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Applications of group theory to conjectures of Artin and Langlands

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In this note, we study conjectures of Artin and Langlands and derive the automorphy of all solvable groups of order at most 200, three groups excepted.

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1. Introduction

Let K/k be a Galois extension of number fields with Galois group G. A long-standing conjecture of Artin asserts that every Artin L-function attached to a non-trivial irreducible character of G extends to an entire function. Via his celebrated reciprocity law, Artin showed that his conjecture is valid for all Artin L-functions attached to 1-dimensional characters. From this and the induction invariance property of Artin L-functions, Artin established his conjecture when G is an M-group, namely, all irreducible characters of G are induced from 1-dimensional characters of subgroups of G.

Inspired by Artin reciprocity, Langlands further conjectured that for every character χ of G, χ corresponds to an automorphic representation π of $GL_{\chi(1)}(\mathbb{A}_k)$, where $\mathbb{A}_k$ denotes the adèle ring of k; and if χ is irreducible, then π will be cuspidal. This is often called the Langlands reciprocity conjecture or the strong Artin conjecture. Indeed, Artin's conjecture follows from Langlands' conjecture and the theory of automorphic L-functions.

By the works of Iwasawa and Tate, one knows that Langlands' conjecture for $GL(1)$ is essentially Artin reciprocity. The next big step was taken by Langlands [14] and Tunnell [21] who proved the Langlands reciprocity conjecture for all (irreducible) 2-dimensional representations with solvable image. In a slightly different vein, from their theory of base change and automorphic induction, Arthur and Clozel [1] derived Langlands reciprocity for all nilpotent Galois extensions. Moreover, they showed every accessible character of a solvable group is of automorphic type. (We recall that an irreducible character χ of G is called accessible if χ is an integral combination of characters induced from linear characters of subnormal subgroups of G, and that a character χ of G is said to be of automorphic type if for any Galois extension K/k with Galois group G, Langlands reciprocity is valid for the Artin L-function $L(s,\chi,K/k)$. A group G is said to be of automorphic type if all its characters are.)

More recently, Ramakrishnan [20] derived the automorphy of solvable Artin representations of $GO(4)$-type. Also, Khare and Wintenberger [12] proved Serre's modularity conjecture and then deduced Langlands reciprocity for any odd irreducible 2-dimensional representation over $\mathbb{Q}$ with non-solvable image. We will discuss some of these deep results of the Langlands program in more detail in Sec. 3.

In [27], the author showed that Artin's conjecture holds if K/k is *nearly supersolvable*, i.e. $G = \mathrm{Gal}(K/k)$ admits a normal subgroup N with G/N supersolvable such that all irreducible characters of N are of degree at most 2. Moreover, the author [26] extended the above-mentioned result of Arthur and Clozel to *nearly nilpotent* extensions. More precisely, if G is nearly nilpotent, i.e. G admits a normal subgroup N with G/N nilpotent such that all irreducible characters of N are of degree at most 2, then G is of automorphic type. As an application, the Langlands reciprocity conjecture is established for all Galois extensions of square-free degree as well as all non-A_5 extensions of degree ≤ 60.

Now, let us consider a finite set S of natural numbers. We shall call an irreducible character χ of G S-accessible if χ is an integral sum of characters induced from irreducible characters ψ_i of subnormal subgroups of G where each $\psi_i(1)$ belongs to S. In particular, for $S = \{1\}$, this notion gives accessible characters introduced by Arthur and Clozel. In Sec. 3, we deduce a mild generalization of the earlier-mentioned result of Arthur and Clozel as follows. (We, however, note that the first part of this proposition is known by experts, at least, implicitly.)

Proposition 1.1. *Suppose G is solvable. If $\chi \in \mathrm{Irr}(G)$ is a $\{1,2\}$-accessible character, then Langlands reciprocity holds for χ. Also, if $|G|$ is not divisible by* 36 *and $\chi \in \mathrm{Irr}(G)$ is a $\{1,2,3\}$-accessible character, then χ is of automorphic type.*

By studying the monomiality of finite groups, van der Waall [22, 23] proved the Artin conjecture for all groups of order ≤ 100, 24 groups excepted. This work has been extended by van der Waall [24, 25] himself to groups of order between 100 and 200 by giving necessary and sufficient conditions for these groups being monomial.

In light of the work of van der Waall, we are interested in investigating groups of order between 60 and 200. Our method, referred to as a method of "low-dimensional groups", was utilized in [26, 27], which also allows us to deduce a result concerning the automorphy of groups of cube-free order (see Proposition 3.13 below) from the above proposition. Our main result is the following theorem.

Theorem 1.2. *Let G be a solvable group of order at most* 200. *If $|G|$ is not equal to* $108, 160$, *nor* 168, *then G is of automorphic type.*

In fact, among solvable groups of order at most 200, there are exactly three exceptional groups G not being shown to be of automorphic type. The exceptional group of order 108 has GAP ID [108, 15] and is not monomial. For the remaining instances, G has GAP ID [160, 234] or [168, 43]. These two groups are monomial, and hence Artin's conjecture is already known.

2. Group-Theoretic Preliminaries

In this section, we will recall some results from the theory of groups. First, in this note, G always denotes a finite group, and H and N denote a subgroup and a normal subgroup of G, respectively. The direct product of n-copies of G will be denoted by G^n, and $\mathbf{Z}(G)$ will stand for the center of G. We let $\mathrm{Irr}(G)$ be the set of irreducible characters of G, and set $\mathrm{cd}(G) := \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$. For any finite set π of primes, G_π denotes a Hall π-subgroup of G. The cyclic group of order m, the Klein four-group, and the quaternion group of order 8 will be denoted as C_m, V_4, and Q, respectively.

A classical result of Hölder asserts that a (finite) group of square-free order must be meta-cyclic. From this, the author [26] deduced that all groups of square-free order are of automorphic type. In 2005, Dietrich and Eick [6] studied the class of groups of cube-free order and, in particular, characterized non-solvable groups of cube-free order. Their work was extended by Qiao and Li [19] who gave a description of the class of solvable groups of cube-free order as follows.

Proposition 2.1 ([19]). *Let G be a solvable group of cube-free order. Then one of the following holds.*

(1) $G = (C_a \times C_b^2) \rtimes (C_c \times C_d^2)$, *or* $(C_2^2 \times C_a \times C_b^2) \rtimes (C_c \times C_d^2)$; *or*
(2) $G = A \rtimes B \rtimes G_{\{2\}} = (C_a \times C_b^2) \rtimes (C_c \times C_d^2) \rtimes G_{\{2\}}$,

where a, b, c, and d are suitable odd integers such that $(a,b) = (c,d) = 1$, ac is cube-free, bd is square-free, prime divisors of ab are not less than prime divisors of cd, and B contains a Sylow 3*-subgroup of G.*

We remark that Qiao and Li showed that the first case happens if a Hall $\{2,3\}$-subgroup $G_{\{2,3\}} = G_{\{2\}} \rtimes G_{\{3\}}$ of G is non-abelian (cf. [19, Lemma 3.8]).

