu_{\pi\times\pi'}(v, i, j)),$$

with

$$|\mathfrak{Re}(\mu_{\pi\times\pi'}(v, i, j))| \leq 1 - \frac{1}{n^2+1} - \frac{1}{n'^2+1}.$$

Also, for infinite unramified v for both π or π',

$$\{\mu_{\pi\times\pi'}(v, i, j)\} = \{\mu_\pi(v, i) + \mu_{\pi'}(v, j)\}.$$

Define the complete Rankin–Selberg L-function by

$$\Lambda(s, \pi \times \pi') = L(s, \pi \times \pi') \prod_{v|\infty} L(s, \pi_v \times \pi'_v).$$

It has been shown that $\Lambda(s, \pi\times\pi')$ extends to an meromorphic function, of order 1, over $\mathbb{C}$; from the normalisation of central characters of both π and π', it follows that $\Lambda(s, \pi\times\pi')$ is entire if and only if $\pi' \not\cong \check{\pi}$. If $\Lambda(s, \pi \times \pi')$ admits poles, then such poles are simple and at $s = 0, 1$. Also, Shahidi proved that the Rankin–Selberg L-function is non-vanishing on $\mathfrak{Re}(s) = 1$, and extended narrow zero-free regions for such L-functions are obtained by Brumley [9] (see also [21]). Furthermore, $\Lambda(s, \pi \times \pi')$ satisfies the functional equation

$$\Lambda(s, \pi \times \pi') = W(\pi \times \pi') q(\pi \times \pi')^{1/2-s} \Lambda(1-s, \check{\pi} \times \check{\pi}'),$$

where $W(\pi \times \pi')$ is the root number of $\pi \times \pi'$, a complex number of modulus 1, and $q(\pi \times \pi')$ is the conductor of $\pi \times \pi'$. The analytic conductor of $\pi \times \pi'$ is set to be

$$\mathfrak{q}(t, \pi \times \pi') = q(\pi \times \pi') \prod_{v|\infty} \prod_{j=1}^{n} \prod_{j'=1}^{n'} (1 + |it + \mu_{\pi \times \pi'}(v, j, j')|).$$

We will often write $\mathfrak{q}(\pi \times \pi') = \mathfrak{q}(0, \pi \times \pi')$.

To derive effective Chebotarev–Sato–Tate distributions (see Section 4), it is crucial to bound the conductors of Rankin–Selberg L-functions. To bound the Rankin–Selberg conductors at finite places, one may use a result of Bushnell–Henniart [11]. For the conductors at infinite places, one may invoke [21, Lemma A.2]. (We note that Brumley has commented that such a bound has been claimed and utilised without proof in many sources; [21, Lemma A.2] finally provides a proof.) From these bounds, as done in [9, Eq. (8)], it can be shown that

$$\mathfrak{q}(t, \pi \times \pi') \leq \mathfrak{q}(\pi)^{n'} \mathfrak{q}(\pi')^{n} (1 + |t|)^{nn'[k:\mathbb{Q}]}. \tag{2.2}$$

2.3. Symmetric power L-functions of Hilbert modular forms

Let k be a totally real number field, and let π be a cuspidal representation corresponding to a non-CM Hilbert modular form of k. (Recall that for any totally real number field k, if π is cuspidal and with essentially square-integrable π_∞, then π is associated to Hilbert modular forms of weights ≥ 2 at all infinite places of k, and vice versa.)

By writing the local L-function of π at finite unramified v as

$$L(s, \pi_v) = (1 - \alpha_v N v^{-s})^{-1} (1 - \beta_v N v^{-s})^{-1},$$

Langlands conjectured that for each $m \geq 2$, there exists an automorphic representation $\operatorname{Sym}^m \pi$ of $\mathrm{GL}_{m+1}(\mathbb{A}_k)$ such that

$$L(s, (\operatorname{Sym}^m \pi)_v) = \prod_{j=0}^{m} (1 - \alpha_v^j \beta_v^{m-j} N v^{-s})^{-1}.$$

(Indeed, that the local Langlands correspondence predicts that there are appropriate local L-functions $L(s, (\operatorname{Sym}^m \pi)_v)$ for all places v.) We note that for $m = 1$, $\operatorname{Sym}^1 \pi = \pi$; and we set $L(s, \operatorname{Sym}^0 \pi) = \zeta_k(s)$, the Dedekind zeta function of k. We also remark that by the works of Shimura, Gelbart–Jacquet, Shahidi, and Kim, such a functoriality conjecture is true for $m \leq 4$.

By the work of Blasius [5], rooting in Deligne's proof of the classical Ramanujan conjecture, RPC holds for π, which in particular, demands that for any unramified v,

$$|\alpha_v| = |\beta_v| = 1.$$

Also, we shall remark that Barnet-Lamb, Gee, Geraghty, and Taylor [4], in fact, showed that any irreducible, totally odd, essentially self-dual, regular, weakly compatible system of ℓ-adic representations of the absolute Galois group of a totally real number field is potentially automorphic. (This implies the potential automorphy of symmetric powers of the cuspidal automorphic representations associated to Hilbert modular forms in our consideration.) However, as we cannot afford to provide all the necessary details here, we will only consider the instance of Hilbert modular forms and indicate the reader to the references mentioned in the introduction.

For $k = \mathbb{Q}$ and π corresponding to a normalised Hecke eigenform, Serre (see [32, Section 5]) derived the bound $q(\mathrm{Sym}^m \pi) = O(q(\pi)^{am})$ for some a not depending on m. In light of this, for cuspidal π, we shall say *Serre's bound* holds for π if there is a constant a that does not depend on m such that

$$q(\mathrm{Sym}^m \pi) = O(q(\pi)^{am}). \tag{2.3}$$

This bound was used by V.K. Murty in [27]. In general, Rouse in [32, Lemma 2.1] gave a bound $q(\mathrm{Sym}^m \pi) = O(q(\pi)^{am^3})$ via Bushnell–Henniart's bound [11].

