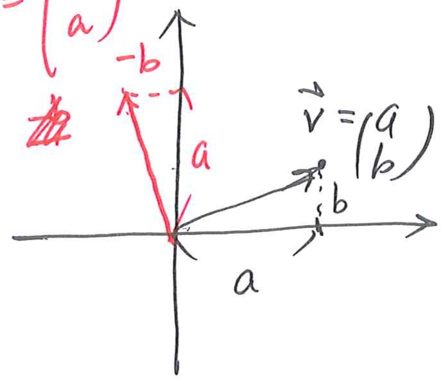
Linear algebra: diagonalization.

矩陣可視為一種變換

$$\vec{v} \longmapsto A\vec{v}$$

$\vec{v}$ 變成 $A\vec{v}$.

$$A\vec{v} = \begin{pmatrix} -b \\ a \end{pmatrix}$$



Exmp. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} a \\ b \end{pmatrix} \longmapsto \begin{pmatrix} -b \\ a \end{pmatrix}$$

逆時針轉 90°.

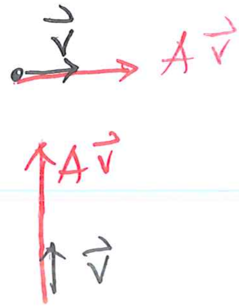
Exmp. $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

$$\begin{pmatrix} a \\ b \end{pmatrix} \longmapsto \begin{pmatrix} 2a \\ 3b \end{pmatrix}$$

x 方向 ×2 , y 方向 ×3.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$



Suppose for some $\underset{\neq 0}{\vec{v}} \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$

$$A\vec{v} = \lambda \vec{v}.$$

Then $\vec{v}$ is an eigenvector (特徵向量) with respect to the eigenvalue λ (特徵值).

$$\vec{v} \longmapsto A\vec{v} = \lambda \vec{v}.$$

$\vec{v}$ 經過 A 變換以後~~被~~被放大 λ 倍.

- 若 $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$, 則 $A^T = \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ a_{12} & & a_{m2} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{pmatrix}$

transpose
轉置.

Exmp. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

$B = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $B^T = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = B$.

- a ~~mtr~~ matrix A is <u>symmetric</u> if $A = A^T$.

Thm. Let A be a $n \times n$ symmetric matrix.
Then there are $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$
$\lambda_1, \ldots, \lambda_n \in \mathbb{R}$

such that
$$\begin{cases} \cdot\ A\vec{v}_i = \lambda_i \vec{v}_i & \text{for all } i. \\ \cdot\ \vec{v}_i \cdot \vec{v}_i = 1 & \text{for all } i \quad \leftarrow \text{unit length} \\ \cdot\ \vec{v}_i \cdot \vec{v}_j = 0 & \text{for all } i \neq j. \leftarrow \text{mutually orthogonal.} \end{cases}$$

Exmp. $\vec{v}_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\vec{v}_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\vec{v}_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ or $\vec{v}_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\vec{v}_2\ \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $\vec{v}_3\ \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

- We say $\{\vec{v}_1, \dots, \vec{v}_n\}$ is <u>orthonormal</u> if each $\vec{v}_i$ has unit length and they are mutually orthogonal.

- If $\{\vec{v}_1, \dots, \vec{v}_n\}$ is orthogonal and $A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}$,

then $A^T A = A A^T = I$ 單位矩陣. $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$$\underset{A^T}{\begin{pmatrix} — & \vec{v}_1 & — \\ & \vdots & \\ — & \vec{v}_n & — \end{pmatrix}} \underset{A}{\begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}} = (\vec{v}_i \cdot \vec{v}_j) = \underset{I.}{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}$$

$$A^T A = I \Rightarrow A^T = \underset{\text{inverse, 反矩陣.}}{A^{-1}} \Rightarrow A A^T = I.$$

Properties: $(I \sim 1,\ A^{-1} \sim \frac{1}{A})$

- $I \cdot A = A$, $A \cdot I = A$.
- $AB = AC \Rightarrow A^{-1}AB = A^{-1}AC$
  $\Rightarrow IB = IC \Rightarrow B = C$.
- $BA = CA \Rightarrow BAA^{-1} = CAA^{-1}$
  $\Rightarrow BI = CI \Rightarrow B = C$.
- $BC = BAA^{-1}C = BA^{-1}AC$

Consequence of eigenvectors.

A: sym matrix

$v_1, \ldots, v_n$ eigen~~values~~ vectors

$\lambda_1, \ldots, \lambda_n$ eigenvalues.

$$\begin{cases} A\vec{v}_i = \lambda_i \vec{v}_i & \text{for all } i. \\ \{\vec{v}_1, \ldots, \vec{v}_n\} & \text{is orthonormal.} \end{cases}$$

$\Rightarrow$. Let $Q = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix} \Rightarrow Q^TQ = QQ^T = I \Leftrightarrow Q^T = Q^{-1}$

$$AQ = A\begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ A\vec{v}_1 & \cdots & A\vec{v}_n \\ | & & | \end{pmatrix}$$

$$= \begin{pmatrix} | & & | \\ \lambda_1\vec{v}_1 & \cdots & \lambda_n\vec{v}_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

$$= Q \cdot \underbrace{D}_{\text{對角線矩陣}}$$

$$\boxed{\begin{aligned} AQ = QD &\Leftrightarrow Q^{-1}AQ = D \\ &\Leftrightarrow A = QDQ^{-1} \end{aligned}}$$

對角化.

$Q^{-1} = Q^T$ in this case.

Exmp. (對角化的好處)

If $A = QDQ^{-1}$,

then $A^{100} = (QDQ^{-1})(QDQ^{-1}) \cdots (QDQ^{-1})$

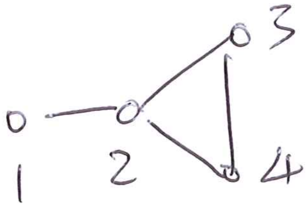
$= QD(Q^{-1}Q)D(Q^{-1}Q) \cdots (Q^{-1}Q)DQ^{-1}$

$= QD^{100}Q^{-1}$

Spectral embeding

Recall $L(G)_{i,j} = \begin{cases} \deg(i) & \text{if } i=j \\ -1 & \text{if } i \sim j \leftarrow \text{ij 相鄰.} \\ 0 & \text{o.w.} \end{cases}$

Laplacian matrix.



G

$$\longrightarrow \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

$L(G)$

Algorithm

graph dimension

Input: G, d

Output: positions of ~~..~~ vertices of G. in $\mathbb{R}^d$

1. Compute $L(G)$.
2. Compute $\{\vec{v}_1, \ldots, \vec{v}_n\}$, the orthonormal eigenvectors. $\lambda_1, \ldots, \lambda_n$
3. Let $Y = \underbrace{\begin{pmatrix} | & & | \\ \vec{v}_2 & \cdots & \vec{v}_{d+1} \\ | & & | \end{pmatrix}}_{\text{d 個}} \Big\}\text{n 維} = \begin{pmatrix} - & \vec{y}_1 & - \\ & \vdots & \\ - & \vec{y}_n & - \end{pmatrix} \Big\}\text{n 個}$ with $\underbrace{\quad\quad}_{\text{d 維.}}$
4. The positions of vertex i is $\vec{y}_i$.