We now recall some concepts of relative M-groups and relative SM-groups (cf. [8 and 9, Chap. 6]), which will help us to study the Langlands reciprocity conjecture later.

Definition 2.2. Let G be a finite group, and N be a normal subgroup of G. A character χ of G is called a relative M-character (respectively, a relative

SM-character) with respect to N if there exists a subgroup (respectively, a subnormal subgroup) H with $N \leq H \leq G$ and an irreducible character $\psi \in \mathrm{Irr}(H)$ such that $\mathrm{Ind}_H^G \psi = \chi$ and $\psi|_N \in \mathrm{Irr}(N)$. If every irreducible character of G is a relative M-character (respectively, a relative SM-character) with respect to N, then G is said to be a relative M-group (respectively, a relative SM-group) with respect to N.

We note that if N is normal in G and G/N is nilpotent or supersolvable, then G is a relative M-group with respect to N. In general, one has the following result due to Price (cf. [2, Theorem 7.63; 9, Theorem 6.22]).

Proposition 2.3. *Let G be a finite group, and N be a normal subgroup of G such that G/N is solvable. Suppose that every chief factor of every non-trivial subgroup of G/N has order equal to an odd power of some prime. Then G is a relative M-group with respect to N.*

Based on Proposition 2.3, Horváth [8] gave a sufficient condition for groups being relative SM-groups as follows.

Proposition 2.4 ([8, Proposition 2.7]). *Let G be a finite group and N be a normal subgroup of G such that G/N is nilpotent. Then G is a relative SM-group with respect to N.*

We note that Horváth omitted the proof and remarked that it is similar to the proof of Proposition 2.3. However, for the sake of completeness, we give below a simple proof as an immediate consequence of Proposition 2.3.

Proof. By Proposition 2.3, we already know that each $\chi \in \mathrm{Irr}(G)$ is a relative M-character with respect to N, i.e. there exists a subgroup H with $N \leq H \leq G$ and an irreducible character $\psi \in \mathrm{Irr}(H)$ such that $\mathrm{Ind}_H^G \psi = \chi$ and $\psi|_N \in \mathrm{Irr}(N)$. Now as G/N is nilpotent, all its subgroups are subnormal. In particular, we have an invariant series

$$H/N = \overline{H_0} \trianglelefteq \overline{H_1} \trianglelefteq \cdots \trianglelefteq \overline{H_m} = G/N,$$

where for each i, $\overline{H_i}$ is a normal subgroup of $\overline{H_{i+1}}$. Now by lifting this series (with respect to N), we can see that H is subnormal in G. In other words, each χ is a relative SM-character with respect to N, and hence G is a relative SM-group with respect to N. $\square$

Recall that a Dedekind group is a group G such that every subgroup of G is normal. In light of the above proof, we deduce the following variant that may be of interest.

Proposition 2.5. *Let G be a finite group and N be a normal subgroup of G such that G/N is a Dedekind group. Then for every $\chi \in \mathrm{Irr}(G)$, there exists a normal subgroup H of G with $N \leq H$ and an irreducible character $\psi \in \mathrm{Irr}(H)$ such that $\mathrm{Ind}_H^G \psi = \chi$ and $\psi|_N \in \mathrm{Irr}(N)$.*

As a consequence, any irreducible character of a metabelian group G is induced from a 1-dimensional character of a normal subgroup of G. To end this section, we further collect more results from the representation theory of finite groups.

Lemma 2.6 ([9, pp. 28]). *Let G be a finite group and $\mathbf{Z}(G)$ its center. Then for every irreducible character χ of G, one has*

$$\chi(1)^2 \leq [G : \mathbf{Z}(G)].$$

Via this lemma, Proposition 2.3, Sylow's theory (or the computer algebra package [7]), one has the following lemma.

Lemma 2.7. *If G is of order $1, 2, 4, 3$, or 9, then $\mathrm{cd}(G) = \{1\}$. If G is of order $8, 16, 6, 18$, or 28, then $\mathrm{cd}(G) \subseteq \{1, 2\}$. If $|G|$ is $12, 24$, or 36, then $\mathrm{cd}(G) \subseteq \{1, 2, 3, 4\}$ where $4 \in \mathrm{cd}(G)$ only if $|G'| = 9$.*

We also invoke the following result of Isaacs.

Lemma 2.8 ([9, Theorems 12.5, 12.6 and 12.15]). *If G is a finite group with $|\,\mathrm{cd}(G)| \leq 3$, then G must be solvable.*

Let ρ be an irreducible representation of G. As the finite subgroups of $PGL_3(\mathbb{C})$ have been classified by Blichfeldt [3, 18], one has the following.

Lemma 2.9 ([3, 18]). *If ρ is primitive, 3-dimensional, and with solvable projective image $\overline{G}$ in $PGL_3(\mathbb{C})$, then $\overline{G}$ is of order 36, 72, or 216.*

3. Artin L-Functions and Automorphic Representations

Let K/k be a Galois extension of number fields with Galois group G, and χ an irreducible character of G. As mentioned in the beginning, Langlands conjectured that there exists a cuspidal automorphic representation $\pi \in \mathfrak{A}(GL_{\chi(1)}(\mathbb{A}_k))$ such that

$$L(s, \chi, K/k) = L(s, \pi),$$

where $L(s, \pi)$ denotes the automorphic L-function attached to π. Now, let H be a subgroup of G and assume $\chi = \mathrm{Ind}_H^G \psi$ for some $\psi \in \mathrm{Irr}(H)$. According to the Langlands reciprocity conjecture, there should be a cuspidal automorphic representation $\Pi \in \mathfrak{A}(GL_{\psi(1)}(\mathbb{A}_{K^H}))$ corresponding to ψ. Now, the induction invariance property of Artin L-functions, i.e.

$$L(s, \chi, K/k) = L(s, \mathrm{Ind}_H^G \psi, K/k) = L(s, \psi, K/K^H),$$

suggests that there should be a map sending Π to π. This conjectural map is called the automorphic induction and has been established by Arthur and Clozel [1] for extensions of prime degree as stated in the following theorem.

Theorem 3.1 ([1, Arthur and Clozel]). *Let K/k be a Galois extension of number fields of prime degree p, and Π denote an automorphic representation induced* *from cuspidal of* $GL_n(\mathbb{A}_K)$ (*or, in particular, a cuspidal automorphic representation of* $GL_n(\mathbb{A}_K)$). *Then the automorphic induction* $I(\Pi)$ *of* Π *exists as an automorphic representation of* $GL_{np}(\mathbb{A}_k)$.