For the infinite places, if $k = \mathbb{Q}$, Serre gave the gamma factors of the symmetric power L-functions associated to the Ramanujan τ-function. In general, Moreno and Shahidi [26] derived an explicit description, which implies that

$$\mathfrak{Re}(\mu_{\mathrm{Sym}^m \pi}(i)) \leq 0, \ \ |\mu_{\mathrm{Sym}^m \pi}(i)| \leq (m+1) \max_j |\mu_\pi(j)|. \tag{2.4}$$

2.4. Artin L-functions

To end this section, we recall some properties concerning the conductors of Artin L-functions.

Let K/k be a Galois extension of number fields with Galois group G. Let χ be a character of G, and $L_v(s, \chi, K/k)$ denote the local Artin L-function attached to χ at v. For each infinite place v of k, $L_v(s, \chi, K/k)$ is defined by

$$L_v(s, \chi, K/k) = \begin{cases} \Gamma_{\mathbb{C}}(s)^{\chi(1)} & \text{if } v \text{ is complex,} \\ \Gamma_{\mathbb{R}}(s)^{n^+(\chi)} \Gamma_{\mathbb{R}}(s+1)^{n^-(\chi)} & \text{if } v \text{ is real,} \end{cases}$$

where $n^+(\chi)$ and $n^-(\chi)$ denote the dimensions of $(+)$-eigenspace and $(-)$-eigenspace of complex conjugation for χ, respectively, and hence $n^+(\chi) + n^-(\chi) = \chi(1)$.

The conductor of the Artin L-function attached to χ is defined as

$$A_\chi = d_k^{\chi(1)} N\mathfrak{f}(\chi), \tag{2.5}$$

where d_k is the absolute discriminant of k and $\mathfrak{f}(\chi)$ denotes the (global) Artin conductor of χ. By Hensel's estimate (for complete details, see [29] and [35]), one has

$$\log A_\chi = \chi(1)\log d_k + \log N\mathfrak{f}(\chi) \ll n_k \chi(1) \log \mathcal{M}(K/k), \tag{2.6}$$

where the implied constant is absolute, $\mathcal{M}(K/k)$ is defined as

$$\mathcal{M}(K/k) = n d_k^{1/n_k} \prod_{p \in P(K/k)} p, \tag{2.7}$$

$n_k = [k : \mathbb{Q}]$, $n = [K : k]$, and $P(K/k)$ denotes the set of rational primes p for which there is a prime v of k with $v|p$ so that v is ramified in K.

Finally, we remark that the Langlands reciprocity conjecture predicts that for any $\chi \in \mathrm{Irr}(G)$, there is a cuspidal π_χ of $\mathrm{GL}_{\chi(1)}(\mathbb{A}_k)$ such that

$$L(s, (\pi_\chi)_v) = L_v(s, \chi, K/k).$$

(Here, as later, $\mathrm{Irr}(G)$ stands for the set of irreducible characters of G.) For such an instance, we will often write the analytic conductor of π_χ as

$$\mathfrak{q}(t, \chi) = \mathfrak{q}(t, \pi_\chi),$$

and set $\mathfrak{q}(\chi) = \mathfrak{q}(0, \chi)$. (We note that in the language introduced in Section 2.1, $q(\pi_\chi) = A_\chi$ and each $\mu_{\pi_\chi}(v, i)$ is either 0 or 1.)

3. Chebotarev–Sato–Tate distributions

Let k be a number field. In light of the potential automorphy discussed previously, we consider the following automorphy hypotheses concerning representations related to a given Galois representation ρ of the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$ of k. Let π be a representation, of degree n, obtained from ρ by a representation-theoretic operation (such as symmetric or exterior powers) and satisfy the following hypotheses.

- (*Potential automorphy*, denoted $\mathrm{PA}(L, k)$): there exists a (finite) Galois extension L/k so that $\pi|_L$ is associated to a cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_L)$.
- (*Automorphic descent*, denoted $\mathrm{AD}(L, k)$): if L/k is a Galois extension and $\pi|_L$ is associated to a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_L)$, then for any solvable Galois subextension L/F, containing k, one knows how to associate $\pi|_F$ with a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$.

Under the above automorphy hypotheses, we have the following proposition.

Proposition 3.1. *Let K/k be a Galois extension of number fields with Galois group G. Let ρ_1 and ρ_2 be irreducible Galois representations of $\mathrm{Gal}(\overline{k}/k)$. Assume there is a finite Galois extension L/K such that $\mathrm{Sym}^{m_1}\rho_1$ and $\mathrm{Sym}^{m_2}\rho_2$ satisfy PA(L,k) and AD(L,k). Then for every $\chi \in \mathrm{Irr}(G)$ and idèle class character ψ of k, the L-functions $L(s, \mathrm{Sym}^{m_1}\rho_1 \otimes \chi \otimes \psi)$ and $L(s, \mathrm{Sym}^{m_1}\rho_1 \otimes \mathrm{Sym}^{m_2}\rho_2 \otimes \chi \otimes \psi)$ extend to meromorphic functions over $\mathbb{C}$, which are holomorphic and non-vanishing on $\mathfrak{Re}(s) \geq 1$, with a possible pole at $s = 1$ that appears only if $\mathrm{Sym}^{m_2}\rho_2$ and the contragredient of $\mathrm{Sym}^{m_1}\rho_1$ (after certain base change) are twist-equivalent.*

Proof. The proof makes a use of the "Brauer–Taylor reduction" as follows. Observing L/k is a Galois extension and regrading $\chi \in \mathrm{Irr}(G)$ as a character of $\widetilde{G} := \mathrm{Gal}(L/k)$, the Brauer induction theorem asserts that

$$\chi = \sum_i n_i \,\mathrm{Ind}_{H_i}^{\widetilde{G}} \psi_i,$$

where for each i, $n_i \in \mathbb{Z}$, H_i is a nilpotent subgroup of $\widetilde{G}$, and ψ_i is an abelian character of H_i. By Artin reciprocity, each ψ_i can be seen as an idèle class character of L^{H_i}. Furthermore, the automorphic descent hypothesis AD(L,k) tells us that $(\mathrm{Sym}^{m_1}\rho_1)|_{L^{H_i}}$ and $(\mathrm{Sym}^{m_2}\rho_2)|_{L^{H_i}}$ are cuspidal over L^{H_i} (since L/L^{H_i} is nilpotent and hence solvable).

