Sample ~~Sol~~ Solutions for Sample Questions 2

1. Assume
$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

Compute
$$A(u+v) = A \begin{pmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{pmatrix} = \begin{pmatrix} a_{11}(u_1+v_1) + a_{12}(u_2+v_2) + \cdots + a_{1n}(u_n+v_n) \\ \vdots \\ a_{m1}(u_1+v_1) + a_{m2}(u_2+v_2) + \cdots + a_{mn}(u_n+v_n) \end{pmatrix}$$

$$Au + Av = \begin{pmatrix} a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n \\ \vdots \\ a_{m1}u_1 + a_{m2}u_2 + \cdots + a_{mn}u_n \end{pmatrix} + \begin{pmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ \vdots \\ a_{m1}\overset{v_1}{\cancel{u_1}} + a_{m2}\overset{v_2}{\cancel{u_2}} + \cdots + a_{mn}\overset{v_n}{\cancel{u_n}} \end{pmatrix}$$

So $A(u+v) = Au + Av$.

Compute
$$A(rv) = A \begin{pmatrix} rv_1 \\ \vdots \\ rv_n \end{pmatrix} = \begin{pmatrix} a_{11}rv_1 + \cdots + a_{1n}rv_n \\ \vdots \\ a_{m1}rv_1 + \cdots + a_{mn}rv_n \end{pmatrix}$$

$$r(Av) = r \cdot \begin{pmatrix} a_{11}v_1 + \cdots + a_{1n}v_n \\ \vdots \\ a_{m1}v_1 + \cdots + a_{mn}v_n \end{pmatrix}.$$

So $A(rv) = r(Av)$.

2. "prove $A \subseteq B$"

Suppose $v = c_1 v_1 + c_2 v_2 + c_3 v_3 \in A$.

$$\Rightarrow v = \begin{pmatrix} -c_1 - c_2 - c_3 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Compute $x + y + z + w = (-c_1 - c_2 - c_3) + c_1 + c_2 + c_3 = 0$.

$\Rightarrow v \in B$.

"prove $B \subseteq A$".

Suppose $v \in B$. Then v is a solution of $x + y + z + w = 0$.

free

Solve $x + y + z + w = 0$.

Let $y = c_1$ Then $x = -c_1 - c_2 - c_3$.

$z = c_2$

$w = c_3$

$$\Rightarrow v \text{ can be written as } \begin{pmatrix} -c_1 - c_2 - c_3 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$= c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in A.$$

3.

Consider the three types of row operations.

①. $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \xrightarrow{\rho_i \leftrightarrow \rho_j} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ change the positions only.

② $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \xrightarrow{k\rho_i} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ replace 0 by $k \cdot 0 = 0$.

③. $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \xrightarrow{k\rho_i + \rho_j} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ replace 0 by $k \cdot 0 + 0 = 0$.

So $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \xrightarrow[\text{row operation}]{\text{any}} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

4.

augmented matrix $\left(\begin{array}{cc|c} 3 & 6 & 18 \\ 1 & 2 & 6 \end{array}\right) \longrightarrow$ echelon form $\left(\begin{array}{cc|c} 3 & 6 & 18 \\ 0 & 0 & 0 \end{array}\right)$

$$\begin{array}{c} \uparrow \\ \text{free.} \end{array}$$

particular solution:

set $y=0$

$3x \quad = 18 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} = \vec{p}.$

homogeneous solution. $\boxed{\text{Solve } R\vec{v} = \vec{0} \text{ !!!}}$

set $y=1$, solve $\left(\begin{array}{cc|c} 3 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$3x + 6 = 0 \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \vec{\beta}_1.$

$\Rightarrow$ general solution

$$= \left\{ \vec{p} + c_1 \vec{\beta}_1 \;\middle|\; c_1 \in R \right\}.$$

5. augmented matrix

$$\left(\begin{array}{cccc|c} 0 & 1 & 2 & -1 & 3 \\ 1 & 2 & 1 & 0 & 4 \\ 1 & 1 & -1 & 1 & 1 \end{array}\right) \xrightarrow{\rho_1 \leftrightarrow \rho_3} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 1 & 2 & 1 & 0 & 4 \\ 0 & 1 & 2 & -1 & 3 \end{array}\right)$$

$$\xrightarrow{-\rho_1+\rho_2} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & -1 & 3 \end{array}\right) \xrightarrow{-\rho_2+\rho_3} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ & & \uparrow & \uparrow & \\ & & \text{free} & & \end{array}\right).$$

Solve particular solution $\vec{p}$.

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ & 1 & 2 & -1 & 3 \\ & & \uparrow & \uparrow & \\ & & 0 & 0 & \end{array}\right) \Rightarrow \vec{p} = \begin{pmatrix} -2 \\ 3 \\ 0 \\ 0 \end{pmatrix}$$

Solve for homogeneous solution $\vec{\beta_1}$, $\vec{\beta_2}$.

