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RESEARCH

Almost all primes satisfy the Atkin–Serre conjecture and are not extremal

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Abstract

Let $f(z) = \sum_{n=1}^{\infty} a_f(n)e^{2\pi inz}$ be a non-CM holomorphic cuspidal newform of trivial nebentypus and even integral weight $k \geq 2$. Deligne's proof of the Weil conjectures shows that $|a_f(p)| \leq 2p^{\frac{k-1}{2}}$ for all primes p. We prove for 100% of primes p that $2p^{\frac{k-1}{2}} \log\log p/\sqrt{\log p} < |a_f(p)| < \lfloor 2p^{\frac{k-1}{2}} \rfloor$. Our proof gives an effective upper bound for the size of the exceptional set. The lower bound shows that the Atkin–Serre conjecture is satisfied for 100% of primes, and the upper bound shows that $|a_f(p)|$ is as large as possible (i.e., p is extremal for f) for 0% of primes. Our proofs use the effective form of the Sato–Tate conjecture proved by the second author, which relies on the recent proof of the automorphy of the symmetric powers of f due to Newton and Thorne.

1 Introduction and statement of main results

In this note, we study properties of Fourier coefficients of newforms. A *newform* f of weight k, level q, and trivial nebentypus, given by

$$f(z) = \sum_{n=1}^{\infty} a_f(n)e^{2\pi inz} \in S_k^{\text{new}}(\Gamma_0(q)), \tag{1}$$

is a normalized cusp form (so that $a_f(1) = 1$) that is an eigenform of all the Hecke operators and all of the Atkin–Lehner involutions $|_k W(q)$ and $|_k W(Q_p)$ for each prime $p|q$ (see [9, Sect. 2.5]). Throughout, we assume that f is non-CM, i.e., there is no imaginary quadratic field K with the property that p is inert in K if and only if $a_f(p) = 0$ (for $p \nmid q$).

The study of the Fourier coefficients $a_f(n)$ of newforms is a central topic in the theory of modular forms. A key motivating case arises from newforms associated to non-CM elliptic curves via modularity. Indeed, any elliptic curve $E/\mathbb{Q}$ (CM or non-CM) of conductor q has an associated newform

$$f_E(z) = \sum_{n=1}^{\infty} a_E(n)e^{2\pi inz} \in S_2^{\text{new}}(\Gamma_0(q)).$$

This newform encodes information about points on the elliptic curve; for each prime $p \nmid q$, we have the identity $a_E(p) = p+1-\#E(\mathbb{F}_p)$. There are also higher-weight newforms which arise naturally, perhaps most notably

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) e^{2\pi inz} \in S_{12}^{\text{new}}(\Gamma_0(1)),$$

whose coefficients are given by the Ramanujan tau function.

As a consequence of Deligne's proof of the Weil conjectures, for each prime p, there exists $\theta_p \in [0, \pi]$ such that

$$a_f(p) = 2p^{\frac{k-1}{2}} \cos\theta_p.$$

It is fruitful to study the distribution of $\cos\theta_p$ as p varies. The definitive conjecture in this direction is the Sato–Tate conjecture (as extended by Serre [12]), which was proved by Barnet-Lamb, Geraghty, Harris, and Taylor:

Theorem (Sato–Tate Conjecture [1]) *If $H : [-1, 1] \to \mathbb{C}$ is Riemann integrable, then*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} H(\cos\theta_p) = \int_{-1}^{1} H(t)d\mu_{\text{ST}}, \qquad d\mu_{\text{ST}} := \frac{2}{\pi}\sqrt{1-t^2}\, dt, \tag{2}$$

where $\pi(x) := \#\{p \le x\}$ is the usual prime counting function.[1]

In a recent breakthrough, Newton and Thorne [7,8] proved that for all integers $n \ge 1$, the n-th symmetric power L-function $L(s, \text{Sym}^n f)$ associated to f is the L-function of an automorphic representation of $\text{GL}_{n+1}(\mathbb{A}_\mathbb{Q})$, where $\mathbb{A}_\mathbb{Q}$ denotes the ring of adeles over $\mathbb{Q}$. The zeros of these L-functions dictate the distribution of the primes p such that θ_p lie in a given interval, much like how the zeros of Dirichlet L-functions dictate the distribution of primes in arithmetic progressions. Using the results of Newton and Thorne, the second author in [13] proved a strong version of (2) with an effective error term. The purpose of this note is to apply these results to make improvements toward important conjectures concerning the Fourier coefficients $a_f(p)$.