By the base change of GL(1), $\psi|_{L^{H_i}}$ is an idèle class character of L^{H_i} and so is $\phi_i := \psi_i \otimes \psi|_{L^{H_i}}$. Now the functoriality of $\mathrm{GL}(n)\times\mathrm{GL}(1)$ tells us that $(\mathrm{Sym}^{m_1}\rho_1)|_{L^{H_i}} \otimes \phi_i$ is cuspidal over L^{H_i} for every i. Thus, it follows from the Rankin–Selberg theory that

$$L(s, (\mathrm{Sym}^{m_1}\rho_1)|_{L^{H_i}} \otimes \phi_i), \;\; L(s, (\mathrm{Sym}^{m_1}\rho_1)|_{L^{H_i}} \otimes (\mathrm{Sym}^{m_2}\rho_2)|_{L^{H_i}} \otimes \phi_i)$$

are holomorphic and non-vanishing on $\mathfrak{Re}(s) \geq 1$ except for a possible pole at $s = 1$ that appears only if $(\mathrm{Sym}^{m_2}\rho_2)|_{L^{H_i}} \otimes \phi_i$ is the contragredient of $(\mathrm{Sym}^{m_1}\rho_1)|_{L^{H_i}}$ for some i. $\square$

We note that the automorphy hypotheses above cover the consideration ℓ-adic representations (and their symmetric powers) arising from non-CM elliptic curves defined over totally real number fields and Hilbert modular forms (see, e.g., [2–4]). Also, as mentioned in Section 2.2, the non-vanishing of Rankin–Selberg L-functions on $\mathfrak{Re}(s) = 1$ is derived by Shahidi. We further remark that if ρ_1 and ρ_2 arise from non-CM Hilbert modular forms that are not twist-equivalent, then the only pole at $s = 1$ appears when the corresponding L-function is $\zeta_k(s)$, the Dedekind zeta function of k (cf. [30, Theorem 2] and [31, Section 5]). Herein, following Serre [34], one can apply the Wiener–Ikehara Tauberian theorem to deduce Theorem 1.1, immediately. However, for the benefit of the reader, we sketch the argument below.

Proof of Theorem 1.1. Recall that π_1 and π_2 are (corresponding to) non-CM Hilbert modular forms of k that are not twist-equivalent. We fix a Galois extension K/k of totally real number fields with Galois group $G_1 \times G_2$ such that K^{G_1}/k is totally real and G_1 is abelian. (We again note that each π_i is associated to a compatible system of ℓ-adic representations.) From Serre's formalism, it suffices to show that for any $\chi \in \mathrm{Irr}(\mathrm{Gal}(K^{G_1}/k))$, $\psi \in \mathrm{Irr}(\mathrm{Gal}(K^{G_2}/k))$, and $m \in \mathbb{N}$, the L-function

$$L(s, \mathrm{Sym}^m \pi_1 \otimes \chi \otimes \psi)$$

is holomorphic and non-vanishing on $\mathfrak{Re}(s) \geq 1$ except for the case that both χ and ψ are trivial and $m = 0$, i.e., the Dedekind zeta function of k.

Now as $\mathrm{Gal}(K^{G_2}/k) = G_1$ is abelian, Artin reciprocity asserts that each ψ can be seen as an idèle class character of k. Also, since K^{G_1}/k is totally real, for every m, there is a totally real Galois extension L_m/k such that $(\mathrm{Sym}^m \pi_1)|_{L_m}$ is automorphic and cuspidal over L_m (we remark that one may choose L_m such that L_m/K^{G_1} is Galois by an analogous argument used in [44, Section 2], which is, in fact, a consequence of [4, Theorem 4.5.1], and can be applied not only for Hilbert modular forms but also for a more general family of (finite collections of) ℓ-adic representations). By the work of Barnet-Lamb, Gee, Geraghty, and Taylor [2,4] (together with Arthur–Clozel's base change), $(\mathrm{Sym}^m \pi_1)|_F$ is cuspidal whenever L_m/F is solvable and Galois. Thus, applying Proposition 3.1 to K^{G_1}/k and ψ completes the proof for this case.

For the pair (π_1, π_2), we then consider the L-function

$$L(s, \mathrm{Sym}^{m_1} \pi_1 \otimes \mathrm{Sym}^{m_2} \pi_2 \otimes \chi \otimes \psi).$$

Again, from a similar argument as used in [44, Section 2], one can deduce the simultaneously potential automorphy of $\mathrm{Sym}^{m_1} \pi_1$ and $\mathrm{Sym}^{m_2} \pi_2$ over a totally real L_{m_1,m_2} from the results of Barnet-Lamb et al. (i.e., $(\mathrm{Sym}^{m_1} \pi_1)|_{L_{m_1,m_2}}$ and $(\mathrm{Sym}^{m_2} \pi_2)|_{L_{m_1,m_2}}$ are cuspidal over L_{m_1,m_2}), and again each $(\mathrm{Sym}^{m_i} \pi_i)|_F$ is cuspidal whenever $L_{m_1,m_2}/F$, containing k, is solvable. Hence, we conclude the proof by applying Proposition 3.1. □

We shall first remark that although referring to as the potential automorphy, what was really established is the *potential cuspidality* for the symmetric powers. Also, from the above reduction argument, it follows that the corresponding L-functions satisfy the expected functional equations.

