Chromatic numbers of distance graphs with distance sets missing multiples

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Abstract

Given positive integers \( m, k \) and \( s \) with \( m > ks \), let \( D_{m,k,s} \) represents the set \( \{1, 2, \ldots, m\} - \{k, 2k, \ldots, sk\} \). The distance graph \( G(Z, D_{m,k,s}) \) has as vertex set all integers \( Z \) and edges connecting \( i \) and \( j \) whenever \( |i - j| \in D_{m,k,s} \). This paper investigates the chromatic numbers of the distance graphs \( G(Z, D_{m,k,s}) \).

It is proved that if \( m \geq (s+1)k \), then \( \chi(G(Z, D_{m,k,s}) \leq [(m+sk+1)/(s+1)]+1 \), while the lower bound \( \chi(G(Z, D_{m,k,s}) \geq [(m + sk + 1)/(s + 1)] \) was known. This upper bound improves previous known upper bounds.

1 Introduction

Given a set \( D \) of positive integers, the distance graph \( G(Z, D) \) has all integers as vertices; and two vertices are adjacent if their difference falls within \( D \), that is, the vertex set is \( Z \) and the edge set is \( \{uv : |u - v| \in D\} \). We call \( D \) the distance set. The chromatic number of \( G(Z, D) \) is denoted by \( \chi(Z, D) \).

For different types of distance sets \( D \), the problem of determining \( \chi(Z, D) \) has been studied extensively (see [2, 3, 4, 5, 6, 7, 8, 11, 16, 14, 17].) For instance, suppose \( D \) is a subset of prime numbers and \( \{2, 3\} \in D \), Eggleton, Erdős and Skilton [8] proved \( \chi(Z, D) \) is either 3 or 4. The problem of classifying \( G(Z, D) \) with distance sets \( D \) of primes into chromatic number 3 or 4 was studied by Eggleton, Erdős and Skilton [8],
by Voigt [15], and by Voigt and Walther [16]. However, a complete classification is not obtained yet.

The case that \( D \) contains at most three integers were studied by Eggleton, Erdős and Skilton [5], Chen, Chang, and Huang [3], Voigt [14], and Zhu [17]. The chromatic number of such distance graphs has now been completely determined [17].

Given integers \( m, k \) and \( s \) with \( m > ks \), let \( D_{m,k,s} \) denote the distance set \( D_{m,k,s} = \{1, 2, 3, \ldots, m\} - \{k, 2k, 3k, \ldots, sk\} \). This article studies the chromatic number \( \chi(Z, D_{m,k,s}) \) of \( G(Z, D_{m,k,s}) \).

For \( s = 1 \), the chromatic number of \( G(Z, D_{m,k,1}) \) was first studied by Eggleton, Erdős and Skilton [5], in which \( \chi(Z, D_{m,k,1}) \) was solved as \( k = 1 \) and partially solved as \( k = 2 \). The same results for the case \( k = 1 \) were also obtained in [11] by a different approach. If \( k \) is an odd number, or \( k = 2 \), or \( k = 4 \), then \( \chi(Z, D_{m,k,1}) \) were determined in [12]. Finally, the exact values of \( \chi(Z, D_{m,k,1}) \) for all \( m \) and \( k \) were determined in [2]. For \( s = 2 \), the chromatic number of \( G(Z, D_{m,k,1}) \) was recently determined in [13]. Some results concerning the problem for general \( s \) were also obtained in [13]. In this paper, we extend the study of \( \chi(Z, D_{m,k,s}) \) for general values of \( s \).

Note that the chromatic number is easy to determine if \( m < (s + 1)k \): Define a coloring \( f \) of \( G(Z, D_{m,k,s}) \) by: for any \( x \in Z \), \( f(x) = y \pmod k \), \( 1 \leq y \leq k \). Since \( D_{m,k,s} \) contains no multiples of \( k \), it can be easily verified that \( f \) is a proper coloring. Thus, \( \chi(Z, D_{m,k,s}) \leq k \). As any consecutive \( k \) vertices in \( G(Z, D_{m,k,s}) \) form a complete graph, \( \chi(Z, D_{m,k,s}) \geq k \). This implies \( \chi(Z, D_{m,k,s}) = k \), if \( m < (s + 1)k \). Therefore, throughout the article, we shall assume \( m \geq (s + 1)k \).

We prove in this paper that for any integers \( m, k, s \), \( \chi(Z, D_{m,k,s}) \leq \lceil (m + sk + 1)/(s + 1) \rceil + 1 \) hold for arbitrary \( s \). Combined with the lower bound \( \chi(Z, D_{m,k,s}) \geq \lfloor (m + sk + 1)/(s + 1) \rfloor \) obtained in [12], we conclude that for any values of \( m, k, s \), either \( \chi(Z, D_{m,k,s}) = \lceil (m + sk + 1)/(s + 1) \rceil + 1 \) or \( \chi(Z, D_{m,k,s}) = \lfloor (m + sk + 1)/(s + 1) \rfloor \).
2 An upper bound

We shall use extensively the pre-coloring method introduced in [13] (a simpler version of this coloring method was used in [2]).

It is known and easy to verify, for any distance set \( D \), \( \chi(Z, D) = \chi(Z^+ \cup \{0\}, D) \), where \( G(Z^+ \cup \{0\}, D) \) is the subgraph of \( G(Z, D) \) induced by the set of non-negative integers \( Z^+ \cup \{0\} \). Therefore, to color the graph \( G(Z, D_{m,k,s}) \), it suffices to color the subgraph of \( G(Z, D_{m,k,s}) \) induced by \( Z^+ \cup \{0\} \).