1.1 Improvements toward the Atkin–Serre conjecture

When $k = 2$ and f corresponds with a non-CM elliptic curve $E/\mathbb{Q}$, Elkies [3] proved that $a_f(p) = 0$ for infinitely many p (thus E has infinitely many supersingular primes). In contrast, when $k \ge 4$ we expect that for any fixed $t \in \mathbb{R}$, $a_f(p) = t$ holds for only finitely many primes p. This statement is quantified by a deep conjecture of Atkin and Serre:

Conjecture (Atkin–Serre [11]) *Let $f \in S_k^{new}(\Gamma_0(q))$ be a non-CM newform of weight $k \ge 4$. For each $\epsilon > 0$, there exist constants $c_{\epsilon,f} > 0$ and $c'_{\epsilon,f} > 0$ such that if $p > c'_{\epsilon,f}$, then*

$$|a_f(p)| \ge c_{\epsilon,f}\, p^{\frac{k-3}{2}-\epsilon}. \tag{3}$$

Rouse [10] proved that for each non-CM newform $f \in S_k^{\text{new}}(\Gamma_0(q))$ of even integral weight $k \ge 4$, the bound (3) holds for almost all primes p, conditional on the generalized Riemann hypothesis for the symmetric power L-functions associated to f. Unconditionally, M. Ram Murty, V. Kumar Murty, and Saradha [5] and V. Kumar Murty [6] proved that there exists a density one subset of the primes for which $|a_f(p)| \ge (\log p)^{1-\epsilon}$ for any fixed $\epsilon > 0$. Thus $|a_f(p)|$ cannot be "too small" often, but in a much weaker sense than Atkin and Serre predicted. Our first result establishes (3) unconditionally for almost all primes p, a noticeable improvement over the result of Murty, Murty, and Saradha.

[1] Recall that the prime number theorem states that $\pi(x) \sim x/\log x$.

Theorem 1.1 *Let*

$$f(z)=\sum_{n=1}^{\infty}a_f(n)e^{2\pi inz}\in S_k^{\mathrm{new}}(\Gamma_0(q))$$

be a holomorphic normalized cuspidal non-CM newform of even integral weight $k\geq 2$ on $\Gamma_0(q)$ with trivial nebentypus. There exists an absolute and effectively computable constant $c_1>0$ such that for $x\geq 3$,

$$\#\left\{x<p\leq 2x\colon |a_f(p)|\leq 2p^{\frac{k-1}{2}}\frac{\log\log p}{\sqrt{\log p}}\right\}\leq c_1\frac{x\log(kq\log x)}{(\log x)^{3/2}}.$$

Thus the set of primes p such that (3) *holds at p forms a density one subset of the primes.*

1.2 Unconditional bounds on extremal primes

Theorem 1.1 shows that it is rare for the Fourier coefficients of f to be "small" on the primes. In the opposite direction, one can ask how often the Fourier coefficients of f are "large" on the primes. To make this precise, consider first the holomorphic newforms $f\in S_k^{\mathrm{new}}(\Gamma_0(q))$ such that $a_f(n)\in\mathbb{Z}$ for all $n\geq 1$. In this case, since $|a_f(p)|\leq 2p^{\frac{k-1}{2}}$, the largest value that $|a_f(p)|$ could assume is $\lfloor 2p^{\frac{k-1}{2}}\rfloor$, where $\lfloor r\rfloor=\max\{t\in\mathbb{Z}\colon t\leq r\}$ is the usual floor function. If $|a_f(p)|$ assumes this value, we call p *extremal* for f. James, Tran, Trinh, Wertheimer, and Zantout [4] conjectured that if f is a non-CM newform of weight 2 corresponding with an elliptic curve over $\mathbb{Q}$, then

$$\#\{p\leq x\colon a_f(p)=\lfloor 2\sqrt{p}\rfloor\}\sim\frac{8}{3\pi}\frac{x^{\frac{1}{4}}}{\log x}. \tag{4}$$

David, Gafni, Malik, Prabhu, and Turnage-Butterbaugh [2] proved that (4) is $\ll_f\sqrt{x}$ under the generalized Riemann hypothesis for the symmetric power L-functions of f. We prove a nontrivial unconditional upper bound in a more general context than elliptic curves.

Theorem 1.2 *Let $f\in S_k^{\mathrm{new}}(\Gamma_0(q))$ be a newform as in Theorem 1.1. There exists an absolute and effectively computable constant $c_3>0$ such that if $x\geq 16$, then*

$$\#\{x<p\leq 2x\colon |a_f(p)|\geq\lfloor 2p^{\frac{k-1}{2}}\rfloor\}\leq c_3\frac{x(\log(kq\log x))^2}{(\log x)^2}.$$

Notation

The Vinogradov notation $F\ll G$ will be used to denote the existence of an effectively computable positive constant c (not necessarily the same in each occurrence) such that $|F|\leq c|G|$ in the range indicated. We write $F=G+O(H)$ to denote that $|F-G|\ll H$.

2 Preliminary lemmas

The main results of this paper follow from the work of Newton and Thorne [7,8] and of the second author [13]. For simplicity, it is assumed that the nebentypus is trivial so that $a_f(n)\in\mathbb{R}$ for all $n\geq 1$; with additional effort, this assumption can be removed.