Secondly, in [28], the authors established a similar Chebotarev–Sato–Tate distribution for pairs of elliptic curves (E_1, E_2) by assuming that E_2 is of CM. Indeed, they applied the potential automorphy result, established in [16,38], to deduce the potential automorphy for the *odd* symmetric powers of ℓ-adic representations ρ arising from E_1, and then invoked the Clebsch–Gordan branching rule for SL_2, i.e.,

$$\mathrm{Sym}^{m-1} \rho \oplus \mathrm{Sym}^{m+1} \rho = \mathrm{Sym}^m \rho \otimes \mathrm{Sym}^1 \rho,$$

to handle the *even* symmetric powers. As the Rankin–Selberg theory for quadruple products of automorphic representations is still unknown, the authors [28] were forced to make the CM-assumption. Nevertheless, as now the (simultaneously) potential automorphy is valid for *all* the symmetric powers of the representations in our consideration (in other words, the construction of L_{m_1,m_2} above became possible), the CM-assumption can be dropped. Indeed, this was also pointed out by Harris [19, pp. 716–719]. (We shall note that as remarked in the abstract of the article, [19, Theorems 2.4 and 2.5] are no longer conditional, which reflects the notable progress that has been made.)

Thirdly, comparing our result with Murty–Murty's version, it may be noticed that the solvability assumption on the Galois extensions has been removed. We shall briefly explain this refinement for the case of the ℓ-adic representations ρ arising from elliptic curves over a totally real number field k as they were considered in [28]. Assume M/k is a Galois extension of number fields. In the argument of [28], the authors first applied the automorphy result from [16,38] to obtain a field L so that $\rho|_L$ is automorphic and L/k is finite and Galois. Under further the assumption on the solvability of M/k, Murty–Murty used the base change of Arthur–Clozel to deduce the automorphy of $\rho|_{LM}$. In contrast, from the argument of Virdol [44] as used previously, we instead choose L, *containing* M, such that $\rho|_L$ is automorphic, which relies on the assumption that M is totally real. (Note that M here corresponds to K^{G_1} in the argument above.) Herein, we avoid the use of Arthur–Clozel's base change, which requires the solvability of M/k.

Nonetheless, the solvability is still used subtly and implicitly as follows. Note that in our argument, we used the Brauer induction theorem to decompose $\chi \in \mathrm{Irr}(\mathrm{Gal}(L/k))$ into the 1-dimensional representations ψ_i over L^{H_i} for some nilpotent $H_i \leq \mathrm{Gal}(L/k)$. It is crucial that the nilpotency of H_i implies that H_i is solvable, and hence L/L^{H_i} is a solvable Galois extension, which allows one to complete the reduction argument by invoking Arthur–Clozel's base change to deduce the automorphy of $\rho|_{L^{H_i}}$ from $\rho|_L$.

Finally, recalling that $\mathrm{Gal}(K/k) = G_1 \times G_2$, we shall address the assumption that $G_1 = \mathrm{Gal}(K^{G_2}/k)$ is abelian. Note that any irreducible character of $G_1 \times G_2$ is of the form $\chi \otimes \psi$ for some $\chi \in \mathrm{Irr}(G_2)$ and $\psi \in \mathrm{Irr}(G_1)$. As discussed above, we used the reduction argument to handle the Galois representations arising from $G_2 = \mathrm{Gal}(K^{G_1}/k)$, where the total reality of K^{G_1}/k was utilised. Now as G_1 is abelian, every ψ is 1-dimensional, which by Artin reciprocity, can be regarded as an idèle class character over k. Hence, we can write the L-functions, considered in the proof of Theorem 1.1, in the form required by Proposition 3.1.

4. Effective versions

In this section, we will give two effective versions of the Chebotarev–Sato–Tate distributions under blessings of the "Langlands philosophy" and the generalised Riemann hypothesis. (We remind the reader that RPC is established for Hilbert modular forms by Blasius [5].) As shall be seen below, apart from controlling the analytic conductors of Rankin–Selberg convolutions, the proofs do follow the strategies developed in [10,27,29].

Nonetheless, for the sake of completeness and concept clarity, we shall emphasise the critical steps for the proof of Theorem 1.2 and sketch the proof for the second instance. Furthermore, as pointed out by the anonymous referee, to control the analytic conductors of Rankin–Selberg convolutions in the consideration, we do assume the cuspidality and functoriality of the involved representations (cf. Section 2.2).

Now we are in a position to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Let K/k be a Galois extension of totally real number fields with Galois group G. Let π be a cuspidal representation, arising from a non-CM Hilbert modular form of k, with conductor $q = q(\pi)$. Furthermore, we suppose that all the irreducible characters χ of G and all the symmetric powers $\operatorname{Sym}^m \pi$ of π are cuspidal over k, and that each $L(s, \operatorname{Sym}^m \pi \otimes \chi)$ satisfies GRH. Also, we assume Serre's bound holds for π, namely, $q(\operatorname{Sym}^m \pi) = O(q^{am})$, for some a independent of m.

