Triple homomorphisms of C*-algebras

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In memory of our beloved friend, Kosita Beidar.

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Abstract. In this note, we will discuss what kind of operators between C*-algebras preserves Jordan triple products \( \{a, b, c\} = (ab^*c + cb^*a)/2 \). These include especially isometries and disjointness preserving operators.

Keywords: C*-algebras, Jordan triples, isometries, disjointness preserving operators.

1. Introduction

Recall that a Banach algebra \( A \) is an algebra with a norm \( \| \cdot \| \) such that \( \|ab\| \leq \|a\|\|b\| \), and every Cauchy sequence converges. A complex Banach algebra \( A \) is a C*-algebra if there is an involution \( * \) defined on \( A \) such that \( \|a^*a\| = \|a\|^2 \). A special example is \( B(H) \), the algebra of all bounded linear operators on a (complex) Hilbert space \( H \). By the Gelfand-Naimark-Sakai Theorem, C*-algebras are exactly those norm closed *-subalgebras of \( B(H) \). An abelian C*-algebra \( A \) can also be represented as the algebra \( C_0(X) \) of continuous functions on a locally compact Hausdorff space \( X \) vanishing at infinity. \( X \) is compact if and only if \( A \) is unital.

It is well known that the algebraic structure determines the geometric (norm) structure of a C*-algebra \( A \). Indeed, the norm of a self-adjoint element \( a \) of \( A \) coincides with the spectral radius of \( a \), and the latter is a pure algebraic object. In general, the norm of an arbitrary element \( a \) of \( A \) is equal to \( \|a^*a\|^{1/2} \), and \( a^*a \) is self-adjoint. For an abelian C*-algebra \( A = C_0(X) \), we note that the underlying space \( X \) can be considered as the maximal ideal space of \( A \) consisting of complex homomorphisms (= linear and multiplicative functionals) of \( A \). The topology of \( X \) is the hull-kernel topology, and thus be solely determined by the
algebraic structure of $A$.

In this note, we will discuss how much the algebraic structure can be recovered if we know the norm, or other, structure of a $C^*$-algebra. In particular, isometries and disjointness preserving operators of $C^*$-algebras preserve triple products $\{a, b, c\} = (ab^*c + cb^*a)/2$.

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2. The geometric structure determines the algebraic structure

Suppose $T : A \to B$ is an isometric linear embedding between $C^*$-algebras. That is, $\|Tx\| = \|x\|$ for all $x$ in $A$. We are interested in knowing what kind of algebraic structure $T$ inherits from $A$ to its range, which is in general just a Banach subspace of $B$. We begin with two famous results.

Theorem 2.1. (Banach and Stone; see, e.g., [5]) Let $X$ and $Y$ be locally compact Hausdorff spaces. Let $T : C_0(X) \to C_0(Y)$ be a surjective linear isometry. Then $T$ is a weighted composition operator $Tf = h \cdot f \circ \varphi, \quad \forall f \in C_0(X)$, where $h$ is a continuous scalar function on $Y$ with $|h(y)| \equiv 1$, and $\varphi$ is a homeomorphism from $Y$ onto $X$. Consequently, two abelian $C^*$-algebras are isomorphic as Banach spaces if, and only if, they are isomorphic as $^*$-algebras.

Here is a sketch of the proof. Let $T^* : M(Y) \to M(X)$ be the dual map of $T$, which is again a surjective linear isometry from the Banach space $M(Y) = C_0(Y)^*$ of all bounded Radon measures on $Y$ onto that on $X$. Restricting $T^*$ to the dual unit balls, which are weak* compact and convex, we get an affine homeomorphism. Since the extreme points of the dual unit balls are exactly unimodular scalar multiples of point masses together with zero, $T^*$ sends a point mass $\delta_y$ to $\lambda \delta_x$. Here $y \in Y$, $x \in X$ and $|\lambda| = 1$. We write $x = \varphi(y)$ and $\lambda = h(y)$ to indicate that $x$ and $\lambda$ depend on $y$. It follows that $Tf(y) = T^*(\delta_y)(f) = h(y)\delta_{\varphi(y)}(f) = h(y)f(\varphi(y))$. In other words, $Tf = h \cdot f \circ \varphi, \forall f \in C_0(X)$. It is then routine to see that $h$ is unimodular and continuous on $Y$, and that $\varphi$ is a homeomorphism from $Y$ onto $X$.

Theorem 2.2. (Kadison [6]) Let $A$ and $B$ be $C^*$-algebras. Let $T : A \to B$ be a surjective linear isometry. Then there is a unitary element $u$ in $\tilde{B} = B \oplus \mathbb{C}1$, the unitalization of $B$, and a Jordan $^*$-isomorphism $J : A \to B$ such that $Ta = uJ(a), \quad \forall a \in A$. 

Consequently, two C*-algebras are isomorphic as Banach spaces if, and only if, they are isomorphic as Jordan *-algebras.

