國立中山大學 NATIONAL SUN YAT-SEN UNIVERSITY

線性代數（一） **MATH 103 / GEAI 1215: Linear Algebra I**

期末考 January 6, 2020 **Final Examination**

姓名 Name : solution

學號 Student ID # : ______________

Lecturer:	Jephian Lin 林晉宏
Contents:	cover page,
	8 pages of questions,
	score page at the end
To be answered:	on the test paper
Duration:	**110 minutes**
Total points:	**30 points** + 7 extra points

Do not open this packet until instructed to do so.

Instructions:

- Enter your **Name** and **Student ID #** before you start.
- Using the calculator is not allowed (and not necessary) for this exam.
- Any work necessary to arrive at an answer must be shown on the examination paper. Marks will not be given for final answers that are not supported by appropriate work.
- Clearly indicate your final answer to each question either by **underlining it or circling it**. If multiple answers are shown then no marks will be awarded.
- 可用中文或英文作答

1. [1pt] Write down an example of a system of **linear** equations in variables x, y, and z.

$$\left\{ x + y + z = 0. \right.$$

2. [1pt] Write down an example of a system of equations in variables x, y, and z that is **not a linear system**.

$$\left\{ x^2 + y^2 + z^2 = 0 \right.$$

3. [1pt] Write down an example of a system of **two linear equations** in its **echelon form** that contains **three free variables**.

$$\left\{ \begin{array}{l} x + z + u + w = 0 \\ y + z + u + w = 0 \end{array} \right.$$

$\uparrow \ \uparrow \ \uparrow$
free

4. [1pt] Write down an example of a 4×4 **singular** matrix.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

5. [1pt] Write down an example of a 4×4 **nonsingular** matrix.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[See Midterm 1]

6. [1pt] Let $S \subseteq \mathbb{R}^3$ be a set of vectors. Give an example of S such that $\text{span}(S) = \mathbb{R}^3$ and S is not linearly independent.

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

7. [1pt] Let $S \subseteq \mathbb{R}^3$ be a set of vectors. Give an example of S such that S is linearly independent and $\text{span}(S) \neq \mathbb{R}^3$.

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

8. [1pt] Let $S \subseteq \mathbb{R}^3$ be a set of vectors. Give an example of S such that S is linearly independent and $\text{span}(S) = \mathbb{R}^3$.

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

9. [1pt] Let $V \subseteq \mathbb{R}^3$ be a set of vectors. Give an example of V such that V is a not subspace of $\mathbb{R}^3$.

$$V = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

10. [1pt] Let $V \subseteq \mathbb{R}^3$ be a set of vectors. Give an example of V such that V is a subspace of $\mathbb{R}^3$.

$$V = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

[See Midterm 2]

11. [1pt] Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that f is an isomorphism.

$\cancel{f(x) \in \mathbb{R}}$.

$f(\vec{v}) = \vec{v}$ for all $\vec{v} \in \mathbb{R}^2$.

12. [1pt] Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that f is a homomorphism but not an isomorphism.

$f(\vec{v}) = \vec{0}$ for all $\vec{v} \in R^2$.

13. [1pt] Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that f is not a homomorphism.

$f(\vec{v}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for all $\vec{v} \in \mathbb{R}^2$.

14. [1pt] Suppose V_1 and V_2 are two subspaces of $\mathbb{R}^3$. Give an example of V_1 and V_2 such that they are linearly independent (in terms of subspaces).

$V_1 = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$ $V_2 = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$.

15. [1pt] Suppose V_1 and V_2 are two subspaces of $\mathbb{R}^3$. Give an example of V_1 and V_2 such that they are not linearly independent (in terms of subspaces).

$V_1 = V_2 = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$.

16. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\mathcal{D} = \{\mathbf{u}_1, \mathbf{u}_2\}$ with

$$\mathbf{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix}2\\3\end{bmatrix}, \text{ and } \mathbf{u}_2 = \begin{bmatrix}3\\2\end{bmatrix}.$$

Define a homomorphism $f : \mathbb{R}^3 \to \mathbb{R}^2$ such that $f(\mathbf{v}_1) = 5\mathbf{u}_1$, $f(\mathbf{v}_2) = 7\mathbf{u}_2$, and $f(\mathbf{v}_3) = 9\mathbf{u}_1 + 9\mathbf{u}_2$.

(a) [2pt] Find $\text{Rep}_{\mathcal{B},\mathcal{D}}(f)$.

$$f(\vec{v}_1) = 5\vec{u}_1 + 0\vec{u}_2 \xrightarrow{Rep_D} \begin{pmatrix}5\\0\end{pmatrix}$$

$$f(\vec{v}_2) = 0\vec{u}_1 + 7\vec{u}_2 \xrightarrow{Rep_D} \begin{pmatrix}0\\7\end{pmatrix} \quad \Rightarrow \quad Rep_{B,D}(f) = \begin{pmatrix}5 & 0 & 9\\ 0 & 7 & 9\end{pmatrix}$$

$$f(\vec{v}_3) = 9\vec{u}_1 + 9\vec{u}_2 \xrightarrow{Rep_D} \begin{pmatrix}9\\9\end{pmatrix}$$

(b) [3pt] Find a matrix A such that $f(\mathbf{v}) = A\mathbf{v}$ for any $\mathbf{v} \in \mathbb{R}^3$.

$$f\begin{pmatrix}1\\0\\0\end{pmatrix} = f(\vec{v}_1 - \vec{v}_2) = 5\vec{u}_1 - 7\vec{u}_2 = \begin{pmatrix}-11\\1\end{pmatrix}$$

$$f\begin{pmatrix}0\\1\\0\end{pmatrix} = f(\vec{v}_2 - \vec{v}_3) = 7\vec{u}_2 - 9\vec{u}_1 - 9\vec{u}_2 = -9\vec{u}_1 - 2\vec{u}_2 = \begin{pmatrix}-24\\-31\end{pmatrix}$$

$$f\begin{pmatrix}0\\0\\1\end{pmatrix} = f(\vec{v}_3) = 9\vec{u}_1 + 9\vec{u}_2 = \begin{pmatrix}45\\45\end{pmatrix}$$

$$\Rightarrow \quad A = \begin{pmatrix}-11 & -24 & 45\\ 1 & -31 & 45\end{pmatrix}.$$

Check: $A\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}10\\15\end{pmatrix} = 5\cdot\begin{pmatrix}2\\3\end{pmatrix}$

$A\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}21\\14\end{pmatrix} = 7\begin{pmatrix}3\\2\end{pmatrix}$. ✓

17. [5pt] Let $f : V \to W$ be a homomorphism. Show that $f(X)$ is a subspace of W if X is a subspace of V.

Claim: $f(X) \neq \phi$.

$\vec{y}_1, \vec{y}_2 \in f(X) \Rightarrow \vec{y}_1 + \vec{y}_2 \in f(X)$

$\vec{y}_1 \in f(X) \Rightarrow r\vec{y}_1 \in f(X)$.

① ~~$f(\vec{0}_V) =$~~ $\vec{0}_V \in X$

$f(\vec{0}_V) = 0_W \in f(X)$

$\Rightarrow f(X) \neq \phi$.

②. Let $\vec{y}_1, \vec{y}_2 \in f(X)$

$\vec{y}_1 = f(\vec{x}_1)$, $\vec{y}_2 = f(\vec{x}_2)$ for some $\vec{x}_1, \vec{x}_2 \in X$.

Since X is a subspace, $\vec{x}_1 + \vec{x}_2 \in X$.

$\Rightarrow \vec{y}_1 + \vec{y}_2 = f(\vec{x}_1 + \vec{x}_2) \in f(X)$.

③ Let $\vec{y}_1 \in f(X)$

$\vec{y}_1 = f(\vec{x}_1)$ for some $\vec{x}_1 \in X$.