Setting $D(\chi, m) = (m+1)\chi(1)[k : \mathbb{Q}]^2$ and assuming GRH, the standard analytic number theory machinery (see, e.g., [22, Theorem 5.15]) yields that

$$\sum_{Nv \le x} \chi(\sigma_v) \sum_{j=0}^{m} \alpha_v^j \beta_v^{m-j} \log Nv = \delta(m,\chi)x + O\left(x^{\frac{1}{2}}(\log x)\log(x^{D(\chi,m)}\mathfrak{q}(\chi \otimes \operatorname{Sym}^m \pi))\right),$$

where the implied constant is absolute, $\delta(m,\chi) = 1$ if $m = 0$ and χ is trivial, and $\delta(m,\chi) = 0$ otherwise. Now (2.2), combined with Serre's bound, (2.4) and (2.6), gives

$$\mathfrak{q}(\chi \otimes \operatorname{Sym}^m \pi) \le \mathfrak{q}(\chi)^{m+1}\mathfrak{q}(\operatorname{Sym}^m \pi)^{\chi(1)} \ll A_\chi^{(m+1)} q^{am\chi(1)} m^{D(\chi,m)},$$

where A_χ is the conductor of χ as defined in (2.5). Thus, we have

$$\sum_{Nv \le x} \chi(\sigma_v) \sum_{j=0}^{m} \alpha_v^j \beta_v^{m-j} = \delta(m,\chi)\operatorname{Li}(x) + O\left(x^{\frac{1}{2}} m\chi(1)[k:\mathbb{Q}]^2 \log(xA_\chi qm)\right). \quad (4.1)$$

Following V.K. Murty [27] (see also [10]), we need the following result of Vinogradov [43, Lemma 12]. For any $A, B \in \mathbb{R}$, and δ with $\delta \in (0, \frac{1}{2})$ and $\delta \le B - A \le 1 - \delta$, there is a continuous function $D = D_{A,B}$ on $\mathbb{R}$ of period 1 satisfying the following properties.

- $D(x) = 1$ on $[A + \frac{\delta}{2}, B - \frac{\delta}{2}]$.
- $D(x) = 0$ on $[B + \frac{\delta}{2}, 1 + A - \frac{\delta}{2}]$.
- $0 \le D(x) \le 1$ in the rest of interval.
- The Fourier expansion of $D(x)$ is

$$B - A + \sum_{m \ge 1} (a_m \cos 2\pi m x + b_m \sin 2\pi m x),$$

where $|a_m|$, $|b_m| \le \min\left\{2(B-A), \frac{2}{m\pi}, \frac{2}{m\pi}\left(\frac{r}{m\pi\delta}\right)^r\right\}$ for all $m \ge 1$. Consider the even function $F_{A,B}(\theta) = D(\frac{\theta}{2\pi}) + D(\frac{-\theta}{2\pi})$, which is of period 2π and hence of the form

$$F_{A,B}(\theta) = \sum_{m\in\mathbb{Z}} c_{m,A,B}e^{im\theta},$$

where $c_{0,A,B} = 2a_0 = 2(B-A)$ and $c_{m,A,B} = c_{-m,A,B} = a_m$ for all $m \geq 1$. In addition, by taking $r = 1$, one has, for any integer $M \geq 3$,

$$F_{A,B}(\theta) = \sum_{|m|\leq M} c_{m,A,B}e^{im\theta} + O\left(\frac{1}{\delta M}\right).$$

Furthermore, recalling the characters of SU(2) are of the form

$$\chi_k(\theta) = \sum_{j=0}^{k} e^{i\theta(k-2j)},$$

one has

$$F_{A,B}(\theta) = (c_{0,A,B} - c_{2,A,B}) + \sum_{m=1}^{M-2}(c_{m,A,B} - c_{m+2,A,B})\chi_m(\theta) + O\left(\frac{1}{\delta M}\right).$$

Also, the fourth part of Vinogradov's lemma implies

$$\sum_{m=1}^{M-2} m|c_{m,A,B} - c_{m+2,A,B}| = O\left(\frac{\log M}{\delta}\right).$$

Recalling that $\alpha_v = e^{i\theta_v}$ and $\beta_v = e^{-i\theta_v}$, and applying the above estimate together with (4.1), we then have

$$\begin{aligned}\sum_{Nv\leq x}\chi(\sigma_v)F_{A,B}(\theta_v) = &\left((c_{0,A,B} - c_{2,A,B}) + O\left(\frac{1}{\delta M}\right)\right)\delta(\chi)\,\mathrm{Li}(x)\\ &+ O\left(\frac{\log M}{\delta}x^{\frac{1}{2}}\chi(1)[k:\mathbb{Q}]^2\log(xA_\chi qM)\right),\end{aligned}$$

where $\delta(\chi) = \delta(0,\chi)$. As mentioned in [10], the indicator function χ_I of the interval $I = [2\pi\alpha, 2\pi\beta]$ is bounded from above by $F_{\alpha-\frac{\delta}{2},\beta+\frac{\delta}{2}}$ and from below by $F_{\alpha+\frac{\delta}{2},\beta-\frac{\delta}{2}}$. Also, the quantities $c_{0,\alpha-\frac{\delta}{2},\beta+\frac{\delta}{2}} - c_{2,\alpha-\frac{\delta}{2},\beta+\frac{\delta}{2}}$ and $c_{0,\alpha+\frac{\delta}{2},\beta-\frac{\delta}{2}} - c_{2,\alpha+\frac{\delta}{2},\beta-\frac{\delta}{2}}$ differ from $\mu_{ST}(I)$ by $O(\delta)$. Choosing $M = \lceil\delta^{-2}\rceil$, we now have

$$\begin{aligned}\sum_{Nv\leq x}\chi(\sigma_v)\chi_I(\theta_v) = &(\mu_{ST}(I) + O(\delta))\delta(\chi)\,\mathrm{Li}(x)\\ &+ O\left(\frac{\log\delta^{-2}}{\delta}x^{\frac{1}{2}}\chi(1)[k:\mathbb{Q}]^2\log(xA_\chi q\delta^{-2})\right).\end{aligned}$$

