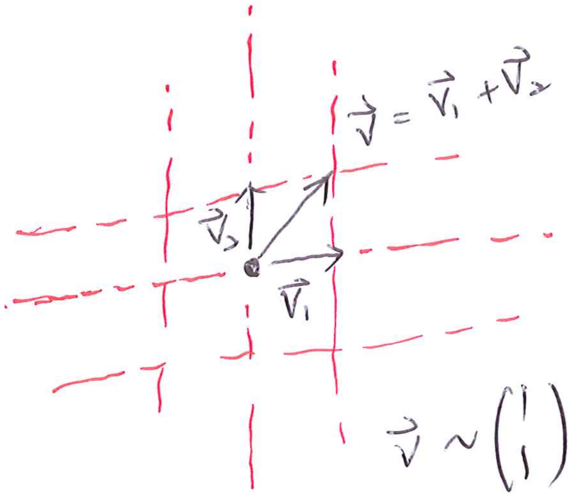
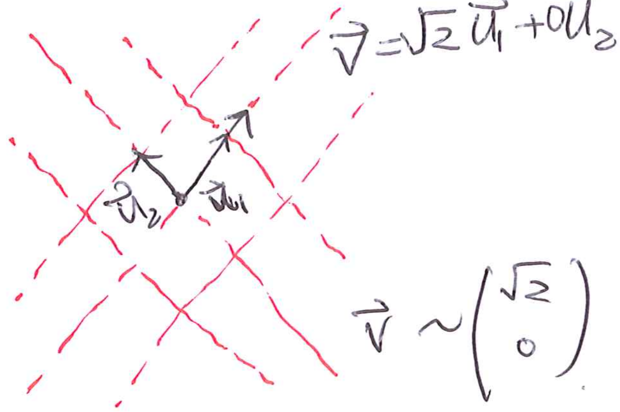
Linear algebra : decompositions.

basis and representation .

basis 基底 ~ 座標軸

representation 表示法 ~ 座標





- If $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is orthonormal in $\mathbb{R}^n$, then it is a basis and any vector $\vec{v} \in \mathbb{R}^n$ can be written as

$$\vec{v} = c_1 \vec{v}_1 + \ldots + c_n \vec{v}_n$$

for some $c_1, \ldots, c_n \in \mathbb{R}$.

- Note that $\vec{v} \cdot \vec{v}_1 = (c_1\vec{v}_1 + \ldots + c_n\vec{v}_n) \cdot \vec{v}_1$

$$= c_1\underbrace{(\vec{v}_1 \cdot \vec{v}_1)}_{1} + c_2\underbrace{(\vec{v}_2 \cdot \vec{v}_1)}_{0} + \ldots + c_n\underbrace{(\vec{v}_n \cdot \vec{v}_1)}_{0}$$

since orthonormal

$$= c_1 .$$

$$\Rightarrow \boxed{c_i = \vec{v} \cdot \vec{v}_i}$$

- Let $Q = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}$

Then $Q^T \vec{v} = \begin{pmatrix} - & \vec{v}_1 & - \\ & \vdots & \\ - & \vec{v}_n & - \end{pmatrix} = \begin{pmatrix} \vec{v}_1 \cdot \vec{v} \\ \vdots \\ \vec{v}_n \cdot \vec{v} \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$.

$$\vec{v} \longmapsto Q^T \vec{v}.$$

標準座標 $\longleftrightarrow$ 用 $\{\vec{v}_1, \cdots, \vec{v}_n\}$ 觀察下的座標.

Recall: If A is symmetric,
then $AQ = QD \Leftrightarrow A = QDQ^T$

$$v \longmapsto A\vec{v}$$

$$v \xrightarrow[\text{換座標}]{} Q^T\vec{v} \longmapsto DQ^T\vec{v} \xrightarrow[\text{換回去}]{} QDQ^T\vec{v}.$$

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \longmapsto \begin{pmatrix} \lambda_1 c_1 \\ \vdots \\ \lambda_n c_n \end{pmatrix}$$

A 把 $\vec{v}_1$ 方向 放大 λ_1 倍
$\vec{v}_2$ λ_2
$\vdots$
$\vec{v}_n$ λ_n

$\{\vec{v}_1, \ldots, \vec{v}_n\}$ 座標下的世界.

← orthonormal

• 内積 $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \cdot \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = c_1 d_1 + \cdots + c_n d_n$.

$$(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n)(d_1\vec{v}_1 + \cdots + d_n\vec{v}_n)$$

$$= c_1 d_1(\vec{v}_1 \cdot \vec{v}_1) + c_2 d_2(\vec{v}_2 \cdot \vec{v}_2) + \cdots + c_n d_n(\vec{v}_n \cdot \vec{v}_n)$$

$$= c_1 d_1 + \cdots + c_n d_n.$$

• 長度 length of $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \sqrt{c_1^2 + \cdots + c_n^2}$.

$$|c_1\vec{v}_1 + \cdots + c_n\vec{v}_n|$$

$$= \sqrt{(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) \cdot (c_1\vec{v}_1 + \cdots + c_n\vec{v}_n)}$$

$$= \sqrt{c_1^2 + \cdots + c_n^2}.$$

$$(x_1 \cdots x_n)\begin{pmatrix} A \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

~~Ray~~ Rayleigh quotient

A: sym mtx

The Rayleigh quotient is of the form $\dfrac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}}$

↑ $\vec{x}$ 長度^2.