For non-normal extensions, one has a theorem due to Jacquet *et al.* [11] below.

Theorem 3.2 ([11]). *Let* K/k *be a non-normal cubic extension of number fields. Let* χ *be an idèle class character of* K. *Then the automorphic induction* $I(\chi)$ *of* χ *exists as an automorphic representation of* $GL_3(\mathbb{A}_k)$.

Thus, by Theorems 3.1 and 3.2, all monomial characters of degree 3 are of automorphic type.

As mentioned in the introduction, some special cases of the Langlands reciprocity conjecture have been established. We first extract the works of Artin, Langlands [14], and Tunnell [21] as follows.

Proposition 3.3. *If a character* χ *of a solvable group* G *is of degree at most* 2, *then* χ *is of automorphic type.*

Let $GO_n(\mathbb{C})$ denote the subgroup of $GL_n(\mathbb{C})$ consisting of orthogonal similitudes, i.e. matrices M such that $M^tM = \lambda_M I$, with $\lambda_M \in \mathbb{C}$. We will say that a $\mathbb{C}$-representation (ρ, V) (of the absolute Galois group of a number field k) is of $GO(n)$-type if and only if $\dim V = n$ and it factors as

$$\rho : \mathrm{Gal}(\overline{k}/k) \to GO_n(\mathbb{C}) \subset GL(V).$$

In his paper [20], Ramakrishnan derived the automorphy of solvable Artin representations of $GO(4)$-type as follows.

Proposition 3.4 ([20]). *Let* k *be a number field and let* ρ *be a continuous* 4-*dimensional representation of* $\mathrm{Gal}(\overline{k}/k)$ *whose image is solvable and lies in* $GO_4(\mathbb{C})$. *Then* ρ *is automorphic.*

One also has below results concerning “symplectic” Galois representations and “hypertetrahedral” Galois representations due to Martin [15, 16]. (Recall that the (mth) symplectic similitude group is defined as

$$GSp_{2m}(\mathbb{C}) = \{M \in GL_{2m}(\mathbb{C}) \,|\, M^tJM = \lambda_M J, \lambda_M \in \mathbb{C}\},$$

where the matrix J is defined as

$$J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix},$$

and I_m is the identity $m \times m$ matrix.)

Proposition 3.5 ([15]). *Let* K/k *be a Galois extension of number fields and* ρ *be an irreducible* 4-*dimensional representation of* $G = \mathrm{Gal}(K/k)$ *into* $GSp_4(\mathbb{C})$. *Assume the projective image of* ρ *in* $PGL_4(\mathbb{C})$ *is isomorphic to* $E_{2^4} \rtimes C_5$, *where* E_{2^4} *denotes the elementary abelian group of order* 2^4. *Then* ρ *is automorphic.*

Proposition 3.6 ([16]). *Let K/k be a Galois extension of number fields and ρ be an irreducible 4-dimensional representation of $G = \mathrm{Gal}(K/k)$. Suppose the projective image $\overline{G}$ of ρ is an extension of A_4 by V_4, i.e. $\overline{G}/V_4$ is isomorphic to A_4. Then ρ is automorphic.*

Moreover, for imprimitive 4-dimensional representations which are essentially self-dual, one has a handy result due to Martin and Ramakrishnan (cf. [17, Proposition 7.2]) below.

Proposition 3.7. *If ρ is an irreducible 4-dimensional representation, which is imprimitive and essentially self-dual, of solvable $G = \mathrm{Gal}(K/k)$, then ρ is automorphic.*

We now consider a character χ of $G = \mathrm{Gal}(K/k)$ which is induced from an irreducible character ψ of a subnormal subgroup H of G. Assume, further, that ψ is automorphic over the fixed field K^H, i.e. there is a cuspidal automorphic representation Π of $GL_{\psi(1)}(\mathbb{A}_{K^H})$ such that

$$L(s, \psi, K/K^H) = L(s, \Pi).$$

Since H is a subnormal subgroup of G, H admits a subgroup series

$$H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_{m-1} \trianglelefteq H_m = G,$$

where for each i, H_i is a normal subgroup of H_{i+1}. As G is finite and solvable, we may require each H_{i+1}/H_i is a cyclic group of prime order. Thus, one has a tower of Galois extensions of prime degree

$$K \supset K^{H_1} \supset \cdots \supset K^{H_{m-1}} \supset k.$$

Now, applying the Arthur–Clozel theorem of automorphic induction, i.e. Theorem 3.1, successively, one can derive that $\mathrm{Ind}_H^G \psi$ corresponds to an automorphic representation over k. In other words, Langlands reciprocity holds for χ. Following Arthur–Clozel's proof of the automorphy of accessible characters of solvable groups (cf. [1, Proposition 7.2]), one has the following lemma.

Lemma 3.8. *Assume $G = \mathrm{Gal}(K/k)$ is solvable and χ is irreducible. If χ is an integral sum of characters induced from irreducible characters, which are of automorphic type, of subnormal subgroups of G, then Langlands reciprocity holds for χ.*

Let S be a finite set of natural numbers. An irreducible character χ of G is called S-accessible if χ is an integral combination of characters induced from irreducible characters ψ_i of subnormal subgroups of G where each $\psi_i(1)$ belongs to S. Moreover, a group is called S-accessible if all its irreducible characters are. For example, $\{1\}$-accessible characters (respectively, groups) are exactly accessible characters (respectively, groups), and nilpotent groups are $\{1\}$-accessible. With this notion and Lemma 3.8 in mind, one has the following corollary (à la Arthur et Clozel).

Corollary 3.9. *Suppose G is solvable. If χ is a $\{1,2\}$-accessible character of G, then Langlands reciprocity holds for χ. Also, if $|G|$ is not divisible by 36 and χ is a $\{1,2,3\}$-accessible character of G, then Langlands reciprocity holds for χ. Thus, if G is nearly nilpotent (or, in particular, metabelian), then G is of automorphic type.*

Proof. It suffices to show that any irreducible character ψ, with $\psi(1) \leq 3$, of any subgroup of G is of automorphic type. As all subgroups of G are solvable, if $\psi(1) \leq 2$, the assertion follows from Proposition 3.3. So we may assume $\psi(1) = 3$. Since 36 does not divide the order of any quotient group of any subgroup of G, Lemma 2.9 tells us that ψ must be monomial and hence of automorphic type. □

We note that Langlands reciprocity for nearly nilpotent Galois extensions follows from the fact that all nearly nilpotent groups are solvable (cf. Lemma 2.8) and $\{1,2\}$-accessible (cf. Proposition 2.4). As remarked in [1], Dade [5] has shown that if G is solvable, then $\{1\}$-accessible characters are monomial. It would be interesting to investigate whether a similar result holds or not. For example, are $\{1,2\}$-accessible characters of a solvable group G all induced from irreducible characters of degree at most 2? We have no clue about this question; and instead of trying to answer this question, we give a criterion for a group being $\{1,2,3\}$-accessible as follows.