Balancing the error terms by setting $\delta = x^{\frac{-1}{4}}\chi(1)^{\frac{1}{2}}[k:\mathbb{Q}](\log x)(\log(xA_\chi q))^{\frac{1}{2}}$, we obtain

$$\sum_{Nv\le x}\chi(\sigma_v)\chi_I(\theta_v) = \mu_{ST}(I)\delta(\chi)\operatorname{Li}(x) + O\left(x^{\frac{3}{4}}\chi(1)^{\frac{1}{2}}[k:\mathbb{Q}](\log(xA_\chi q))^{\frac{1}{2}}\right). \tag{4.2}$$

To obtain the final estimate, we borrow a lemma from [29, Proposition 1.3] stating

$$\sum_C \frac{1}{|C|}\left|\pi(x,\delta_C) - \frac{|C|}{|G|}\pi(x,1_G)\right|^2 = \frac{1}{|G|}\sum_{\chi\ne 1_G}|\pi(x,\chi)|^2, \tag{4.3}$$

where δ_C is the indicator function of C, the sum on the left is over conjugacy classes C of G, and the sum on the right is over the non-trivial irreducible characters of G. (Here, for any class functions h of G, we set $\pi(x,h) = \sum_{Nv\le x} h(\sigma_v)\chi_I(\theta_v)$.)

Now applying the Cauchy–Schwarz inequality and (4.3), we have

$$\begin{aligned}\sum_C \frac{1}{|C|}\left|\pi(x,\delta_C) - \frac{|C|}{|G|}\ell(x)\right|^2 &\le \sum_C \frac{2}{|C|}\left|\pi(x,\delta_C) - \frac{|C|}{|G|}\pi(x,1_G)\right|^2 \\ &\quad + \sum_C \frac{2}{|C|}\left|\frac{|C|}{|G|}\pi(x,1_G) - \frac{|C|}{|G|}\ell(x)\right|^2 \\ &= \frac{2}{|G|}\sum_{\chi\ne 1_G}|\pi(x,\chi)|^2 + \frac{2}{|G|}(\pi(x,1_G)-\ell(x))^2 \\ &\ll \frac{1}{|G|}\sum_{\chi\in\operatorname{Irr}(G)}\chi(1)^2\left(x^{\frac{3}{4}}[k:\mathbb{Q}]^{\frac{3}{2}}(\log(x\mathcal{M}(K/k)q))^{\frac{1}{2}}\right)^2,\end{aligned}$$

where $\ell(x) = \mu_{ST}(I)\operatorname{Li}(x)$, $\mathcal{M}(K/k)$ is defined as in (2.7), and the last estimate follows from the estimates (2.6) and (4.2). Finally, by recalling that $\sum_{\chi\in\operatorname{Irr}(G)}\chi(1)^2 = |G|$, we conclude the proof. □

To end this section, we give a sketch of the proof for Theorem 1.3.

Proof of Theorem 1.3. Let π_1 and π_2 be cuspidal representations, arising from non-CM Hilbert modular forms of k that are non-twist-equivalent, with conductors $q_1 = q(\pi_1)$ and $q_2 = q(\pi_2)$, respectively. We assume that all the irreducible characters χ of $G = \operatorname{Gal}(K/k)$ and all the symmetric powers $\operatorname{Sym}^{m_i}\pi_i$ of each π_i are cuspidal over k, and that each $L(s, \operatorname{Sym}^{m_1}\pi_1\otimes\operatorname{Sym}^{m_2}\pi_2\otimes\chi)$ satisfies GRH and the analytic properties predicted by the functoriality of tensor product. In addition, we assume Serre's bounds for π_1 and π_2.

As before, let $\theta_{1,v}$ and $\theta_{2,v}$ be Frobenius angles of π_1 and π_2 at v, respectively, and let σ_v denote the Frobenius element at v. Also, one can bound the analytic conductor of $\chi\otimes\operatorname{Sym}^{m_1}\pi_1\otimes\operatorname{Sym}^{m_2}\pi_2$ as

$$\mathfrak{q}(\chi \otimes \mathrm{Sym}^{m_1} \pi_1 \otimes \mathrm{Sym}^{m_2} \pi_2) \ll A_\chi^{(m_1+1)(m_2+1)} q_1^{a_1 m_1 (m_2+1)\chi(1)} q_2^{a_2 m_2 (m_1+1)\chi(1)} (m_1+m_2)^{D(\chi,m_1,m_2)},$$

for some a_1 and a_2, where $D(\chi, m_1, m_2) = 2(m_1+1)(m_2+1)\chi(1)[k:\mathbb{Q}]^3$. Thus,

$$\sum_{Nv \leq x} \chi(\sigma_v) \sum_{j=0}^{m_1} \alpha_{1,v}^{j} \beta_{1,v}^{m_1-j} \sum_{j'=0}^{m_2} \alpha_{2,v}^{j'} \beta_{2,v}^{m_2-j'} \ll \delta(m_1,m_2,\chi)\,\mathrm{Li}(x) + O\left(x^{\frac{1}{2}} (m_1+1)(m_2+1)\chi(1)[k:\mathbb{Q}]^3 \log(xA_\chi q_1 q_2 (m_1+m_2))\right),$$

where $\delta(m_1, m_2, \chi) = 1$ if $m_1 = m_2 = 0$ and χ is trivial, and $\delta(m, \chi) = 0$ otherwise. Under the same use of notation as before, Vinogradov's lemma (with $r = 1$) yields that

$$F_{A_1,B_1}(\theta_1) F_{A_2,B_2}(\theta_2) = \sum_{m_1,m_2=0}^{M} d_{m_1,A_1,B_1} d_{m_2,A_2,B_2} \chi_{m_1}(\theta_1) \chi_{m_2}(\theta_2) + O\left(\frac{1}{\delta^2 M}\right),$$

where $d_{m_i,A_i,B_i} = c_{m_i,A_i,B_i} - c_{m_i+2,A_i,B_i}$; the fourth part of Vinogradov's lemma gives

$$\sum_{\substack{m_1,m_2 \\ (m_1,m_2)\neq(0,0)}}^{M-2} m_1 m_2 |d_{m_1,A_1,B_1} d_{m_2,A_2,B_2}| = O\left(\frac{(\log M)^2}{\delta^2}\right).$$