Thm. A: sym mtx; $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ eigenvalues.

Then $\lambda_1 = \min\limits_{\vec{x} \neq \vec{0}} \dfrac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}}$ and $\lambda_n = \max\limits_{\vec{x} \neq \vec{0}} \dfrac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}}$

Reason for the thm.

① $$\frac{(k\vec{x})^T A (k\vec{x})}{(k\vec{x})^T(k\vec{x})} = \frac{k^2 \vec{x}^T A \vec{x}}{k^2 \vec{x}^T \vec{x}} = \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}} \text{ for any } k \neq 0.$$

$$\Rightarrow \min_{\vec{x}\neq 0} \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}} = \min_{\substack{\vec{x} \\ |\vec{x}|=1}} \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}} = \min_{\substack{\vec{x} \\ |\vec{x}|=1}} \vec{x}^T A \vec{x}.$$

② Imagine the $\overbrace{\text{eigenvectors}}^{}$ $\{\vec{v}_1, \ldots, \vec{v}_n\}$.
orthonormal.

Let $\vec{x} = c_1 \vec{v}_1 + \ldots + c_n \vec{v}_n$
with $|\vec{x}|^2 = c_1^2 + \ldots + c_n^2 = 1$.

③ Compute $\vec{x}^T A \vec{x}$

$$= (c_1\vec{v}_1 + \ldots + c_n\vec{v}_n)^T A (c_1\vec{v}_1 + \ldots + c_n\vec{v}_n)$$

$$= (c_1\vec{v}_1 + \ldots + c_n\vec{v}_n)^T (c_1\lambda_1\vec{v}_1 + \ldots + c_n\lambda_n\vec{v}_n)$$

$$= c_1^2\lambda_1 + \ldots + c_n^2\lambda_n$$

Recall $c_1^2 + \ldots + c_n^2 = 1$

min $= \lambda_1$, max $= \lambda_n$

相當於 $\lambda_1, \ldots, \lambda_n$ 出現的機率.

Thm (Courant-Fischer thm; simplified)

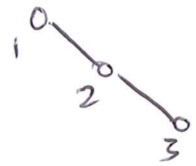
A: sym mtx; $\lambda_1 \leq \cdots \leq \lambda_n$ eigenvalues.

Suppose $\vec{v}_1$ is known. $A\vec{v}_1 = \lambda\vec{v}_1$

Then

$$\lambda_2 = \min_{\substack{\vec{x} \\ |\vec{x}|=1 \\ \vec{x}\cdot\vec{v}_1 = 0}} \vec{x}^T A \vec{x}$$

Laplacian matrix. L



$$\longrightarrow \begin{pmatrix} 1 & -1 & \\ -1 & 2 & -1 \\ & -1 & 1 \end{pmatrix}.$$

$$\cancel{\vec{x}^T A \vec{x}} = (x_1\ x_2\ x_3) \begin{pmatrix} 1 & -1 & \\ -1 & 2 & -1 \\ & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\vec{x}^T L \vec{x}$

$$= x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 - x_2x_3$$

$$= (x_1^2 - 2x_1x_2 + x_2^2) + (x_2^2 - x_2x_3 + x_3^2)$$

$$= (x_1 - x_2)^2 + (x_2 - x_3)^2$$

$\rightarrow$ ~~minimized~~ min $= 0$

when $x_1 = x_2 = x_3$.

$$\Rightarrow \lambda_1 = 0, \quad \vec{v}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Check: $\begin{pmatrix} 1 & -1 & \\ -1 & 2 & -1 \\ & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$ ✓

$A \quad \vec{v}_1 \quad = \quad 0\ \vec{v}_1$

Thm.

G = graph, $L(G) = L$: Laplacian.

Then $\vec{x} L \vec{x} = \sum_{ij \in E(G)} (x_i - x_j)^2 \geq 0$.

$\Rightarrow$ ~~Ao~~ $\begin{cases} \lambda_1 = 0 & \text{for any } L(G). \\ \vec{v}_1 = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \end{cases}$

Thm.

G: graph ; $L = L(G)$: Laplacian.

Let $\vec{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$.

Then $\lambda_2 = \min_{\substack{\vec{x} \\ |\vec{x}|=1 \\ \vec{x}\cdot\vec{1}=\vec{0}}} \vec{x} L \vec{x} = \sum_{i \sim j} (x_i - x_j)^2$.