There are two steps in the pre-coloring method. First, we partition the set of non-negative integers \( Z^+ \cup \{0\} \) into \( s + 1 \) parts by a mapping \( c : Z^+ \cup \{0\} \rightarrow \{0, 1, 2, \cdots, s\} \). Second, for each non-negative integer \( x \), according to the value of \( c(x) \), we assign a color to \( x \) by the rule defined as follows.

**Definition 1** Suppose \( m, k, s \) are positive integers. For a given mapping \( c : Z^+ \cup \{0\} \rightarrow \{0, 1, 2, \cdots, s\} \), define a coloring \( c' \) of \( Z^+ \cup \{0\} \) recursively by:

\[
c'(j) = \begin{cases} 
  j, & \text{if } j < k; \\
  c(j - k), & \text{if } j \geq k \text{ and } c(j) \neq 0; \\
  n, & \text{if } j \geq k \text{ and } c(j) = 0,
\end{cases}
\]

where \( n \) is the smallest non-negative integer (color) not been used in the \( m \) vertices preceding \( j \), that is, \( n = \min \{ t \in Z^+ \cup \{0\} : c'(j - i) \neq t \text{ for } i = 1, 2, \cdots, m \} \).

Note that \( c' \) defined above is uniquely determined by \( c \). We call \( c \) the pre-coloring, and \( c' \) the coloring induced by \( c \). For any \( x \in Z^+ \cup \{0\} \), \( c(x) \) and \( c'(x) \) are called the pre-color and the color of \( x \), respectively.

The following Lemmas are proved in [13]:

**Lemma 2** Suppose \( c \) is a pre-coloring of \( Z^+ \cup \{0\} \). If for any integer \( j \geq sk \), \( c(j), c(j - k), c(j - 2k), \cdots, c(j - sk) \) are all distinct, then the induced coloring \( c' \) is a proper coloring for \( G(Z, D_{m,k,s}) \).
Lemma 3 Suppose $c$ is a pre-coloring and $c'$ is the induced coloring. Then the number of colors used by $c'$ is $k + \ell$, where $\ell$ is the maximum number of vertices with pre-color 0 among any $m - k + 1$ consecutive integers greater than $k$.

Lemma 4 Given integers $m, k$ and $s$, $\chi(Z, D_{m,k,s}) \leq n$ if there exists a pre-coloring $c$ such that the following two conditions are satisfied:

1. for any integer $j \geq sk$, $c(j), c(j + k), c(j + 2k), \ldots, c(j + sk)$ are all distinct, and

2. among any consecutive $m - k + 1$ integers, there are at most $n - k$ vertices with pre-color 0.

Theorem 5 For any integers $m, k, s$ with $m \geq (s + 1)k$, we have $\chi(Z, D_{m,k,s}) \leq \lfloor (m+sk+1)/(s+1) \rfloor + 1$. Moreover, if $m-k+1 \equiv 1 \pmod{s+1}$ then $\chi(Z, D_{m,k,s}) = \lfloor (m+sk+1)/(s+1) \rfloor$.

Proof. It was proved in [13] that $\chi(Z, D_{m,k,s}) \geq \lfloor (m+sk+1)/(s+1) \rfloor$. Therefore by Lemma 4, it suffices to find a pre-coloring $c$ of $Z^+ \cup \{0\}$ such that for any integer $j \geq sk$, $c(j), c(j - k), c(j - 2k), \ldots, c(j - sk)$ are all distinct, and that among any consecutive $m - k + 1$ integers, there are at most $n - k$ vertices with pre-color 0, where $n = \lfloor (m+sk+1)/(s+1) \rfloor$ if $m-k+1 \equiv 1 \pmod{s+1}$ and $n = \lfloor (m+sk+1)/(s+1) \rfloor + 1$ otherwise.

Let $d = (s + 1, k)$, and let $q = (s + 1)k/d$. For any integer $i$, we write $i$ in the form $i = u(s + 1)k + qx + (s + 1)y + z$, where $u, x, y, z$ are non-negative integers such that $x < d, y < k/d$ and $z < s + 1$. Obviously $u, x, y, z$ are uniquely determined by $i$. Let $c(i) = x + z \pmod{s + 1}$. We shall prove that $c$ satisfies the conditions above.

First for any $j$, we prove that $j, j + k, \ldots, j + sk$ have different pre-colors.

Assume to the contrary that $c(j + ak) = c(j + bk)$ for some $0 \leq a < b \leq s$. Suppose $j + ak = u(s + 1)k + qx + (s + 1)y + z$ and that $j + bk = u'(s + 1)k + qx' +$
\[(s + 1)y' + z'.\] Then
\[(b - a)k = (u' - u)(s + 1)k + (x' - x)q + (y' - y)(s + 1) + (z' - z).\]

Since \(d\) divides each of the terms \((b - a)k, (u' - u)(s + 1)k, (x' - x)q, (y' - y)(s + 1)\), it follows that \(d\) divides \(z' - z\). Because \(c(j + ak) = x + z \pmod{s + 1} = c(j + bk) = x' + z' \pmod{s + 1}\), it follows that \(z' - z = x - x' \pmod{s + 1}\). As \(d\) divides both \(z' - z\) and \(s + 1\), we conclude that \(d\) divides \(x - x'\). As \(|x - x'| < d\), it follows that \(x - x' = 0\). Therefore \(z' - z = 0 \pmod{s + 1}\), which implies that \(z = z'\). Thus \((b - a)k - (u' - u)(s + 1)k = (s + 1)(y' - y)\), which implies that \(k\) divides \((s + 1)(y' - y)\).