Lemma 2.1 *Let $f\in S_k^{\mathrm{new}}(\Gamma_0(q))$ be as in the statement of Theorem 1.1, and let $I=[\alpha,\beta]\subseteq[-1,1]$ be a subinterval. If $x\geq 3$, then*

$$\pi_{f,I}(x):=\#\{x<p\leq 2x\colon p\nmid q,\ \cos\theta_p\in I\}=(\pi(2x)-\pi(x))\Big(\mu_{\mathrm{ST}}(I)+O\Big(\frac{\log(kq\log x)}{\sqrt{\log x}}\Big)\Big).$$

Lemma 2.1 is immediate from the work of the second author [13, Theorem 1.1]. (Note that $\pi(2x) - \pi(x) \sim \pi(x) \sim x/\log x$ by the prime number theorem.) We will also use the key technical proposition proved in [13] which leads to Lemma 2.1. In what follows, let U_n be the n-th Chebyshev polynomial defined by

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}, \qquad n \geq 0.$$

The sequence $\{U_n(t)\}_{n=0}^{\infty}$ forms an orthonormal basis of $L^2([-1,1], \mu_{\mathrm{ST}})$.

Lemma 2.2 *Let $f \in S_k^{\mathrm{new}}(\Gamma_0(q))$ be a newform Lemma 2.1. There exist absolute and effectively computable constants $c_5 > 0$ and $c_6 > 0$ such that if $1 \leq n \ll \sqrt{\log x}/\sqrt{\log(kq\log x)}$, then*

$$\Big|\sum_{x<p\leq 2x} U_n(\cos\theta_p)\Big| \ll \frac{x}{\log x}\bigg(n^2 x^{-\frac{1}{c_5 n}} + n^2\bigg(\exp\Big[-c_6\frac{\log x}{n^2\log(kqn)}\Big] + \exp\Big[-c_6\frac{\sqrt{\log x}}{\sqrt{n}}\Big]\bigg)\bigg).$$

Lemma 2.2 follows immediately from [13, Proposition 2.1] and partial summation.

3 Proof of Theorem 1.1

Since $a_f(p) = 2p^{\frac{k-1}{2}}\cos\theta_p$, we see that

$$\#\Big\{x < p \leq 2x\colon |a_f(p)| \leq 2p^{\frac{k-1}{2}}\frac{\log(kq\log p)}{\sqrt{\log p}}\Big\} \leq \#\Big\{x < p \leq 2x\colon |\cos\theta_p| \leq \frac{\log(kq\log x)}{\sqrt{\log x}}\Big\}.$$

Writing $I = [-\frac{\log(kq\log x)}{\sqrt{\log x}}, \frac{\log(kq\log x)}{\sqrt{\log x}}]$, we find via a Taylor expansion that

$$\mu_{\mathrm{ST}}(I) = \frac{4}{\pi}\frac{\log(kq\log x)}{\sqrt{\log x}} + O\Big(\frac{(\log(kq\log x))^3}{(\log x)^{3/2}}\Big).$$

Thus the desired result follows from Lemma 2.1.

4 Proof of Theorem 1.2

For an integer $N \geq 3$ (which we will later take to depend on x), let $I_N = [\cos N^{-1}, 1]$ and $I'_N = [-1, -\cos N^{-1}]$. We take N large enough so that $\cos N^{-1} \leq 1 - x^{-\frac{1}{2}}$. If $x < p \leq 2x$, then we have $x < p < 4p^{k-1}$. With $\{t\} = t - \lfloor t \rfloor$, we deduce the chain of inequalities

$$\cos N^{-1} \leq 1 - x^{-\frac{1}{2}} < 1 - \frac{1}{2p^{\frac{k-1}{2}}} < 1 - \frac{\{2p^{\frac{k-1}{2}}\}}{2p^{\frac{k-1}{2}}} < 1.$$

Therefore, $\#\{x < p \leq 2x\colon |a_f(p)| \geq \lfloor 2p^{\frac{k-1}{2}} \rfloor\}$ equals

$$\begin{aligned}&\#\Big\{x < p \leq 2x\colon \frac{|a_f(p)|}{2p^{\frac{k-1}{2}}} \geq 1 - \frac{\{2p^{\frac{k-1}{2}}\}}{2p^{\frac{k-1}{2}}}\Big\}\\ &\leq \#\{x < p \leq 2x\colon\ \cos\theta_p \in I_N\} + \#\{x < p \leq 2x\colon\ \cos\theta_p \in I'_N\}.\end{aligned}$$

We will estimate $\#\{x < p \leq 2x\colon\ \cos\theta_p \in I_N\}$; the case for I'_N is the same. By the work in [2] (Proposition 2.2, Equation 3.2, and the displayed equation preceding it), we find that if $N \geq 3$, then (note that $U_0(\cos\theta_p)$ is identically 1)

$$\#\{x < p \leq 2x\colon\ \cos\theta_p \in I_N\} \ll \frac{\pi(2x) - \pi(x)}{N^2} + \frac{1}{N^2}\sum_{n=1}^{N}\Big|\sum_{x<p\leq 2x} U_n(\cos\theta_p)\Big|. \tag{5}$$

Theorem 1.2 follows once we apply Lemma 2.2 to (5) with $N = \lfloor \sqrt{\log x} / \log(kq \log x) \rfloor$.

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