

Sample Questions 15

For all problems, let $\mathcal{M}_{m \times n}$ be the space of all 2×3 matrices, and let \mathcal{P}_n be the space of all polynomials of degree at most n .

Let E_{ij} be the 2×3 matrix whose entries are all zeros except that the i, j -entry is one. Then

$$\mathcal{B} = \{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$$

is a basis of $\mathcal{M}_{2 \times 3}$. Suppose $f : \mathcal{M}_{2 \times 3} \rightarrow \mathcal{M}_{2 \times 3}$ is a homomorphism such that $\text{Rep}_{\mathcal{B}, \mathcal{B}}(f)$ equals

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

1. Let $\mathbf{v} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Find $f(\mathbf{v})$.
2. Find a matrix \mathbf{v} such that $f(\mathbf{v}) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ or show such a matrix does not exist.
3. Find the nullspace and the nullity of f .
4. Find the range and the rank of f .

Let $\mathcal{B} = \{1, x\}$ be a basis of \mathcal{P}_2 and $\mathcal{D} = \{1, x, x^2\}$ be a basis of \mathcal{P}_3 . Define $f : \mathcal{P}_2 \rightarrow \mathcal{P}_3$ by $f(p(x)) = (x + 1) \cdot p(x)$. Define $g : \mathcal{P}_2 \rightarrow \mathcal{P}_3$ by $g(p(x)) = (x - 1) \cdot p(x)$.

5. Check that $f + g$ is also a homomorphism from \mathcal{P}_2 to \mathcal{P}_3 defined by $(f + g)(p(x)) = (2x) \cdot p(x)$. Let $\mathbf{A} = \text{Rep}_{\mathcal{B}, \mathcal{D}}(f)$, $\mathbf{B} = \text{Rep}_{\mathcal{B}, \mathcal{D}}(g)$, and $\mathbf{C} = \text{Rep}_{\mathcal{B}, \mathcal{D}}(f + g)$. Find \mathbf{A} , \mathbf{B} , \mathbf{C} and then check if $\mathbf{A} + \mathbf{B} = \mathbf{C}$. Is $f + g$ one-to-one? Is it onto?

Let $\mathcal{B}_n = \{1, \dots, x^n\}$ be a basis of \mathcal{P}_n . Define $f : \mathcal{P}_3 \rightarrow \mathcal{P}_2$ by $f(p(x)) = p'(x)$. Define $g : \mathcal{P}_2 \rightarrow \mathcal{P}_1$ by $g(p(x)) = p'(x)$. That is, both f and g are the differential operator, with different domains and different codomains.

6. Check that $g \circ f$ is a homomorphism from \mathcal{P}_3 to \mathcal{P}_1 defined by $g \circ f(p(x)) = p''(x)$. Let $\mathbf{A} = \text{Rep}_{\mathcal{B}_3, \mathcal{B}_2}(f)$, $\mathbf{B} = \text{Rep}_{\mathcal{B}_2, \mathcal{B}_1}(g)$, and $\mathbf{C} = \text{Rep}_{\mathcal{B}_3, \mathcal{B}_1}(g \circ f)$. Find \mathbf{A} , \mathbf{B} , \mathbf{C} and then check if $\mathbf{BA} = \mathbf{C}$. Is $g \circ f$ one-to-one? Is it onto?

7. Define $\text{id} : \mathcal{P}_2 \rightarrow \mathcal{P}_2$ by $\text{id}(p(x)) = p(x)$. Let $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{D} = \{1, x + 1, (x + 1)^2\}$ be two bases of \mathcal{P}_2 . Find $\text{Rep}_{\mathcal{B}, \mathcal{D}}(\text{id})$. Is id nonsingular?