Since \(\left\lfloor (s - 1)/d, k \right\rfloor = 1\), we conclude that \(k\) divides \(d(y' - y)\). However \(|y' - y| < k/d\), i.e., \(|k(y' - y)| < k\), therefore \(y' - y = 0\). Hence \((b - a)k = (u' - u)(s + 1)k\). This implies that \(u' - u = 0 = b - a\) (as \(0 \leq (b - a) \leq s\)), contrary to the assumption that \(b > a\).

Next we prove that among any consecutive \(m - k + 1\) integers, there are at most \(n - k\) vertices with pre-color 0, where \(n = \left\lfloor (m + sk + 1)/(s + 1) \right\rfloor\) if \(m - k + 1 = 1 \pmod{s + 1}\), and \(n = \left\lceil (m + sk + 1)/(s + 1) \right\rceil + 1\) otherwise.

Divide the set \(Z^+ \cup \{0\}\) into segments \(I_0, I_1, \ldots\), such that \(I_j = \{j(s + 1), j(s + 1) + 1, \ldots, (j + 1)(s + 1) - 1\}\). It follows from the definition of the pre-coloring \(c\) that each segment \(I_j\) contains exactly one element of pre-color 0. Let \(X\) be a set of \(m - k + 1\) consecutive integers. If \(m - k + 1 = 1 \pmod{s + 1}\), then \(X\) intersect at most \(\left\lfloor (m + sk + 1)/(s + 1) \right\rfloor\) of the segments \(I_j\), hence it contains at most \(n = \left\lfloor (m + sk + 1)/(s + 1) \right\rfloor\) vertices of pre-color 0. In general, i.e., if \(m - k + 1 \neq 1 \pmod{s + 1}\), \(X\) intersect at most \(\left\lceil (m + sk + 1)/(s + 1) \right\rceil + 1\) of the segments \(I_j\), hence it contains at most \(\left\lceil (m + sk + 1)/(s + 1) \right\rceil + 1\) vertices of pre-color 0.

It was proved in [13] that for any integers \(m, k, s\) with \(m \geq (s + 1)k\), \(\chi_f(Z, D_{m,k,s}) = (m + sk + 1)/(s + 1)\), which implies that \(\chi(Z, D_{m,k,s}) \geq \left\lfloor (m + sk + 1)/(s + 1) \right\rfloor\). Therefore when \(m \geq (s + 1)k\), \(\chi(Z, D_{m,k,s})\) is equal to either \(\left\lfloor (m + sk + 1)/(s + 1) \right\rfloor\) or \(\left\lceil (m + sk + 1)/(s + 1) \right\rceil + 1\). The results in [13], as well as the results in Section 3 of
this paper, show that both the upper bound and lower bound are sharp. It remains an open problem to determine for which $D_{m,k,s}$ the lower bound is attained, and for which $D_{m,k,s}$ the upper bound is attained. For $s = 1$ and $s = 2$, the problem is completely solved in [2] and [13] respectively. The results in [13] shows that the problem is more difficult when $s + 1$ is not a prime. In the next section, we consider the case $s = 3$, and present some partial solutions.

3 $s=3$

In this section, we consider the case that $s = 3$. We shall divide the discussion into a few cases, according to value of $k \pmod{4}$.

If $k$ is odd, then it follows from a result (Theorem 13) in [13] that $\chi(Z, D_{m,k,3}) = [(m + sk + 1)/(s + 1)]$. In the following, we assume that $k$ is even.

Our next two theorems give the answer for the case $m - k + 1 = 2 \pmod{4}$.

**Theorem 6** If $k = 4t + 2$ for some integer $t$ and $m - k + 1 \neq 0 \pmod{4}$, then $\chi(Z, D_{m,k,3}) = [(m + 3k + 1)/4]$.

**Proof.** Define a pre-coloring $c$ as follows: for any integer $i$, write $i$ in the form $i = 4ku + 2kx + 4y + z$, where $u, x, y, z$ are non-negative integers such that $x \leq 1$, $y \leq 2t$ and $z \leq 3$. If $x = 0$, then let $c(i) = z \pmod{4}$; if $x = 1$, then let $c(x) = 3 + z \pmod{4}$.

In the following we show that for any integer $j \geq 3k$, $c(j), c(j + k), c(j + 2k), c(j + 3k)$ are all distinct, and that among any consecutive $m - k + 1$ integers, there are at most $n - k$ vertices with pre-color 0, where $n = [(m + sk + 1)/(s + 1)]$.

Assume to the contrary that there exist $j \geq 0$ and $0 \leq a < b \leq 3$ such that $c(j + ak) = c(j + bk)$. Suppose $j + ak = 4ku + 2kx + 4y + z$, $j + bk = 4ku' + 2kx' + 4y' + z'$. Then $j + bk - (j + ak) = (b - a)k = 4k(u' - u) + 2k(x' - x) + 4(y' - y) + (z' - z)$. As each of the terms $(b - a)k, 4k(u' - u), 2k(x' - x), 4(y' - y)$ is even, we conclude that
\[ z' - z \] is even, i.e., \( z, z' \) have the same parity. Because \( c(j + ak) = c(j + bk) \), it follows from the definition of \( c \) that \( x = x' \), and hence \( z = z' \). This implies that \( k = 4t + 2 \) divides \( 2(y' - y) \), which implies that \( y' - y = 0 \) (because \( |y' - y| < k/2 \)). Therefore \( b - a = 4(u' - u) \). This implies that \( u' - u = 0 \) (because \( 0 \leq b - a \leq 3 \)), and hence \( b = a \), contrary to our assumption.