Therefore, if $\vec{v}_2 = (x_1, \ldots, x_n)^T$

then $\sum_{i \sim j} (x_i - x_j)^2$ is minimized

subject to $x_1 + x_2 + \ldots + x_n = 0$.

一定有正有負.

Thm (Fiedler's partition thm).

G: graph ; $L = L(G)$: Laplacian

$\vec{v}_2 = (x_1, \ldots, x_n)^T$ the eigenvector with respect to λ_2.

spectral embedding 的第一個座標

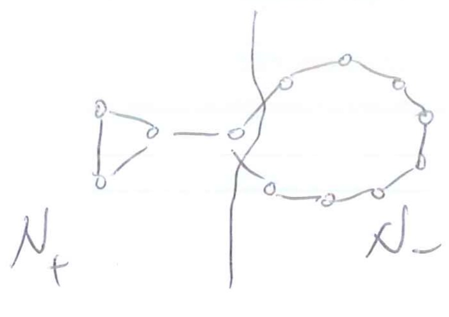
第二小的 eigenvalue

Let $N_+ = \{ i \in V(G) : x_i > 0 \}$.

$N_0 = \{ i \in V(G) : x_i = 0 \}$

$N_- = \{ i \in V(G) : x_i < 0 \}$.

Then both $G[N_+]$, $G[N_+ \cup N_0]$, $G[N_-]$ are connected.



Spectral embedding revisit.

G: graph ; $\underset{\overset{\|}{L}}{L(G)}$: Laplacian

$v_1, \ldots, v_n$ orthonormal eigen~~values~~ vectors

$\lambda_1 \leq \cdots \leq \lambda_n$ eigenvalues.

$$Y = \begin{pmatrix} | & & | \\ \vec{v}_2 & \cdots & \vec{v}_{d+1} \\ | & & | \end{pmatrix} = \begin{pmatrix} - \vec{y}_1 - \\ \vdots \\ - \vec{y}_n - \end{pmatrix}$$

$\vec{v}_1$ 永遠是 $\vec{1}$
所以不~~放~~放.

position of vertex i = $\vec{y}_i \in \mathbb{R}^d$.

Properties

- $\vec{1}^T Y = 0$
- $Y^T Y = I_d$.
- $\mathrm{tr}(Y^T L Y) = \sum_{i \sim j} |\vec{y}_i - \vec{y}_j|^2$

- spectral ~~ebe~~ embedding 的選擇

  符合 $\vec{1}^T Y = 0$, $Y^T Y = I_d$,

  並讓 ~~$\mathrm{tr}(Y^T L Y)$~~

  $\sum_{i \sim j} |\vec{y}_i - \vec{y}_j|$ 最小化.

~~Outer~~ Inner product.

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (1+2+3) = 6$$

Outer product

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}.$$

Fact: $\begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}\begin{pmatrix} — \vec{u}_1^T — \\ \vdots \\ — \vec{u}_n^T — \end{pmatrix}$

$= \vec{v}_1 \vec{u}_1^T + \cdots + \vec{v}_n \vec{u}_n^T.$

Fact: $\begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}\begin{pmatrix} — \vec{u}_1^T — \\ \vdots \\ — \vec{u}_n^T — \end{pmatrix}$

$$= \sum_{i=1}^{n} \underbrace{\lambda_i}_{係數} \underbrace{\vec{v}_i \vec{u}_i^T}_{矩陣}$$

<u>Spectral decomposition</u>

A: sym mtx ; $Q = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}$ ← orthonormal eigenvectors.

$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$

$\Rightarrow A = QDQ^T$

$= \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}\begin{pmatrix} — \vec{v}_1^T — \\ \vdots \\ — \vec{v}_n^T — \end{pmatrix}.$

$$= \sum_{i=1}^{n} \lambda_i \vec{v}_i \vec{v}_i^T$$

好幾層矩陣加在一起

Singular value decomposition

A : $m \times n$ matrix.

columns are orthonormal

Then there are orthogonal matrices U, V
$m \times m$ $n \times n$

and $\Sigma = \left(\begin{array}{ccc|c} \sigma_1 & & 0 & 0 \\ & \ddots & & \\ 0 & & \sigma_r & \\ \hline 0 & & & 0 \end{array}\right)$ with $\sigma_i > 0$.

Σ $m \times n$.

such that $A = U \Sigma V^T$.

Thus, we may write

$$A = \begin{pmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{pmatrix} \begin{pmatrix} \sigma_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_r & \\ 0 & & & 0 \end{pmatrix} \begin{pmatrix} — & \vec{v}_1^T & — \\ & \vdots & \\ — & \vec{v}_n^T & — \end{pmatrix}$$

$$= \sum \sigma_i \vec{u}_i \vec{v}_i^T .$$

若忽略最小的幾個 σ_i

可用來壓縮影像