Lemma 3.10. *Let $G = \mathrm{Gal}(K/k)$. Suppose that $36 \nmid |G|$, and that G admits a normal subgroup N with G/N supersolvable and $\mathrm{cd}(N) \subseteq \{1,2,3\}$. Then the Artin conjecture is true for K/k. Moreover, if G/N is nilpotent, then G is $\{1,2,3\}$-accessible and hence of automorphic type.*

Proof. We first note that Lemma 2.8 asserts that N is solvable and so is G. According to Proposition 2.3, every irreducible character χ of G is induced from an irreducible character ψ of degree at most 3 of a subgroup H of G. If $\psi(1) \leq 2$, ψ is automorphic by Proposition 3.3. On the other hand, for $\psi(1) = 3$, Lemma 2.9 tells us that ψ must be monomial as $|H|$ is not divisible by 36. Thus, ψ is automorphic (over K^H). From this and the induction invariance property of Artin L-functions, Artin's conjecture follows.

Assume, further, that G/N is nilpotent. Then Proposition 2.4 enables us to choose H to be subnormal in G. As now, χ is $\{1,2,3\}$-accessible, Corollary 3.9 yields χ is of automorphic type. □

As mentioned in Sec. 1, these results enable one to study Artin's conjecture via "low-dimensional (sub)groups". For instance, if G is of order 54 or 162, then the Sylow 3-subgroup P (say) is normal in G. Observing that $\mathrm{cd}(P) \subseteq \{1,3\}$ and G/P is of order 2, Lemma 3.10 tells us that G is of automorphic type. In much the same spirit, Corollary 3.9 and Lemma 3.10 give the following results.

Corollary 3.11. *If G is of order pq, p^2q, p^2q^2, $8p$, or $27p$ for some primes p and q, then G is of automorphic type.*

Proof. As all p-groups are nilpotent and hence of automorphic type, we may assume $|G|$ has two distinct prime divisors. By applying the Sylow theorems, G admits a normal Sylow subgroup N (see, for example, [10, Theorems 1.30–1.33 and Problem 1E.1]) unless G is isomorphic to S_4. Thus, for the first three cases, i.e. G is of order pq, p^2q, or p^2q^2, G is metabelian and the automorphy of G follows from Corollary 3.9. Also, if G is not isomorphic to S_4 and $|G|$ is $8p$ or $27p$, N is of order 8, 27, or p. By Lemma 2.6, we can see $\mathrm{cd}(N) \subseteq \{1, 2, 3\}$. Since G/N is nilpotent in these cases, Lemma 3.10 tells us that G is of automorphic type. Finally, the automorphy of S_4 follows from Lemma 3.10 and the fact that $\mathrm{cd}(S_4) = \{1, 2, 3\}$. □

Corollary 3.12. *If G is of order $16p$ for some prime p, then G is of automorphic type.*

Proof. Since 2-groups are clearly of automorphic type, we may assume p is odd. By the Sylow theorems, G has a normal Sylow p-subgroup P unless $p \in \{3, 5, 7\}$. For $p = 5$, if Sylow 5-subgroups of G are not normal, then G must have 16 Sylow 5-subgroups, which give 4×16 non-trivial elements. Thus, the Sylow 2-subgroup is normal in this case. By a more careful argument (see, for example, [4, Lemma 3.6]), if $|G| = 112$, then G also admits a normal Sylow subgroup. Therefore, for $p \neq 3$, G is nearly nilpotent and of automorphic type.

We note that the automorphy of groups of order 48 was obtained by the author [26] via a computational method. In light of this, we shall give another proof for the case $|G| = 112$ as follows. First, by GAP [7], it is easy to check that $|G'| \in \{1, 2, 4, 7, 8, 14, 28\}$. Note that if $|G'| \neq 28$, then $\mathrm{cd}(G') \subseteq \{1, 2\}$. Also, it can be examined by [7] that if G' has order 28, then it has GAP ID [28, 2] and is abelian. Thus, all groups of order 112 are of automorphic type by Corollary 3.9. □

Furthermore, by the work of Qiao and Li (Proposition 2.1), we have the following proposition.

Proposition 3.13. *Assume $G = \mathrm{Gal}(K/k)$ is of cube-free order. Then Langlands reciprocity holds for K/k whenever $|G|$ is odd or G is a solvable group with a non-abelian Hall $\{2, 3\}$-subgroup $G_{\{2,3\}} = G_{\{2\}} \rtimes G_{\{3\}}$.*

Proof. By the celebrated Feit–Thompson theorem, if $|G|$ is odd, then G must be solvable. Thus, by Proposition 2.1, if $|G|$ is odd or G is a solvable group with a non-abelian Hall $\{2, 3\}$-subgroup $G_{\{2,3\}} = G_{\{2\}} \rtimes G_{\{3\}}$, then G is metabelian. Hence, the Langlands reciprocity conjecture follows from Corollary 3.9. □

We remark that for prime p with $3 \mid p+1$, Qiao and Li in [19] gave the following examples of groups which are not metabelian.

(1) $C_p^2 \rtimes S_3$.
(2) $C_p^2 \rtimes C_3 \rtimes C_4$.

Observe that these groups contain normal subgroups isomorphic to $C_p^2 \rtimes C_3$, and that $\mathrm{cd}(C_p^2 \rtimes C_3) \subseteq \{1,3\}$. Applying Lemma 3.10, we know that these groups are of automorphic type.

4. Groups of Order at Most 100

In [22, 23], van der Waall applied group-theoretic methods to show that all groups of order ≤ 100, 24 groups excepted, are monomial. Moreover, van der Waall gave a description of the 24 exceptional groups that are non-monomial.

In light of the work of van der Waall, we will show that all groups, except A_5, of order at most 100, are of automorphic type. We first note that the author [26] has established Langlands reciprocity for all non-A_5 Galois extensions of degree ≤ 60. Now, by Arthur–Clozel's theorem on the automorphy of nilpotent groups (cf. [1, Theorem 7.1]), all groups of prime power order are of automorphic type. Thus, G is of automorphic type if $|G| \in \{61, 67, 71, 73, 79, 83, 89, 97, 64, 81\}$. There are 10 classes of groups.

For any distinct primes p and q, Corollary 3.11 tells us that any group of order pq, p^2q, or p^2q^2 is of automorphic type. Also, the earlier-mentioned result of Hölder asserts that a group of square-free order is meta-cyclic. Therefore, by Corollary 3.9, if G has order 62, 63, 65, 66, 68, 69, 70, 74, 75, 76, 77, 78, 82, 85, 86, 87, 91, 92, 93, 94, 95, 98, 99, or 100, then G is of automorphic type. Here we have 24 classes of groups.