By a similar argument as in the previous proof, we have

$$\sum_{Nv \leq x} \chi(\sigma_v) F_{A_1,B_1}(\theta_{v_1}) F_{A_2,B_2}(\theta_{v_2}) = \left(d_{0,A_1,B_1} d_{0,A_2,B_2} + O\left(\frac{1}{\delta^2 M}\right)\right) \delta(\chi)\,\mathrm{Li}(x) + O\left(\frac{(\log M)^2}{\delta^2} x^{\frac{1}{2}} \chi(1)[k:\mathbb{Q}]^3 \log(xA_\chi q_1 q_2 M)\right),$$

where $\delta(\chi) = \delta(0, 0, \chi)$. Moreover, balancing everything by taking

$$\delta = x^{\frac{-1}{6}} [k:\mathbb{Q}] \chi^{\frac{1}{3}}(1) (\log x) (\log(xA_\chi q_1 q_2))^{\frac{1}{3}}, \quad M = \lceil \delta^{-3} \rceil,$$

it follows that

$$\sum_{Nv \leq x} \chi(\sigma_v) \chi_{I_1}(\theta_{1,v}) \chi_{I_2}(\theta_{2,v}) = \mu_{ST}(I_1) \mu_{ST}(I_2) \delta(\chi)\,\mathrm{Li}(x) + O\left(x^{\frac{5}{6}} [k:\mathbb{Q}] \chi^{\frac{1}{3}}(1) (\log(xA_\chi q_1 q_2))^{\frac{1}{3}}\right).$$

Finally, applying [29, Proposition 1.3], and observing $\frac{1}{|G|}\sum_{\chi\in\mathrm{Irr}(G)} \chi(1)^{\frac{4}{3}} \leq 1$, we obtain

$$\sum_{C} \frac{1}{|C|} \left| \pi_{C,I_1,I_2}(x) - \frac{|C|}{|G|} \mu_{ST}(I_1)\mu_{ST}(I_2) \operatorname{Li}(x) \right|^2 \ll \left(x^{\frac{5}{6}} n_k^{\frac{4}{3}} (\log(x\mathcal{M}(K/k)q_1q_2))^{\frac{1}{3}} \right)^2,$$

where $\pi_{C,I_1,I_2}(x)$, n_k, and $\mathcal{M}(K/k)$ are defined as before. $\square$

5. Final remarks

We first remark that subject to the modularity conjecture our Chebotarev–Sato–Tate theorems cover the cases of elliptic curves defined over any totally real number field k, which is the case whenever $k = \mathbb{Q}$ (by Wiles and Breuil–Conrad–Diamond–Taylor) and k is a real quadratic field (by Freitas, Le Hung, and Siksek [17]).

We also note that one may adapt the method developed in [33] to obtain an extra log-saving for Theorems 1.2 and 1.3. Furthermore, it would be interesting to obtain error terms for the Chebotarev–Sato–Tate distributions unconditionally. However, as the fields obtained by the current potential automorphy result are ineffective, it seems difficult to move forward in this direction at present. Nevertheless, under the Langlands functoriality conjecture, it shall be possible to establish effective Chebotarev–Sato–Tate distributions by the standard analytic machinery as in [42].

Last but not least, as the proofs of Theorems 1.2 and 1.3 require (conjectural) Serre's bound on the conductors of symmetric powers of GL(2)-forms, it may be of interest to improve the bound obtained by Rouse [32], and to extend Serre's argument to Hilbert modular forms. We shall reserve these directions for future study.

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References

[1] J. Arthur, L. Clozel, Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula, Ann. of Math. Studies, vol. 120, Princeton Univ. Press, 1990.
[2] T. Barnet-Lamb, T. Gee, D. Geraghty, The Sato–Tate conjecture for Hilbert modular forms, J. Amer. Math. Soc. 24 (2) (2011) 411–469.
[3] T. Barnet-Lamb, D. Geraghty, M. Harris, R. Taylor, A family of Calabi–Yau varieties and potential automorphy II, Publ. Res. Inst. Math. Sci. 47 (2011) 29–98.
[4] T. Barnet-Lamb, T. Gee, D. Geraghty, R. Taylor, Potential automorphy and change of weight, Ann. Math. 179 (2014) 501–609.
[5] D. Blasius, Hilbert modular forms and the Ramanujan conjecture, in: Noncommutative Geometry and Number Theory, in: Aspects Math., vol. E37, Friedr. Vieweg, Wiesbaden, 2006, pp. 35–56.
[6] V. Blomer, F. Brumley, On the Ramanujan conjecture over number fields, Ann. of Math. 174 (2011) 581–605.
[7] V. Blomer, F. Brumley, The role of the Ramanujan conjecture in analytic number theory, Bull. Amer. Math. Soc. (N.S.) 50 (2) (2013) 267–320.

[8] R. Brauer, On Artin's L-series with general group characters, Ann. Math. 48 (1947) 502–514.

[9] F. Brumley, Effective multiplicity one on GL_n and narrow zero-free regions for Rankin–Selberg L-functions, Amer. J. Math. 128 (6) (2006) 1455–1474.

[10] A. Bucur, K.S. Kedlaya, An application of the effective Sato–Tate conjecture, in: Frobenius Distributions on Curves, in: Contemporary Mathematics, Amer. Math. Soc., Providence, RI, 2015.