Recall that a Jordan *-isomorphism $J$ preserves linear sums, involutions and Jordan products: $a \circ b = (a b + b a)/2$. It is easy to see that the abelian case can also be written in this form with $u = h$ and $Jf = f \circ \varphi$. In general, the product of a pair of elements in $A$ can be decomposed into two parts $a \circ b = a \circ b + [a, b]$, the sum of the Jordan product and the Lie product $[a, b] = (ab - ba)/2$. It is plain that $a \circ b = b \circ a$ is commutative and $[a, b] = -[b, a]$ is anti-commutative. However they are not associative. The Kadison theorem states that the norm structure of a C*-algebra determines completely its Jordan structure.

It is interesting to note that Jordan products are determined by squares: $a \circ b = (a + b)^2 - a^2 - b^2$, $\forall a, b \in A$.

A similar algebraic structure exists in C*-algebras, namely, the Jordan triple products: $\{a, b, c\} = ab^* c + cb^* a$.

There is also a polar identity for triples: $\{a, b, c\} = \frac{1}{8} \sum_{\alpha^2 = 1} \sum_{\beta^4 = 1} \alpha \beta \{a + \alpha b + \beta c\}^{(3)}$.

Hence, a linear map $T$ between C*-algebras preserves triple products if and only if it preserves cubes $a^{(3)} = \{a, a, a\} = aa^* a$.

Kaup [7] rephrased Kadison theorem: a linear surjection between C*-algebras $T : A \rightarrow B$ is an isometry if and only if it preserves triple products. A geometric proof of the Kadison Theorem is given by Dang, Friedman and Russo [2]. It goes first to note that a norm exposed face of the dual unit ball $U_{B^*}$ is of the form $F_u = \{ \varphi \in B^* : \|\varphi\| = \varphi(u) \leq 1 \}$ for a unique partial isometry $u \in B^{**}$. For two $\varphi, \psi$ in $B^*$, they are said to be orthogonal to each other if they have polar decompositions $\varphi = u |\varphi|, \psi = |\psi| v$ such that $u \perp v$, i.e., $u^* v = v^* u = 0$. This amounts to say that $\|\varphi \pm \psi\| = \|\varphi\| + \|\psi\|$. Two faces $F_u, F_v$ are orthogonal if and only if $u \perp v$. Then they verify that the adjoint $T^*$ of the surjective linear isometry $T$ maps faces to faces and preserves orthogonality. Consequently, $T$ sends orthogonal partial isometries to orthogonal partial isometries. By the spectral theory, every element $a$ in $A$ can be approximated in norm by a finite linear sum of orthogonal partial isometries $\sum_j \lambda_j u_j$. Then its cube $a^{(3)}$ can also be approximated by $\sum_j \lambda_j^{(3)} u_j$. It follows that $T(a^{(3)})$ and $(T a)^{(3)}$ can both be approximated by $\sum_j \lambda_j^{(3)} T u_j$. Hence $T(a^{(3)}) = (T a)^{(3)}$, and thus $T$ preserves triple products by the polar identity.

We note that the above (geometric) proof of the Kadison theorem quite depends on the fact the range of the isometry is again a C*-algebra. Extending
the Holsztynski theorem [3, 5], Chu and Wong [1] studied non-surjective linear isometries between C*-algebras.

**Theorem 2.3.** (Chu and Wong [1]) Let $A$ and $B$ be C*-algebras and let $T$ be a linear isometry from $A$ into $B$. There is a largest closed projection $p$ in $B^{**}$ such that $T(\cdot)p : A \rightarrow B^{**}$ is a Jordan triple homomorphism and

$$T(ab^*c + cb^*a)p = T(a)T(b)^*T(c)p + T(c)T(b)^*T(a)p, \quad \forall a, b, c \in A.$$ 

When $A$ is abelian, we have $\|T(a)p\| = \|a\|$ for all $a$ in $A$. In particular, $T$ reduces locally to a Jordan triple isomorphism on the JB*-triple generated by any $a$ in $A$, by a closed projection $p_a$.

Beside the triple technique, the proof of above theorem also makes use of the concept of representing elements in a C*-algebra as special sections of a continuous field of Hilbert spaces developed in [8]. It is still geometric.

### 3. Disjointness preserving operators are triple homomorphisms

In this section, we do not assume the operator $T$ is isometric. Although the following statement might have been known to experts, we provide a new and short proof here as we do not find any in the literature. For simplicity of notations, we also write $T$ for its bidual map $T^{**} : A^{**} \rightarrow B^{**}$.

**Theorem 3.1.** Let $T : A \rightarrow B$ be a bounded linear map between C*-algebras. Then $T$ is a triple homomorphism if and only if $T$ sends partial isometries to partial isometries.