Also, Corollaries 3.11 and 3.12 yield that every group of order 80 or 88 is of automorphic type. On the other hand, any group of order 90 has a normal subgroup of order 45, which is abelian. As a result, all groups of order 90 are metabelian and of automorphic type.

Moreover, for G of order 84, by Proposition 2.1, G is either metabelian or of the form $G = C_7 \rtimes C_3 \rtimes G_{\{2\}}$. By Proposition 2.3, it is easy to see that $\mathrm{cd}(C_7 \rtimes C_3) \subseteq \{1,3\}$. As $36 \nmid |G|$, Lemma 3.10 asserts that G is of automorphic type. Thus, it remains to consider groups of order 72 or 96.

4.1. *The case* $|G| = 72$

Consider a group G of order 72. van der Waall [23, Theorem II. 5.1 and Lemma II. 5.3] showed that G is not monomial if and only if G' is the quaternion group of order 8. Moreover, the derived subgroup G' cannot have order 24. Hence, any non-monomial group of order 72 must be nearly nilpotent and of automorphic type. For G monomial, G' must have order 1, 2, 4, 3, 6, 12, 9, 18, or 36. If G' is of order 1, 2, 4, 3, 6, 9, or 18, we know that $\mathrm{cd}(G') \subseteq \{1,2\}$, and so G is nearly nilpotent.

Now, we assume G' is of order 12 or 36. By Lemma 2.7, one has $\mathrm{cd}(G') \subseteq \{1,2,3,4\}$. On the other hand, Proposition 2.4 tells us that for every $\chi \in \mathrm{Irr}(G)$, there exists a subnormal subgroup H with $G' \leq H \leq G$ and an irreducible character $\psi \in \mathrm{Irr}(H)$ such that $\mathrm{Ind}_H^G \psi = \chi$ and $\psi|_{G'} \in \mathrm{Irr}(G')$. Since every proper subgroup of a group of order 72 has been shown to be of automorphic type, if $H \neq G$, then

Theorem 3.1 yields χ is of automorphic type. Thus, we may assume $H = G$. As G is monomial and solvable, if $\chi(1) \leq 3$, then χ is of automorphic type. Furthermore, if $4 \in \mathrm{cd}(G')$, Lemma 2.7 tells us that G' must be of order 36 and G'' is of order 9. Thus, G/G'' is a 2-group and G is nearly nilpotent.

4.2. *The case* $|G| = 96$

For the last case, $|G| = 96$, if G' is of order 1, 2, 4, 8, 16, 3, 6, 12, or 24, then as above, we have $\mathrm{cd}(G') \subseteq \{1, 2, 3\}$. Hence, by Lemma 3.10, G is of automorphic type.

Let $|G'|$ be 48. Then [23, Theorem II. 6.2] tells us that G'' is of order 16 and abelian. As now G/G'' is supersolvable, Proposition 2.3 asserts that for every $\chi \in \mathrm{Irr}(G)$, there exists a subgroup H with $G'' \leq H \leq G$ and an irreducible character $\psi \in \mathrm{Irr}(H)$ such that $\mathrm{Ind}_H^G \psi = \chi$ and $\psi|_{G''} \in \mathrm{Irr}(G'')$, and hence $\mathrm{cd}(G) \subseteq \{1, 2, 3, 6\}$. We note that if $\chi(1) = 6$, then it must be induced from a linear character of G'', which is normal in G. Thus, Theorem 3.1 implies that χ is of automorphic type. Again, as G is monomial and solvable, if $\chi(1) \leq 3$, then χ is of automorphic type.

Now, it remains to consider the case $|G'| = 32$. Let $\Phi(G')$ stand for the Frattini subgroup of G', i.e. the intersection of all maximal subgroups of G'. We recall that a p-group is termed extra-special if its center, derived subgroup, and Frattini subgroup all coincide. By the classification due to van der Waall [23, Theorem II. 6.5], we have either:

(1) G is not monomial if and only if $\mathbf{Z}(G')$ is of order 8 and $\Phi(G')$ is of order 8 or 2 [23, Cases (4-a) and (4-b)]; or
(2) G is monomial if and only if $G' = Q * Q$, the extra-special group of order 32 of $(+)$-type [23, Case (4-d-2)].

For the first case, we note that if $|\mathbf{Z}(G')| = |\Phi(G')| = 8$, then van der Waall showed that $|G''| = 2$, which implies that $G' \in \Gamma_2$. (Here, Γ_2 is the Hall-Senior family of groups with the derived subgroups isomorphic to C_2 and the inner automorphism groups isomorphic to V_4.) It can be checked, by using the computer algebra package [7] (or even rather easily, but more tediously, by hand), that it has GAP ID [32, 2] and $\mathrm{cd}(G') = \{1, 2\}$ in this case. On the other hand, if $|\mathbf{Z}(G')| = 8$ and $|\Phi(G')| = 2$, then van der Waall proved that $G' \simeq C_2^2 \times Q$, which gives $\mathrm{cd}(G') = \{1, 2\}$. Thus, G is nearly nilpotent and of automorphic type.

For the second case, van der Waall (see [23, pp. 125–126]) showed that for every irreducible representation ρ of G, either

(1) ρ can be regarded as a representation of $G/\mathbf{Z}(G')$; or
(2) ρ is faithful, monomial, and of dimension 4.

As remarked by van der Waall, $G/\mathbf{Z}(G')$ has the abelian derived subgroup and hence is monomial. We further note that this remark, in fact, tells us that $G/\mathbf{Z}(G')$ is metabelian and hence of automorphic type.

Finally, we assume ρ is faithful. Thus, $\rho(G)$ is a solvable subgroup of order 96 of $GL_4(\mathbb{C})$. Since any scalar matrix in $\rho(G)$ lies in its center $\mathbf{Z}(\rho(G))$ and Schur's lemma implies that $\mathbf{Z}(\rho(G))$ is contained inside the scalar matrices, the projective image of ρ in $PGL_4(\mathbb{C})$ is isomorphic to $\rho(G)/\mathbf{Z}(\rho(G)) \simeq G/\mathbf{Z}(G)$. Since $G = (Q*Q) \rtimes C_3$, as may be checked in GAP [7] for example, one can deduce that $G/\mathbf{Z}(G)$ is isomorphic to $V_4 \rtimes A_4$. By a result of Martin, Proposition 3.6, ρ is of automorphic type which completes the proof.