Next we show that among any consecutive \( m - k + 1 \) integers, there are at most \( n - k \) vertices with pre-color 0, where \( n = \lceil (m + sk + 1)/(s + 1) \rceil \). Let \( X \) be a set of \( m - k + 1 \) consecutive integers. We divide the set \( Z^+ \cup \{0\} \) into segments \( I_j \), where \( I_j = \{4j, 4j + 1, 4j + 2, 4j + 3\} \). Then each \( I_j \) has exactly one vertex of pre-color 0. Indeed, the pre-colors of the elements of \( I_j \) are either 0123 or 3012 (in that order). The set \( X \) intersects either with \( n - k \) of the segments \( I_j \), or with \( n - k + 1 \) of the segments \( I_j \). In the former case, of course \( X \) contains at most \( n - k \) elements of pre-color 0. In the latter case, assume the \( n - k + 1 \) segments \( I_j \) that intersects \( X \) are \( I_{q}, I_{q+1}, \ldots, I_{q+n-k} \). Since \( |X| = m - k + 1 \not\equiv 0 \pmod{4} \), we conclude that \( |X \cap I_q| \leq 2 \). Note that the pre-colors of the elements of segment \( I_q \) are either 0123 or 3012, hence none of the last two elements of \( I_q \) are of pre-color 0. Therefore \( X \cap I_q \) contains no vertex of pre-color 0. So \( X \) contains at most \( n - k \) vertices of pre-color 0.

**Theorem 7** Suppose \( k = 4t + 2 \) for some integer \( t \), and that \( m - k + 1 = 4p \) for some integer \( p \). Then \( \chi(Z, D_{m,k,3}) = \lceil (m + 3k + 1)/4 \rceil \) if \( p \) is even, and \( \chi(Z, D_{m,k,3}) = \lceil (m + 3k + 1)/4 \rceil + 1 \) if \( p \) is odd.

**Proof.** First consider the case that \( p \) is odd. Assume to the contrary that \( \chi(Z, D_{m,k,3}) = (m + 3k + 1)/4 \). For any two integers \( i \) and \( j \), let \( G[i, j] \) be the subgraph of \( G(Z, D_{m,k,3}) \) induced by vertices \( \{i + 1, i + 2, \ldots, j\} \). Then for any integer \( i \), the graph \( G[i, i + m + 3k + 1] \) has \( m + 3k + 1 \) vertices and maximum independent set of size 4. Since \( f \) is an \( (m + 3k + 1)/4 \)-coloring, exactly 4 vertices of \( G[i, i + m + 3k + 1] \)
are colored by the same color. It follows that \( f(i) = f(i + m + 3k + 1) \) for any integer \( i \).

Define a circulant graph \( G \) on the set \( \{0, 1, \cdots, m + 3k\} \) with generating set \( D_{m,k,3} \), that is, \( ij \) is an edge of \( G \) if and only if \( j - i \pmod{m + 3k + 1} \in D_{m,k,3} \) or \( i - j \pmod{m + 3k + 1} \in D_{m,k,3} \). The argument in the previous paragraph shows that \( f \) induces a proper \( n \)-coloring of \( G \). Moreover, each color class consists of 4 vertices in \( G \). It is not difficult to verify that all 4-independent sets of \( G \) are of the form \( \{i, i + k, \cdots, i + 3k\} \) (here each number is calculated by modulo \( m + 3k + 1 \)).

Let \( d = (k, m + 3k + 1) \) and \( u = (m + 3k + 1)/d \). Divide the vertex set of \( G \) into \( d \) subsets of the form \( \{i, i + k, i + 2k, \cdots, i + (u - 1)k\} \pmod{4} \), each of size \( u \). Then each of these \( d \) subsets is the union of some color classes of size 4, so 4 divides \( u \). However, this would implies that \( m - k + 1 \) is a multiple of 8, because \( d \) is certainly even.

Suppose \( p \) is even. Let \( d, u \) be as defined in the previous section. Then 4 divides \( u \). Indeed, as \( k = 4t + 2 \) and \( m + 3k + 1 = m - k + 1 + 4k = 4(p + k) \), we know that \( d = 2a \) for some odd integer \( a \). Hence \( u = (m + 3k + 1)/d = 4(p + k)/d \) is a multiple of 4. One can easily define a proper \( u \)-coloring \( f \) on \( G \) by using \( u/4 \) colors to each of the subsets \( \{i, i + k, i + 2k, \cdots, i + (u - 1)k\} \pmod{4} \): the first 4 vertices in a subset use one color and the next 4 vertices use the next, and continue the process until all vertices are colored. It is easy to check that \( f \) is a proper coloring of \( G \). Furthermore, \( f \) can be extended to a proper coloring of \( G(Z, D_{m,k,3}) \) by letting \( f'(y) = f(x), x = y \pmod{m + 3k + 1} \). Therefore, \( G(Z, D_{m,k,3}) \) is \( u \)-colorable, where \( u = (m + 3k + 1)/4 \). This completes the proof of Theorem 7.

