Sample Questions 11

1. Let $\mathbf{v} = (1,1,1)^\top$. Define $V_1 = \text{span}\{\mathbf{v}\}$ and
$$V_2 = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot \mathbf{v} = 0\}.$$
For each $\mathbf{x}$, find $\mathbf{x}_1 \in V_1$ and $\mathbf{x}_2 \in V_2$ such that $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$.

2. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ be the columns of
$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$
Let $V_1 = \text{span}(\{\mathbf{v}_1, \mathbf{v}_3\})$ and $V_2 = \text{span}(\{\mathbf{v}_2, \mathbf{v}_4\})$. Determine whether $\{V_1, V_2\}$ is linearly independent or not.

3. Let V_1 and V_2 be the same as the previous question. Find a basis for $V_1 + V_2$.

4. Determine whether f is an isomorphism.
   (a) $f : \mathbb{R}^3 \to \mathbb{R}^3$ by $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto \begin{bmatrix} a \\ a \\ a \end{bmatrix}$
   (b) $f : \mathcal{M}_{2\times 2} \to \mathbb{R}^4$ by
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a+b+c+d \\ a+b+c \\ a+b \\ a \end{bmatrix}$$
   (c) $f : \mathbb{R} \to \mathbb{R}$ by $x \mapsto x^3$

5. Let V be the plane in $\mathbb{R}^3$ defined by the equation $x+y+z=0$. It is known that V can also be written as
$$\left\{ a \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$
Find an isomorphism from V to $\mathbb{R}^2$. [Hint: The Rep function.]

6. Find a such that $\mathcal{M}_{m\times n} \equiv \mathbb{R}^a$ and find b such that $\mathcal{S}_n = \mathbb{R}^b$.

7. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis of $\mathbb{R}^n$. If a vector $\mathbf{y}$ has the representation
$$\text{Rep}_{\mathcal{B}}(\mathbf{y}) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$
How do you recover $\mathbf{y}$ from $\text{Rep}_{\mathcal{B}}(\mathbf{y})$? Indeed, find a matrix $\mathbf{A}$ such that
$$\mathbf{y} = \mathbf{A}\,\text{Rep}_{\mathcal{B}}(\mathbf{y}).$$
Conversely, how do you find $\text{Rep}_{\mathcal{B}}(\mathbf{y})$ from $\mathbf{y}$? Find a matrix $\mathbf{B}$ such that
$$\text{Rep}_{\mathcal{B}}(\mathbf{y}) = B\mathbf{y}.$$