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \, \cancel{*} \\ & 1 & 2 & -1 & 0 \, \cancel{*} \\ & & \uparrow & \uparrow & \\ & & 1 & 0 & \end{array}\right) \Rightarrow \vec{\beta_1} = \begin{pmatrix} \cancel{*}\,3 \\ \cancel{*}\,-2 \\ 1 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ & 1 & 2 & -1 & 0 \\ & & \uparrow & \uparrow & \\ & & 0 & 1 & \end{array}\right) \Rightarrow \vec{\beta_2} = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \text{gen sol} = \left\{ \vec{p} + c_1\vec{\beta_1} + c_2\vec{\beta_2} \;\middle|\; c_1, c_2 \in \mathbb{R} \right\}.$$

6.

$$\left(\begin{array}{ccccc|c}1&1&1&1&1&1\\2&2&2&2&2&2\end{array}\right)\xrightarrow{-2\rho_1+\rho_2}\begin{array}{c}\left(\begin{array}{ccccc|c}1&1&1&1&1&1\\ &&&&&\end{array}\right)\\ \begin{array}{ccccc}&\uparrow&\uparrow&\uparrow&\uparrow\end{array}\\ \text{free}\end{array}$$

① Solve $\vec{p}$ by $R\vec{v}=\vec{r}$

$$\begin{array}{c}\left(\begin{array}{ccccc|c}1&1&1&1&1&1\\ &&&&&\end{array}\right)\\ \begin{array}{ccccc}&\uparrow&\uparrow&\uparrow&\uparrow\\ &0&0&0&0\end{array}\end{array}\Rightarrow\vec{p}=\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}$$

② Solve $R\vec{v}=\vec{0}$ for $\vec{\beta_1},\ldots,\vec{\beta_4}$.

$$\begin{array}{c}\left(\begin{array}{ccccc|c}1&1&1&1&1&0\\ &&&&&\end{array}\right)\\ \begin{array}{ccccc}&\uparrow&\uparrow&\uparrow&\uparrow\\ &1&0&0&0\end{array}\end{array}\Rightarrow\vec{\beta_1}=\begin{pmatrix}-1\\1\\0\\0\\0\end{pmatrix}$$

$$\begin{array}{c}\left(\begin{array}{ccccc|c}1&1&1&1&1&0\\ &&&&&\end{array}\right)\\ \begin{array}{ccccc}&\uparrow&\uparrow&\uparrow&\uparrow\\ &0&1&0&0\end{array}\end{array}\longrightarrow\vec{\beta_2}=\begin{pmatrix}-1\\0\\1\\0\\0\end{pmatrix}$$

$$\begin{array}{c}\left(\begin{array}{ccccc|c}1&1&1&1&1&0\\ &&&&&\end{array}\right)\\ \begin{array}{ccccc}&\uparrow&\uparrow&\uparrow&\uparrow\\ &0&0&1&0\end{array}\end{array}\Rightarrow\vec{\beta_3}=\begin{pmatrix}-1\\0\\0\\1\\0\end{pmatrix}$$

$$\begin{array}{c}\left(\begin{array}{ccccc|c}1&1&1&1&1&0\\ &&&&&\end{array}\right)\\ \begin{array}{ccccc}&\uparrow&\uparrow&\uparrow&\uparrow\\ &0&0&0&1\end{array}\end{array}\Rightarrow\vec{\beta_4}=\begin{pmatrix}-1\\0\\0\\0\\1\end{pmatrix}$$

So

gen sol

$$=\left\{\vec{p}+c_1\vec{\beta_1}+c_2\vec{\beta_2}+c_3\vec{\beta_3}+c_4\vec{\beta_4}\;\middle|\;c_1,c_2,c_3,c_4\in\mathbb{R}\right\}.$$

7.

(a). $\left(\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow{\rho_1 \leftrightarrow \rho_2} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{array}\right)$

$\xrightarrow[-\rho_1+\rho_4]{-\rho_1+\rho_3} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right) \xrightarrow[-\rho_2+\rho_4]{-\rho_2+\rho_3} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & -1 & 0 \\ 0 & 0 & -1 & -2 & 0 \end{array}\right)$

$\xrightarrow{-\frac{1}{2}\rho_3+\rho_4} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 0 \\ & & -2 & 1 & 0 \\ & & & -1.5 & 0 \end{array}\right)$ $\Rightarrow$ unique solution

$\Rightarrow$ nonsingular.

[So keep writing $\left|\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}\right)$ is boring.].

(b) $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{pmatrix} \xrightarrow[-\rho_1+\rho_2]{\substack{-\rho_3+\rho_4 \\ -\rho_2+\rho_3}} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix}$

$\xrightarrow[-\rho_2+\rho_4]{-\rho_2+\rho_3} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_2} \begin{pmatrix} 4 & 4 & 4 & 4 \\ & 1 & 2 & 3 \\ & & & \\ & & & \end{pmatrix}$

free.

$\Rightarrow$ many solution

$\Rightarrow$ singular.