5. Groups of Order Between 100 and 200

In this section, we will further investigate the automorphy of groups of order between 100 and 200. First, as discussed before, by the results of Arthur–Clozel and Hölder and Corollaries 3.11 and 3.12, if G is of prime power order, square-free order, or order pq, p^2q, p^2q^2, $8p$, $16p$, or $27p$, then G is of automorphic type

Moreover, for G of order 126, 132, 140, 150, 156, 180, or 198, we know that if G is non-solvable then $G \simeq A_5 \times C_3$. On the other hand, by Proposition 2.1, G is either metabelian or of the form $G = G_{\{2\}'} \rtimes G_{\{2\}}$. We note that $G_{\{2\}'}$ is either of order 63, 33, 35, 75, 39, 45, or 99. If $|G_{\{2\}'}|$ is 33, 35, 45, or 99, then $G_{\{2\}'}$ is abelian and G is hence metabelian. Also, as we can require $G_{\{2\}'} = A \rtimes B$ where B contains a Sylow 3-subgroup of G, if G is of order 63, 75, or 39, then $\mathrm{cd}(G_{\{2\}'}) \subseteq \{1, 3\}$. Since none of 126, 150, and 156 is divisible by 36, Lemma 3.10 yields these groups are of automorphic type.

For $|G| = 162$, we know that G is supersolvable and its Sylow 3-subgroup P (say) is normal. By Lemma 2.6, $\mathrm{cd}(P) \subseteq \{1, 3\}$. Again, Lemma 3.10 asserts G is of automorphic type.

For $|G| = 200$, Sylow's theory tells us that the Sylow 5-subgroup of G is normal. Therefore, G is nearly nilpotent.

Thus, we only have to consider the following cases:

(1) $|G| = 108 = 2^2 \cdot 3^3$
(2) $|G| = 120 = 2^3 \cdot 3 \cdot 5$
(3) $|G| = 144 = 2^4 \cdot 3^2$
(4) $|G| = 160 = 2^5 \cdot 5$
(5) $|G| = 168 = 2^3 \cdot 3 \cdot 7$
(6) $|G| = 192 = 2^6 \cdot 3$

5.1. *The case* $|G| = 108$

As remarked by van der Waall [24], G is monomial unless $|G'| = 27$ or $|G'| = 54$ where the latter possibility cannot occur. Also, as shown by van der Waall, there is exactly one non-monomial group G of order 108 which has GAP ID [108, 15] and $\mathrm{cd}(G) = \{1, 3, 4\}$. This non-monomial group presents an example of 3-dimensional Galois representations that Artin's conjecture is unknown. We further note that if $|G'|$ is 1, 2, 4, 3, 6, 12, 9, 18, 36, or 27, then $\mathrm{cd}(G') \subseteq \{1, 2, 3, 4\}$ where $4 \in \mathrm{cd}(G')$

only if $|G'| = 36$ and $|G''| = 9$. However, as may be checked in GAP, G with $|G'| = 36$ does not exist.

Thus, unless $|G'|$ is 27, G is monomial. For $\mathrm{cd}(G') \subseteq \{1, 2, 3\}$, applying Proposition 2.4, for every $\chi \in \mathrm{Irr}(G)$, there is a subnormal subgroup H with $G' \leq H$ and $\psi \in \mathrm{Irr}(H)$ such that $\mathrm{Ind}_H^G \psi = \chi$ and $\psi|_{G'} \in \mathrm{Irr}(G')$. As all proper subgroups of G have been shown to be of automorphic type, Theorem 3.1 tells us that if $H \neq G$, then χ is of automorphic type. So we may assume $H = G$. Now as G is monomial, the works of Artin, Langlands–Tunnell, Arthur–Clozel, and Jacquet–Piatetski–Shapiro–Shalika imply that χ is of automorphic type.

Also, if $|G'| = 27$, G has the derived subgroup isomorphic to either C_{27}, C_3^3, $C_9 \times C_3$, or $C_3^2 \rtimes C_3$. Clearly, for the first three cases, G is metabelian and of automorphic type.

If $G' \simeq C_3^2 \rtimes C_3$, then $\mathrm{cd}(G') \subseteq \{1, 3\}$. Thus, Proposition 2.4 implies that $\mathrm{cd}(G)$ is contained in $\{1, 2, 3, 4, 6\}$. Moreover, for $\chi \in \mathrm{Irr}(G)$ of degree 2, 4, or 6, χ must be induced from a character of a subnormal subgroup of G. Again, as all proper subgroups of G have been shown to be of automorphic type, Theorem 3.1 yields χ is of automorphic type. Also, if G is monomial, then its irreducible characters of degree 3 must be of automorphic type.

5.2. *The case* $|G| = 120$

For non-monomial G of order 120, van der Waall [24] showed that G is either $C_5 \times SL_2(\mathbb{F}_3)$, $C_2 \times A_5$, S_5, or $SL_2(\mathbb{F}_5)$, where the later three cases are only non-solvable groups of order 120. We also note that the functoriality of $GL(n) \times GL(1)$ tells us that $C_5 \times SL_2(\mathbb{F}_3)$ is of automorphic type.

Now we may assume G is monomial (and solvable) and look at its derived subgroup. As shown by van der Waall, $|G'|$ cannot be 24 nor 40. In other words, G' is possibly of order 1, 2, 4, 8, 3, 6, 12, 5, 10, 20, 15, 30, or 60. It is not hard to see that as before, $\mathrm{cd}(G') \subseteq \{1, 2, 3\}$ unless $|G'|$ is 20 or 60. However, as one can check by GAP or hand, if G' is of order 20, then G' is abelian. Also, if $|G'| = 60$, then $G' \simeq C_5 \times A_4$. Thus, $\mathrm{cd}(G') = \{1, 3\}$ in this case. Now, the automorphy of G follows from Lemma 3.10.

5.3. *The case* $|G| = 144$

Suppose $|G|$ is 144. By [24, Theorem 4.1], $|G'|$ cannot be 48. Also, if G' is of order 1, 2, 4, 8, 16, 3, 6, 12, 9, or 18, then as above, we have $\mathrm{cd}(G') \subseteq \{1, 2, 3\}$. Since G is either monomial (unless $8 \mid |G'|$) or nearly nilpotent in these cases, the argument used in the second paragraph of Sec. 5.1 yields the automorphy of G.

On the other hand, if $|G'| = 36$, $\mathrm{cd}(G') \subseteq \{1, 2, 3, 4\}$ where $4 \in \mathrm{cd}(G')$ only if G'' is of order 9. Similarly, if $4 \notin \mathrm{cd}(G')$, G is of automorphic type. Otherwise, G/G'' is of order 16 and G is nearly nilpotent.