In the following we consider the case that \( k \equiv 0 \pmod{4} \). Suppose \( k = 4^a k' \), where \( a \geq 1 \) and \( k' \neq 0 \pmod{4} \). Suppose \( m + 3k + 1 = 4^b q \), where \( b \geq 0 \) and \( q \neq 0 \pmod{4} \). The following theorem is a special case of Theorem 3 of [13]:

**Theorem 8** If \( a < b \) and \( k' \) is odd, then \( \chi(Z, D_{m,k,3}) = (m + 3k + 1)/4 \). If \( 0 < b \leq a \),
then \( \chi(Z, D_{m,k,3}) = (m + 3k + 1)/4 + 1 \).

By this theorem, it remains to consider the following two cases:

1. \( a < b \) and \( k' \) is even;

2. \( b = 0 \).

**Theorem 9** Suppose \( a < b \) and \( k' \) is even. If \( b \geq a + 2 \) or \( q \) is odd, then \( \chi(Z, D_{m,k,3}) = (m + 3k + 1)/4 + 1 \). Otherwise \( \chi(Z, D_{m,k,3}) = (m + 3k + 1)/4 + 1 \).

**Proof.** The proof of this result is parallel to the proof of Theorem 7. First we consider the case that \( b = a + 1 \) and \( q \) is even. Let \( n = (m + 3k + 1)/4 \). We shall prove that \( G(Z, D_{m,k,3}) \) is not \( n \)-colorable. Assume to the contrary, there exists an \( n \)-coloring \( f \) of \( G(Z, D_{m,k,3}) \).

For any two integers \( i \) and \( j \), let \( G[i, j] \) be the subgraph of \( G(Z, D_{m,k,3}) \) induced by vertices \( \{i+1, i+2, \cdots, j\} \). Then for any integer \( i \), the graph \( G[i, i + m + 3k + 1] \) has \( m + 3k + 1 \) vertices and maximum independent set of size 4. Since \( f \) is an \((m + 3k + 1)/4\)-coloring, exactly 4 vertices of \( G[i, i + m + 3k + 1] \) are colored by the same color. It follows that \( f(i) = f(i + m + 3k + 1) \) for any integer \( i \).

Define a circulant graph \( G \) on the set \( \{0, 1, \cdots, m + 3k\} \) with generating set \( D_{m,k,3} \), that is, \( ij \) is an edge of \( G \) if and only if \( j - i \pmod{m + 3k + 1} \in D_{m,k,3} \) or \( i - j \pmod{m + 3k + 1} \in D_{m,k,3} \). The argument in the previous paragraph shows that \( f \) induces a proper \( n \)-coloring of \( G \). Moreover, each color class consists of 4 vertices in \( G \). It is not difficult to verify that all 4-independent sets of \( G \) are of the form \( \{i, i + k, \cdots, i + 3k\} \) (here each number is calculated by modulo \( m + 3k + 1 \)).

Let \( d = (k, m + 3k + 1) \) and \( u = (m + 3k + 1)/d \). Since \( k', q \) are both even, it follows that \( d = 2 \cdot 4^r d'' \) for some odd integer \( d'' \). As \( b = a + 1 \), this implies that \( u \) is not a multiple of 4.

Divide the vertex set of \( G \) into \( d \) subsets of the form \( \{i, i + k, i + 2k, \cdots, i + (u - 1)k\} \pmod{4} \), each of size \( u \). Then each of these \( d \) subsets is the union of some
color classes of size 4. However this is impossible because \( u \) is not a multiple of 4.

On the other hand, if \( b \geq a + 2 \) or \( q \) is odd, then the integer \( u \) defined as above is a multiple of 4. In this case, one can easily partition each of the \( d \) sets 
\[ \{ i, i + k, i + 2k, \ldots, i + (u - 1)k \} \pmod{4} \]
into independent sets of size 4. This implies that the circulant graph \( G \) defined as above is indeed \( n \)-colorable, and hence \( G(Z, D_{m,k,3}) \) is \( n \)-colorable.

For the remaining part of the paper, we assume that \( b = 0 \). If \( m + 3k + 1 \equiv 1 \pmod{4} \), then it follows from Theorem 5 that \( \chi(Z, D_{m,k,3}) = (m + 3k + 1)/4 \).

**Theorem 10** If \( m + 3k + 1 \equiv 2 \pmod{4} \), then \( \chi(Z, D_{m,k,3}) = \lceil (m + 3k + 1)/4 \rceil \).

**Proof.** Let \( n = \lceil (m + 3k + 1)/4 \rceil \). Suppose \( m + 3k + 1 = 4ck + d \), where \( c, d \) are integers such that \( 0 < d < 4k \) (since \( b = 0 \), we know that \( d \neq 0 \)). If \( d \leq 2k \), then define a pre-coloring \( c \) of \( Z^+ \cup \{0\} \) as follows:

For any non-negative integer \( i \), write \( i \) in the form \( i = kx + y \), where \( x, y \) are non-negative integers such that \( y \leq k - 1 \). Let \( c(i) = x + y \pmod{4} \). We show that for any integer \( j \), the vertices \( j, j + k, j + 2k, j + 3k \) have distinct pre-colors, and that any \( m - k + 1 \) consecutive integers contains at most \( n - k \) integers of pre-color 0.