Since X is a subspace, $r\vec{x}_1 \in X$

$\Rightarrow r\vec{y}_1 = f(r\vec{x}_1) \in f(X)$.

18. [5pt] Let $f : V \to W$ be a homomorphism. Show that f is one-to-one if and only if the null space of f is $\{\mathbf{0}\}$.

Claim: one-to-one $\Rightarrow$ nullspace $(f) = \{\vec{0}\}$.

Since f is a homomorphism,

$f(\vec{0}_V) = \vec{0}_W$. $\Rightarrow$ nullspace$(f) \supseteq \{\vec{0}\}$.

Suppose $f(\vec{x}) = \vec{0}_W$.

Since f is one-to-one,

$f(\vec{0}_V) = \vec{0}_W = f(\vec{x})$

implies $\vec{x} = \vec{0}_V$. $\Rightarrow$ nullspace $(f) = \{\vec{0}\}$.

Claim: nullspace $(f) = \{\vec{0}\}$ $\Rightarrow$ one-to-one.

Suppose $\vec{x}, \vec{y} \in V$ and $\vec{x} \neq \vec{y}$.

$\Rightarrow$ $\vec{x} - \vec{y} \neq \vec{0}$.

Since nullspace$(f) = \{\vec{0}\}$,

$f(\vec{x} - \vec{y}) \neq \vec{0}$.

$\Rightarrow$ $f(\vec{x}) - f(\vec{y}) \neq \vec{0}$

$\Rightarrow$ $f(\vec{x}) \neq f(\vec{y})$.

19. Let E_{ij} be the 2×3 matrix whose entries are all zeros except that the i, j-entry is one. Then

$$\mathcal{B} = \{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$$

is a basis of $\mathcal{M}_{2\times 3}$, the space of all 2×3 real matrices. Suppose $f : \mathcal{M}_{2\times 3} \to \mathcal{M}_{2\times 3}$ is a homomorphism such that $\text{Rep}_{\mathcal{B},\mathcal{B}}(f)$ equals

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

(a) [extra 1pt] Let $M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Find $f(M)$.

$$\text{Rep}_{\mathcal{B}}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$$

$$A\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 4 \\ 0 \\ 5 \end{pmatrix}$$

$$\Rightarrow f(M) = 1E_{11} + 0E_{12} + 2E_{13} + 4E_{21} + 0E_{22} + 5E_{23} = \begin{pmatrix} 1 & 0 & 2 \\ 4 & 0 & 5 \end{pmatrix}$$

(b) [extra 2pt] Find the range of f.

$$\text{Colspace}(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

$$\Rightarrow \text{range}(f) = \text{span}\{E_{11}, E_{13}, E_{21}, E_{23}\}.$$

(c) [extra 2pt] Find the nullspace of f.

$$\text{nullspace}(A) = \text{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

$$\Rightarrow \text{nullspace}(f) = \text{span}\{E_{13}, E_{23}\}.$$

20. [extra 2pt] Recall that $\mathcal{L}(V, W)$ is the space of all homomorphisms from V to W. Let $V = \mathcal{M}_{4\times 5}$ be the space of all 4×5 real matrices. Let $W = \mathcal{P}_{100}$ be the space of all polynomials with real coefficients and of degree at most 100. Answer the following questions:

(a) What is the zero vector in $\mathcal{L}(V, W)$?

(b) What is the dimension of V?

(c) What is the dimension of W?

(d) What is the dimension of $\mathcal{L}(V, W)$?

(a) $f : V \longrightarrow W$

~~$f(\vec{v}) = \vec{0}$ for all $\vec{v}$.~~

$f(A) = 0$ for all $A \in \mathcal{M}_{4\times 5}$.

(b) $4 \times 5 = 20$

(c) 101

(d) $20 \times 101 = 2020$.

[END]

Page	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	2	
Total	30 (+7)	