According to [24, Theorem 4.6], if G' is equal to 24, then G' is isomorphic to $SL_2(\mathbb{F}_3)$ or $C_3 \times Q$. For the latter possibility, G is nearly nilpotent. Therefore, we may assume $G' \simeq SL_2(\mathbb{F}_3)$. In this case, G'' is of order 8 and $\mathrm{cd}(G') = \{1, 2, 3\}$. Via Proposition 2.4, for any $\chi \in \mathrm{Irr}(G)$, there exist a subnormal group H containing G' and $\psi \in \mathrm{Irr}(H)$ with $\psi(1) \le 3$ such that $\chi = \mathrm{Ind}_H^G \psi$. If H is a proper subgroup, then Arthur–Clozel's automorphic induction and the fact that we have derived the automorphy for all proper subgroups of G imply that χ is of automorphic type. Thus, it suffices to consider the case when $H = G$ and $\chi(1) \le 3$. Furthermore, as G/G'' is supersolvable, Proposition 2.3 tells us that if $\chi \in \mathrm{Irr}(G)$ is of degree 3, then it is monomial and automorphic. Hence, for $\chi(1) \le 3$, χ is of automorphic type.

Finally, assume $|G'| = 72$. By [24, Theorem 4.3], G' is either $C_3 \times SL_2(\mathbb{F}_3)$ or $Q \rtimes C_9$. In both cases, G/G'' is supersolvable and $|G''| = 8$. Using GAP, we find that $\mathrm{cd}(G) = \{1, 2, 3, 4, 6\}$. As above, Proposition 2.3 implies that if $\chi \in \mathrm{Irr}(G)$ is of degree at most 3, then χ is of automorphic type. On the other hand, for $\chi(1) = 4$, as $|G/G''| = 18$ and $4 \nmid 18$, Proposition 2.3 yields that it is induced from a 2-dimensional character of a normal subgroup, and hence of automorphic type. Thus, we may assume $\chi(1) = 6$. As $\mathrm{cd}(C_3 \times SL_2(\mathbb{F}_3)) = \mathrm{cd}(Q \rtimes C_9) = \{1, 2, 3\}$ and $|G/G'| = 2$, Proposition 2.4 asserts that χ must be induced from a 3-dimensional character of G'. Since G' is normal in G and has been shown to be of automorphic type, applying Theorem 3.1 completes the proof.

5.4. *The case* $|G| = 160$

Assume G is solvable and of order 160. By [24, Theorem 5.2], if $|G'| = 40$, then G' is cyclic. Similarly, one can show that if $|G'| = 20$, then G' is abelian (which can be examined by GAP as well). Therefore, if $|G'|$ is not 32 nor 80, then $\mathrm{cd}(G') \subseteq \{1, 2\}$ and G is nearly nilpotent.

For $|G'| = 80$, van der Waall derived that the Sylow 2-subgroup P of G' is normal and elementary abelian. Hence, G/P is supersolvable, and G is monomial. However, there is a 5-dimensional character of G which is not induced from any (sub)normal subgroup of G. Indeed, this group has GAP ID [160, 234].

If G' is of order 32, van der Waall showed that $G' = Q * D$, the extra-special group of order 32 of $(-)$-type. Moreover, for every irreducible representation ρ of G, either

(1) ρ can be regarded as a representation of $G/\mathbf{Z}(G')$; or
(2) ρ is non-monomial and of dimension 4.

As mentioned by van der Waall, $\mathbf{Z}(G') = G''$ is of order 2. Thus, $G/\mathbf{Z}(G')$ is metabelian and hence of automorphic type. Therefore, we may just consider the second case. First, if ρ is not faithful, then it can be seen as a representation of $G/\operatorname{Ker}(\rho)$. Since all groups of order dividing (but not equal to) 160 have been shown to be of automorphic type, ρ is of automorphic type.

Thus, we may assume ρ is faithful, and $\rho(G)$ is a solvable subgroup of order 160 of $GL_4(\mathbb{C})$. By Schur's lemma, the projective image of ρ in $PGL_4(\mathbb{C})$ is isomorphic to $\rho(G)/\mathbf{Z}(\rho(G)) \simeq G/\mathbf{Z}(G)$. Since $G = (Q * D) \rtimes C_3$, as may be checked in GAP [7] for example, one can deduce that $G/\mathbf{Z}(G)$ is isomorphic to $E_{2^4} \rtimes C_5$. By Martin's result (Proposition 3.5), ρ is of automorphic type.

5.5. *The case* $|G| = 168$

First of all, we know that there is only one non-solvable group, $GL_3(\mathbb{F}_2)$, of order 168, and hence we may assume G is solvable. For G' of order 1, 2, 4, 8, 3, 6, 12, 24, 7, 14, 28, or 21, $\mathrm{cd}(G') \subseteq \{1, 2, 3\}$. Also, if $|G'| = 42$ or $|G'| = 84$, then it can be checked by [7] that either G' is abelian or $\mathrm{cd}(G') = \{1, 3\}$ (in fact, their GAP ID are [42, 6] and [84, 10], respectively). Since $36 \nmid 168$, Lemma 3.10 asserts that G is of automorphic type in these cases.

Now it remains to consider the case $|G'| = 56$. By [25, Theorem 7.3], G'' is either of order 8 or 2. For the latter case, van der Waall noted that G' is isomorphic to $C_7 \times Q$; and hence G is nearly nilpotent. On the other hand, if $|G''| = 8$, then it is elementary abelian and G is monomial. However, there exists a 7-dimensional character of G which is not induced from any (sub)normal subgroup of G. In fact, this group G has GAP ID [168, 43], where $\mathrm{cd}(G) = \{1, 3, 7\}$.

5.6. *The case* $|G| = 192$

In this section, G means a group of order 192. As before, if G' is of order 1, 2, 4, 8, 16, 3, 6, 12, or 24, then $\mathrm{cd}(G') \subseteq \{1, 2, 3\}$. For $|G'| = 48$, it can be checked that $\mathrm{cd}(G') \subseteq \{1, 2, 3\}$ by using [25, Theorem 8.4] or GAP. Moreover, as $36 \nmid 192$, Lemma 3.10 asserts that G is of automorphic type in these cases.

It remains to consider that $|G'|$ is equal to 32, 64, or 96.

Now assume $|G'| = 32$. Then [25, Theorem 8.6] says

(1) G is not monomial if and only if G' is isomorphic to $C_2^2 \times Q$ or
$$\langle a, b \,|\, a^4 = b^4 = c^2 = 1, [a, b] = c, [a, c] = [b, c] = 1\rangle.$$
(2) G is monomial if and only if $G' = Q * Q$, the extra-special group of order 32 of (+)-type.