Assume to the contrary that \( c(j + uk) = c(j + vk) \) for some integers \( j, u, v \) such that \( 0 \leq u < v \leq 3 \). Assume that \( j + uk = kx + y \) and \( j + vk = kx' + y' \). Then 
\[
(v - u)k = (x' - x)k + (y' - y).
\]
It follows that \( k \) divides \( y' - y \), which implies that \( y' - y = 0 \) (because \( |y' - y| \leq k - 1 \)). As \( c(j + uk) = c(j + vk) \), it follows that \( x' \equiv x \pmod{4} \). This implies that \( v - u \equiv 0 \pmod{4} \), contrary to the assumption that \( 0 \leq u < v \leq 3 \).

Next, let \( X \) be a set of \( m - k + 1 \) consecutive integers. We divide the set \( Z^+ \cup \{0\} \) into segments \( I_j \), where \( I_j = \{4j, 4j + 1, 4j + 2, 4j + 3\} \). Each of the segments \( I_j \) contains exactly one integer of pre-color 0. Since \( \lceil (m - k + 1)/4 \rceil = n \), we know that \( X \) either intersects with \( n - k \) of the segments \( I_j \), or \( n - k + 1 \) of the segments \( I_j \). If \( X \) intersect
with \( n - k \) of the segments \( I_j \), then of course \( X \) contains at most \( n - k \) integers of pre-color 0. Assume that \( X \) intersect with \( n - k + 1 \) segments \( I_j \). Since \( m + 3k + 1 \equiv 2 \pmod{4} \), it follows that \( m - k + 1 \equiv 2 \pmod{4} \). Thus \( m - k + 1 = 4(n - k) - 2 \). Let \( I_i, I_{i+1}, \ldots, I_{i+n-k} \) be the segments that intersect \( X \). Then \( |X \cap I_i| = |X \cap I_{i+n-k}| = 1 \). If the last element of \( I_i \) does not have pre-color 0, or the first element of \( I_{i+n-k} \) does not have pre-color 0, then \( X \) contains at most \( n - k \) elements of pre-color 0. Assume that the last element of \( I_i \) has pre-color 0, and the first element of \( I_{i+n-k} \) also have pre-color 0. Then by the definition of \( c \), \( 4i = kx + y \) for some \( x \equiv 1 \pmod{4} \) and \( y \leq k - 1 \), and that \( 4(i + (n - k)) = kx' + y' \) for some \( x' \equiv 0 \pmod{4} \) and \( y' \leq k - 1 \). Now \( m - k + 1 = 4(i + (n - k)) - 4i + (y' - y) - 2 = k(x' - x) + (y' - y) - 2 \), where \( x' - x \equiv 3 \pmod{4} \). Hence if \( m - k + 1 = 4(c - 1)k + d \) for some integers \( c, d \) such that \( d \leq 4k \), then \( d > 2k \) (because \( |y' - y| \leq k - 2 \), and that \( m - k + 1 \) is not a multiple of \( k \)), contrary to the assumption that \( d \leq 2k \).

If \( d > 2k \), then define a pre-coloring \( c \) of \( Z^+ \cup \{0\} \) as follows:

For any non-negative integer \( i \), write \( i \) in the form \( i = kx + y \), where \( x, y \) are non-negative integers such that \( y \leq k - 1 \).

1. If \( x \equiv 0 \pmod{4} \), then let \( c(i) = y \pmod{4} \).

2. If \( x \equiv 1 \pmod{4} \), then let \( c(i) = 3 + y \pmod{4} \).

3. If \( x \equiv 2 \pmod{4} \), then let \( c(i) = 2 + y \pmod{4} \).

4. If \( x \equiv 3 \pmod{4} \), then let \( c(i) = 1 + y \pmod{4} \).

We show that for any integer \( j \), the vertices \( j, j + k, j + 2k, j + 3k \) have distinct pre-colors, and that a set of any \( m - k + 1 \) consecutive integers contains at most \( n - k \)
integers of pre-color 0. The proof is similar to the proof for the case that $d \leq 2k$, and we omit the details.

**Theorem 11** Suppose $m + 3k + 1 \equiv 3 \pmod{4}$ and that $m + 3k + 1 = 4k + d$, where $d \leq 4k$. If $d \leq k$ or $d \geq 3k$, then $\chi(Z, D_{m,k,3}) = [\frac{m + 3k + 1}{4}]$.

The proof of Theorem 11 is parallel to the proof of Theorem 10. We omit the details.

The case that $s = 3$, $m + 3k + 1 \equiv 3 \pmod{4}$ and that $m + 3k + 1 = 4k + d$ for some $k < d < 3k$ remains unsolved.

**Remark** Since the circulation of this manuscript, the problem of determining the chromatic number and circular chromatic number of the distance graphs $G(Z, D_{m,k,3})$ has been completely solved in [10] and [19].

**References**


Omitted details for the proof of Theorem 10:

Assume to the contrary that \( c(j + uk) = c(j + vk) \) for some integers \( j, u, v \) such that \( 0 \leq u < v \leq 3 \). Assume that \( j + uk = kx + y \) and \( j + vk = kx' + y' \). Then \((v-u)k = (x'-x)k + (y'-y)\). It follows that \( k \) divides \( y' - y \), which implies that \( y' - y = 0 \) (because \(|y' - y| \leq k - 1\)). As \( c(j + uk) = c(j + vk) \), it follows that \( x' \equiv x \) (mod 4). This implies that \( v - u \equiv 0 \) (mod 4), contrary to the assumption that \( 0 \leq u < v \leq 3 \).

