Sample Questions 7

1. Let $\mathcal{M}_{2\times 2}$ be the vector space of all 2×2 matrices. Determine whether S is a subspace in $\mathcal{M}_{2\times 2}$ or not. If yes, write S as the span of some finite set of vectors.

   (a) $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R} \right\}$

   (b) $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a + b = 5 \right\}$

   (c) $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a + b = 0 \in \mathbb{R} \right\}$

2. Determine whether $\mathbf{v} \in \operatorname{span}(S)$.

   (a) $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

   (b) $\mathbf{v} = x - x^3$, $S = \{x^2, 2x + x^2, x + x^3\}$

   (c) $\mathbf{v} = \begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix}, S = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix} \right\}$

3. Determine whether $\operatorname{span}(S) = \mathbb{R}^3$.

   (a) $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$

   (b) $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \right\}$

   (c) $S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

4. Let $\mathcal{F}$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. A function $f \in \mathcal{F}$ is *even* if $f(-x) = f(x)$ for all $x \in \mathbb{R}$, and is *odd* if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Show that the set of all even functions is a subspace in $\mathcal{F}$, and the set of all odd functions is also a subspace in $\mathcal{F}$.

5. Every homogeneous linear equation can be written as

$$\begin{bmatrix} — & \mathbf{v}_1 & — \\ — & \vdots & — \\ — & \mathbf{v}_m & — \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

   Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$. Then the solutions are

$$\{\mathbf{c} \in \mathbb{R}^n : \mathbf{c} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{v} \in S\}.$$

   Show that this set is the same as

$$\{\mathbf{c} \in \mathbb{R}^n : \mathbf{c} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{v} \in \operatorname{span}(S)\}.$$

   [Actually, if S' is obtained from S by row operations, then $\operatorname{span}(S) = \operatorname{span}(S')$.]

6. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$,

$$\mathbf{A} = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ | & | & | \end{bmatrix}, \text{ and } \mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

   For a given vector $\mathbf{b} \in \mathbb{R}^m$, show that $\mathbf{Ac} = \mathbf{b}$ has a solution if and only if $\mathbf{b} \in \operatorname{span}(S)$.

7. Let S, $\mathbf{A}$, and $\mathbf{c}$ be the same as that in Question 6. Show that S is linearly independent if and only if $\mathbf{Ac} = \mathbf{0}$ has a unique solution (the trivial solution).