[11] C.J. Bushnell, G. Henniart, An upper bound on conductors for pairs, J. Number Theory 65 (2) (1997) 183–196.

[12] L. Clozel, The Sato–Tate conjecture, Curr. Dev. Math. 2006 (2008) 1–34.

[13] L. Clozel, J.A. Thorne, Level-raising and symmetric power functoriality, I, Compos. Math. 150 (2014) 729–748.

[14] L. Clozel, J.A. Thorne, Level raising and symmetric power functoriality, II, Ann. of Math. (2) 181 (2015) 303–359.

[15] L. Clozel, J.A. Thorne, Level-raising and symmetric power functoriality, III, Duke Math. J. 166 (2017) 325–402.

[16] L. Clozel, M. Harris, R. Taylor, Automorphy for some l-adic lifts of automorphic mod l representations, Publ. Math. Inst. Hautes Études Sci. 108 (2008) 1–181.

[17] N. Freitas, B.V. Le Hung, S. Siksek, Elliptic curves over real quadratic fields are modular, Invent. Math. 201 (2015) 159–206.

[18] S. Gelbart, H. Jacquet, A relation between automorphic representations of GL(2) and GL(3), Ann. Sci. Éc. Norm. Supér. 11 (1978) 471–542.

[19] M. Harris, Galois representations, automorphic forms, and the Sato–Tate conjecture, Indian J. Pure Appl. Math. 45 (5) (2014) 707–746.

[20] M. Harris, N. Shepherd-Barron, R. Taylor, A family of Calabi–Yau varieties and potential automorphy, Ann. Math. 171 (2010) 779–813.

[21] P. Humphries, Standard zero-free regions for Rankin–Selberg L-functions via sieve theory, with an appendix by Farrell Brumley, arXiv:1703.05450v3.

[22] H. Iwaniec, E. Kowalski, Analytic Number Theory, Amer. Math. Soc. Colloq. Publ., vol. 53, Amer. Math. Soc., Providence, RI, 2004.

[23] H. Jacquet, I. Piatetski-Shapiro, J. Shalika, Rankin–Selberg convolutions, Amer. J. Math. 105 (1983) 367–464.

[24] H. Kim, Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2, with appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak, J. Amer. Math. Soc. 16 (1) (2003) 139–183.

[25] W. Luo, Z. Rudnick, P. Sarnak, On the generalized Ramanujan conjecture for $\mathrm{GL}(n)$, in: Automorphic Forms, Automorphic Representations, and Arithmetic, in: Proc. Sympos. Pure Math., Part 2, vol. 66, Amer. Math. Soc., Providence, RI, 1999, pp. 301–310.

[26] C.J. Moreno, F. Shahidi, The L-functions $L(s, \mathrm{Sym}^m(r), \pi)$, Canad. Math. Bull. 28 (4) (1985) 405–410.

[27] V.K. Murty, Explicit formulae and the Lang–Trotter conjecture, Rocky Mountain J. Math. 15 (2) (1985) 535–551.

[28] M.R. Murty, V.K. Murty, The Sato–Tate conjecture and generalizations, in: Current Trends in Science: Platinum Jubilee Special, Indian Academy of Sciences, 2009, pp. 639–646.

[29] M.R. Murty, V.K. Murty, N. Saradha, Modular forms and the Chebotarev density theorem, Amer. J. Math. 110 (1988) 253–281.

[30] C.S. Rajan, On strong multiplicity one for ℓ-adic representations, Int. Math. Res. Not. 1998 (1998) 161–172.

[31] C.S. Rajan, Recovering modular forms and representations from tensor and symmetric powers, in: R. Tandon (Ed.), Algebra and Number Theory, Proceedings of the Silver Jubilee Conference, University of Hyderabad, Hindustan Book Agency, 2005, pp. 281–298.

[32] J. Rouse, Atkin–Serre type conjectures for automorphic representations on GL(2), Math. Res. Lett. 14 (2) (2007) 189–204.

[33] J. Rouse, J. Thorner, The explicit Sato–Tate conjecture and densities pertaining to Lehmer-type questions, Trans. Amer. Math. Soc. 369 (5) (2017) 3575–3604.

[34] J.-P. Serre, Abelian ℓ-Adic Representations and Elliptic Curves, W.A. Benjamin, 1968.

[35] J.-P. Serre, Quelques applications du théorème de densité de Chebotarev, Publ. Math. Inst. Hautes Études Sci. 54 (1981) 123–201.

[36] F. Shahidi, Third symmetric power L-functions for GL(2), Compos. Math. 70 (1989) 245–273.

[37] G. Shimura, On the holomorphy of certain Dirichlet series, Proc. Lond. Math. Soc. 31 (1975) 79–98.

[38] R. Taylor, Automorphy for some l-adic lifts of automorphic mod l representations. II, Publ. Math. Inst. Hautes Études Sci. 108 (2008) 183–239.
[39] J.A. Thorne, On the automorphy of l-adic Galois representations with small residual image, with an appendix by R. Guralnick, F. Herzig, R. Taylor, and J. Thorne, J. Inst. Math. Jussieu 11 (2012) 855–920.
[40] J.A. Thorne, Raising the level for GL_n, Forum Math. Sigma 2 (2014) e16.
[41] J.A. Thorne, Automorphy lifting for residually reducible l-adic Galois representations, J. Amer. Math. Soc. 28 (2015) 785–870.
[42] J. Thorner, The error term in the Sato–Tate conjecture, Arch. Math. 103 (2) (2014) 147–156.
[43] I.M. Vinogradov, The Method of Trigonometrical Sums in the Theory of Numbers, Interscience, London, 1947.
[44] C. Virdol, Base change and the Birch and Swinnerton–Dyer conjecture, Funct. Approx. Comment. Math. 46 (2) (2012) 189–194.