Next, let \( X \) be a set of \( m - k + 1 \) consecutive integers. We divide the set \( \mathbb{Z}^+ \cup \{0\} \) into segments \( I_j \), where \( I_j = \{4j, 4j+1, 4j+2, 4j+3\} \). Each of the segments \( I_j \) contains exactly one integer of per-color 0. Since \( \lceil (m-k+1)/4 \rceil = n \), we know that \( X \) either intersects with \( n-k \) of the segments \( I_j \), or \( n-k+1 \) of the segments \( I_j \). If \( X \) intersect with \( n-k \) of the segments \( I_j \), then of course \( X \) contains at most \( n-k \) integers of pre-color 0. Assume that \( X \) intersect with \( n-k+1 \) segments \( I_j \). Since \( m + 3k + 1 \equiv 2 \) (mod 4), it follows that \( m - k + 1 \equiv 2 \) (mod 4). Thus \( m - k + 1 = 4(n - k) - 2 \). Let \( I_i, I_{i+1}, \ldots, I_{i+n-k} \) be the segments that intersect \( X \). Then \(|X \cap I_i| = |X \cap I_{i+n-k}| = 1\). If the last element of \( I_i \) does not have pre-color 0, or the first element of \( I_{i+n-k} \) does not have pre-color 0, then \( X \) contains at most \( n-k \) elements of pre-color 0. Assume that the last element of \( I_i \) has pre-color 0, and the first element of \( I_{i+n-k} \) also have pre-color 0. Then by the definition of \( c \), \( 4i = kx + y \) for some \( x \equiv 3 \) (mod 4) and \( y \leq k - 1 \), and that \( 4(i + (n-k)) = kx' + y' \) for some \( x' \equiv 0 \) (mod 4) and \( y' \leq k - 1 \). Now \( m - k + 1 = 4(i + (n-k)) - 4i + (y' - y) - 2 = k(x' - x) + (y' - y) - 2 \), where \( x' - x \equiv 1 \) (mod 4). Hence if \( m - k + 1 = 4(c - 1)k + d \) for some integers \( c, d \) such that \( d \leq 4k \), then \( d < 2k \) (because \(|y' - y| \leq k - 2 \), and that \( m - k + 1 \) is not a multiple of \( k \)), contrary to the assumption that \( d > 2k \).

Proof of Theorem 11:

Let \( n = \lceil (m+3k+1)/4 \rceil \). Suppose \( m + 3k + 1 = 4ck + d \), where \( c, d \) are integers such that \( 0 < d < 4k \) (since \( b = 0 \), we know that \( d \neq 0 \)). If \( d \leq k \), then define a
pre-coloring $c$ of $Z^+ \cup \{0\}$ as follows:

For any non-negative integer $i$, write $i$ in the form $i = kx + y$, where $x,y$ are non-negative integers such that $y \leq k - 1$. Let $c(i) = x + y \pmod{4}$. This is the same pre-coloring as defined in the proof of Theorem 10. Hence for any integer $j$, the vertices $j,j + k, j + 2k, j + 3k$ have distinct pre-colors. We need to show that any $m - k + 1$ consecutive integers contains at most $n - k$ integers of pre-color 0.

Let $X$ be a set of $m - k + 1$ consecutive integers. We divide the set $Z^+ \cup \{0\}$ into segments $I_j$, where $I_j = \{4j, 4j + 1, 4j + 2, 4j + 3\}$. Each of the segments $I_j$ contains exactly one integer of pre-color 0. Since $\lceil (m - k + 1)/4 \rceil = n$, we know that $X$ either intersects with $n - k$ of the segments $I_j$, or $n - k + 1$ of the segments $I_j$. If $X$ intersect with $n - k$ of the segments $I_j$, then of course $X$ contains at most $n - k$ integers of pre-color 0. Assume that $X$ intersect with $n - k + 1$ segments $I_j$. Since $m + 3k + 1 \equiv 3 \pmod{4}$, it follows that $m - k + 1 \equiv 3 \pmod{4}$. Thus $m - k + 1 = 4(n - k) - 1$. Let $I_i, I_{i+1}, \ldots, I_{i+n-k}$ be the segments that intersect $X$. Then $|X \cap I_i| + |X \cap I_{i+n-k}| = 3$.

First we consider the case that $|X \cap I_i| = 2$ and $|X \cap I_{i+n-k}| = 1$.

If the last two elements of $I_i$ does not have pre-color 0, or the first element of $I_{i+n-k}$ does not have pre-color 0, then $X$ contains at most $n - k$ elements of pre-color 0. Assume that one of the last two elements of $I_i$ has pre-color 0, and the first element of $I_{i+n-k}$ also have pre-color 0. Then by the definition of $c$, $4i = kx + y$ for some $x \equiv 1 \pmod{4}$ and $y \leq k - 1$, and that $4(i + (n - k)) = kx' + y'$ for some $x' \equiv 0 \pmod{4}$ and $y' \leq k - 1$. Now $m - k + 1 = 4(i + (n - k)) - 4i + (y' - y) - 2 = k(x' - x) + (y' - y) - 2$, where $x' - x \equiv 3$ or 2 (mod 4). Hence if $m - k + 1 = 4(c - 1)k + d$ for some integers $c, d$ such that $d \leq 4k$, then $d > k$ (because $|y' - y| \leq k - 2$, and that $m - k + 1$ is not a multiple of $k$), contrary to the assumption that $d \leq k$.

Next we consider the case that $|X \cap I_i| = 1$ and $|X \cap I_{i+n-k}| = 2$.