For the first case, G' has GAP ID [32, 2] or [32, 47] and $\mathrm{cd}(G') = \{1, 2\}$. Thus, G is nearly nilpotent. For the second case, G has GAP ID [192, 201], [192, 202], [192, 1508], or [192, 1509]. For the latter two possibilities, $\mathrm{cd}(G) = \{1, 3, 4\}$. By Lemma 2.9, we only need to deal with $\chi \in \mathrm{Irr}(G)$, afforded by ρ, of degree 4. As discussed in Sec. 5.4, if ρ is not faithful, then it can be seen as a representation of $G/\mathrm{Ker}(\rho)$. Since all groups of order dividing (but not equal to) 192 have been shown to be of automorphic type, ρ is of automorphic type. If ρ is faithful, the projective image of ρ in $PGL_4(\mathbb{C})$ is isomorphic to $\rho(G)/\mathbf{Z}(\rho(G)) \simeq G/\mathbf{Z}(G)$. Again, as may be checked in GAP [7] for example, one can deduce that $G/\mathbf{Z}(G)$ is isomorphic to $V_4 \rtimes A_4$. Now, Proposition 3.6 tells us that ρ is of automorphic type.

If G has GAP ID [192, 201] or [192, 202], then $\mathrm{cd}(G) = \{1, 3, 4, 6\}$. As now $\mathrm{cd}(G') = \{1, 4\}$ and G/G' is an abelian group of order 6, if $\chi \in \mathrm{Irr}(G)$ has degree

Table 1. The case $|G| = 192$ with $|G'| = 64$.

G	$\mathrm{cd}(G)$	G'	$\mathrm{cd}(G')$	$G/\mathbf{Z}(G)$
[192, 3]	—	[64, 2]	$\{1\}$	—
[192, 4]	$\{1,2,3,4,6\}$	[64, 19]	$\{1,2,4\}$	$V_4 \cdot A_4$
[192, 1020]	—	[64, 192]	$\{1\}$	—
[192, 1021]	$\{1,2,3,4,6\}$	[64, 224]	$\{1,2,4\}$	$V_4 \rtimes A_4$
[192, 1022]	$\{1,2,3,4,6\}$	[64, 239]	$\{1,2,4\}$	$V_4 \rtimes A_4$
[192, 1023]	$\{1,3,12\}$	[64, 242]	$\{1,4\}$	—
[192, 1024]	$\{1,3,4\}$	[64, 242]	$\{1,4\}$	$V_4 \rtimes A_4$
[192, 1025]	$\{1,3,12\}$	[64, 245]	$\{1,4\}$	—
[192, 1541]	—	[64, 267]	$\{1\}$	—

3, it is monomial. Also, if $\chi(1) = 6$, then GAP library tells us that χ is not faithful and hence of automorphic type. On the other hand, in each case, all irreducible 4-dimensional characters of G are faithful and $G/\mathbf{Z}(G)$ has GAP ID [96, 70]. Although here we no longer have any analogue of a result of Martin (Proposition 3.6), GAP tells us that all irreducible 4-dimensional representations of these two groups are of $GO(4)$-type, and thus of automorphic type by Proposition 3.4.

We now consider the case $|G'| = 64$. By GAP, we have Table 1.

For the first, third, and last cases, G is metabelian and hence of automorphic type. For the remaining cases, as $|G/G'| = 3$, if $\chi \in \mathrm{Irr}(G)$ is of order 3, 6, or 12, then it must be induced from G' and hence of automorphic type. Also, by Proposition 3.3, we now only need to consider the case $\chi(1) = 4$. By a similar argument as above, it suffices to consider faithful irreducible 4-dimensional representations ρ of G. For such instances, $G/\mathbf{Z}(G)$ must be an extension of A_4 by V_4 and Proposition 3.6 implies that ρ is of automorphic type.

Finally, it remains to consider the case $|G'| = 96$. Again, the GAP library gives Table 2.

As mentioned in Table 2 (or by [25, Theorem 8.7]), G'' is of order 32 and G/G'' is supersolvable. Thus, every irreducible 3-dimensional character of G is monomial.

Table 2. The case $|G| = 192$ with $|G'| = 96$.

G	$\mathrm{cd}(G)$	G'	$\mathrm{cd}(G')$	G''	$\mathrm{cd}(G'')$
[192, 180]	$\{1,2,3,4,6\}$	[96, 3]	$\{1,2,3,6\}$	[32, 2]	$\{1,2\}$
[192, 181]	$\{1,2,3,4,6\}$	[96, 3]	$\{1,2,3,6\}$	[32, 2]	$\{1,2\}$
[192, 1489]	$\{1,2,3,4,6\}$	[96, 203]	$\{1,2,3,6\}$	[32, 47]	$\{1,2\}$
[192, 1490]	$\{1,2,3,4,6\}$	[96, 203]	$\{1,2,3,6\}$	[32, 47]	$\{1,2\}$
[192, 1491]	$\{1,2,3,4,6,8\}$	[96, 204]	$\{1,3,4\}$	[32, 49]	$\{1,4\}$
[192, 1492]	$\{1,2,3,4,6,8\}$	[96, 204]	$\{1,3,4\}$	[32, 49]	$\{1,4\}$
[192, 1493]	$\{1,2,3,4,6,8\}$	[96, 204]	$\{1,3,4\}$	[32, 49]	$\{1,4\}$
[192, 1494]	$\{1,2,3,4,6,8\}$	[96, 204]	$\{1,3,4\}$	[32, 49]	$\{1,4\}$

As a consequence, if $\chi \in \mathrm{Irr}(G)$ is of degree at most 3, then it is of automorphic type.

For the first four cases, it is clear that if $\chi \in \mathrm{Irr}(G)$ is of degree 4, then it must be induced from a 2-dimensional character of G' (of index 2 in G), and hence it is of automorphic type. According to GAP, each of these four groups has three irreducible 6-dimensional characters χ_1, χ_2, and χ_3 (say) where χ_1 and χ_2 are tensor products of irreducible 2-dimensional characters and irreducible 3-dimensional characters (of G), and $\chi_3|_{G'}$ is reducible. As all irreducible characters of G of degree at most 3 are of automorphic type, the functoriality of $GL(2) \times GL(3)$, proved by Kim and Shahidi [13], implies that χ_1 and χ_2 are of automorphic type. On the other hand, by Proposition 2.4, χ_3 is induced from a 3-dimensional character of G'. Thus, χ_3 is of automorphic type.

Similarly, for the last four cases, if $\chi \in \mathrm{Irr}(G)$ is of degree 6 or 8, then it must be induced from a character of G' since $|G/G'| = 2$. As any group of order 96 is of automorphic type, Theorem 3.1 implies that χ is of automorphic type in this case. Therefore, it only remains irreducible characters of degree 4. Indeed, as may be checked in GAP, each group has exactly two faithful irreducible 4-dimensional characters. In all these instances, $G/\mathbf{Z}(G)$ has GAP ID [96, 227]. Unfortunately, we do not have any analogue of a result of Martin (again). However, GAP tells us that for these groups, their irreducible 4-dimensional representations are all imprimitive and essentially self-dual (where if G has GAP ID [192, 1491] or [192, 1493], then its irreducible 4-dimensional representations are indeed of $GO(4)$-type). Now, applying a result of Martin and Ramakrishnan (Proposition 3.7) completes the proof.

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