If the last element of $I_i$ does not have pre-color 0, or the first two elements of $I_{i+n-k}$ do not have pre-color 0, then $X$ contains at most $n - k$ elements of pre-color 0.
Assume that the last two element of $I_i$ has pre-color $0$, and one of the first two elements of $I_{i+n-k}$ has pre-color $0$. Then by the definition of $c$, $4i = kx + y$ for some $x \equiv 1 \pmod{4}$ and $y \leq k - 1$, and that $4(i + (n - k)) = kx' + y'$ for some $x' \equiv 0$ or $3 \pmod{4}$ and $y' \leq k - 1$. Now $m - k + 1 = 4i + (n - k) - 4i + (y' - y) - 2 = k(x' - x) + (y' - y) - 2$, where $x' - x \equiv 3$ or $2 \pmod{4}$. Hence if $m - k + 1 = 4(c - 1)k + d$ for some integers $c, d$ such that $d \leq 4k$, then $d > k$ (because $|y' - y| \leq k - 2$, and that $m - k + 1$ is not a multiple of $k$), contrary to the assumption that $d \leq k$.

If $d \geq 3k$, then define a pre-coloring $c$ of $\mathbb{Z}^+ \cup \{0\}$ as follows:

For any non-negative integer $i$, write $i$ in the form $i = kx + y$, where $x, y$ are non-negative integers such that $y \leq k - 1$.

1. If $x \equiv 0 \pmod{4}$, then let $c(i) = y \pmod{4}$.

2. If $x \equiv 1 \pmod{4}$, then let $c(i) = 3 + y \pmod{4}$.

3. If $x \equiv 2 \pmod{4}$, then let $c(i) = 2 + y \pmod{4}$.

4. If $x \equiv 3 \pmod{4}$, then let $c(i) = 1 + y \pmod{4}$.

Again this is one of the pre-coloring defined in the proof of Theorem 10, where it is proved that that for any integer $j$, the vertices $j, j + k, j + 2k, j + 3k$ have distinct pre-colors. We need to show that any set of $m - k + 1$ consecutive integers contains at most $n - k$ integers of pre-color $0$.

Let $X$ be a set of $m - k + 1$ consecutive integers. We divide the set $\mathbb{Z}^+ \cup \{0\}$ into segments $I_j$, where $I_j = \{4j, 4j + 1, 4j + 2, 4j + 3\}$. Each of the segments $I_j$ contains exactly one integer of pre-color $0$. Since $\lfloor (m - k + 1)/4 \rfloor = n$, we know that $X$ either intersects with $n - k$ of the segments $I_j$, or $n - k + 1$ of the segments $I_j$. If
$X$ intersect with $n - k$ of the segments $I_j$, then of course $X$ contains at most $n - k$
integers of pre-color 0.

Assume that $X$ intersect with $n - k + 1$ segments $I_j$. Since $m + 3k + 1 \equiv 3$
(mod 4), it follows that $m - k + 1 \equiv 3$ (mod 4). Thus $m - k + 1 = 4(n - k) - 1$. Let
$I_i, I_{i+1}, \ldots, I_{i+n-k}$ be the segments that intersect $X$. Then $|X \cap I_i| + |X \cap I_{i+n-k}| = 3$.

First we consider the case that $|X \cap I_i| = 2$ and $|X \cap I_{i+n-k}| + 1$.

If the last two elements of $I_i$ does not have pre-color 0, or the first element of
$I_{i+n-k}$ does not have pre-color 0, then $X$ contains at most $n - k$ elements of pre-color 0.
Assume that one of the last two elements of $I_i$ has pre-color 0, and the first element of
$I_{i+n-k}$ also have pre-color 0. Then by the definition of $c$, $4i = kx + y$ for some $x \equiv 2$ or 3
(mod 4) and $y \leq k - 1$, and that $4(i + (n - k)) = kx' + y'$ for some $x' \equiv 0$ (mod 4)
and $y' \leq k - 1$. Now $m - k + 1 = 4(i + (n - k)) - 4i + (y' - y) - 2 = k(x' - x) + (y' - y) - 2$,
where $x' - x \equiv 1$ or 2 (mod 4). Hence if $m - k + 1 = 4(c - 1)k + d$ for some integers
c, d such that $d \leq 4k$, then $d < 3k$ (because $|y' - y| \leq k - 2$, and that $m - k + 1$ is
not a multiple of $k$), contrary to the assumption that $d \geq 3k$.

Next we consider the case that $|X \cap I_i| = 1$ and $|X \cap I_{i+n-k}| = 2$.

If the last element of $I_i$ does not have pre-color 0, or the first two elements of
$I_{i+n-k}$ do not have pre-color 0, then $X$ contains at most $n - k$ elements of pre-color 0.
Assume that the last two element of $I_i$ has pre-color 0, and one of the first two elements
of $I_{i+n-k}$ has pre-color 0. Then by the definition of $c$, $4i = kx + y$ for some $x \equiv 3$
(mod 4) and $y \leq k - 1$, and that $4(i + (n - k)) = kx' + y'$ for some $x' \equiv 0$ or 1 (mod 4)
and $y' \leq k - 1$. Now $m - k + 1 = 4(i + (n - k)) - 4i + (y' - y) - 2 = k(x' - x) + (y' - y) - 2$,
where $x' - x \equiv 3$ or 2 (mod 4). Hence if $m - k + 1 = 4(c - 1)k + d$ for some integers
c, d such that $d \leq 4k$, then $d < 3k$ (because $|y' - y| \leq k - 2$, and that $m - k + 1$ is
not a multiple of $k$), contrary to the assumption that $d \geq 3k$. 

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