**Proof.** One direction is trivial. Suppose $T$ sends partial isometries to partial isometries. Let $u, v$ be two partial isometries in $A$. Observe that they are orthogonal to each other, namely, $u^*v = uv^* = 0$, if and only if they have orthogonal initial spaces and orthogonal range spaces. This amounts to say that $u + \lambda v$ is a partial isometry for all scalar $\lambda$ with $|\lambda| = 1$. Consequently, $T$ sends orthogonal partial isometries to orthogonal partial isometries. For every $a$ in $A$, approximate $a$ in norm by a finite linear sum $\sum_n \lambda_n u_n$ of orthogonal partial isometries. Then its cube $a^{(3)} = aa^*a$ can also be approximated in norm by $\sum_n \lambda_n^{(3)} u_n$. It follows that $Ta$ and $T(a^{(3)})$ can be approximated in norm by $\sum_n \lambda_n Tu_n$ and $\sum_n \lambda_n^{(3)} Tu_n$, respectively. This gives $T(a^{(3)}) = (Ta)^{(3)}$, $\forall a \in A$. By the polar identity, we see that $T$ is a triple homomorphism.

We say that a linear map $T : A \rightarrow B$ between C*-algebras is **disjointness preserving** if

$$a^*b = ab^* = 0 \implies (Ta)^*(Tb) = (Ta)(Tb)^* = 0, \quad \forall a, b \in A.$$
Clearly, $T$ is disjointness preserving if and only if it preserves disjointness of partial isometries. It is clear that every triple homomorphism preserves disjointness. Looking at the well-known abelian case, that is, the Jarosz theorem [4, 5], we see that not every disjointness preserving map is a triple homomorphism. Indeed, let $T : C_0(X) \to C_0(Y)$ be a bounded disjointness preserving linear map between abelian C*-algebras. Then there is a closed subset $Y_0$ of $Y$ on which every $Tf$ vanishes. On $Y_1 = Y \setminus Y_0$ there is a bounded continuous function $h$ and a continuous map $\phi$ from $Y_1$ into $X$ such that $Tf|_{Y_1} = h \cdot f \circ \phi$ for all $f$ in $C_0(X)$. Hence, $T$ is a triple homomorphism if and only if $T_1$ is a partial isometry in $C_0(Y)^{**}$. We end this note with a proof of this fact for the non-abelian case.

**Theorem 3.2.** Let $T : A \to B$ be a bounded linear map between C*-algebras. Then $T$ is a triple homomorphism if and only if $T$ is disjointness preserving and $T_1$ is a partial isometry.

**Proof.** We verify the sufficiency only. By the polar identity it suffices to check that $T$ sends the cube $a^{(3)}$ to the cube $(Ta)^{(3)}$ for every element $a$ of $A$. Identify the JB*-triple of $A$ generated by 1 and $a$ with $C(X)$ (see [7, Corollary 1.15]), where $X$ is some compact set of complex numbers. Denote again by $T$ the bidual map of $T$ from $C(X)^{**}$ into $B^{**}$.

Let $X = \bigcup_n X_n$ be any finite Borel partition of $X$ and pick an arbitrary point $x_n$ from $X_n$. In particular,

$$1 = \sum_n 1_{X_n},$$

where $1_{X_n}$ is the characteristic function of the Borel set $X_n$. For $j \neq k$, we can find two sequences $\{f_m\}_m$ and $\{g_m\}_m$ in $C(X)$ such that $f_{m+p}g_m = 0$ for $m,p = 0,1,\ldots$, $f_m \to 1_{X_j}$ and $g_m \to 1_{X_k}$ pointwisely on $X$. By the weak* continuity of $T$, we see that

$$T(1_{X_j})T(g_m)^* = \lim_{p \to \infty} T(f_{m+p})T(g_m)^* = 0 \quad \text{for all } m = 1,2,\ldots.$$ 

Thus

$$T(1_{X_j})T(1_{X_k})^* = \lim_{m \to \infty} T(1_{X_j})T(g_m)^* = 0.$$

Similarly, we have

$$T(1_{X_j})^*T(1_{X_k}) = 0.$$

Consequently, for each $j$ we have

$$T(1)T(1_{X_j})^*T(1) = \sum_{m,n} T(1_{X_n})T(1_{X_j})^*T(1_{X_m}) = (T(1_{X_j}))^{(3)}.$$

This gives

$$\sum_n T(1_{X_n}) = T1 = (T1)^{(3)} = \sum_n (T(1_{X_n}))^{(3)}.$$
Multiplying the above identity on the left by $T(1_{X_n})^*$ and $(T(1_{X_n}))^{(3)}^*$ respectively, we see that

$$(T(1_{X_n}) - (T(1_{X_n}))^{(3)})^* (T(1_{X_n}) - (T(1_{X_n}))^{(3)}) = 0.$$ 

Hence $T(1_{X_n})$ is a partial isometry for each $n$ and orthogonal to the others. It follows that

$$(T(f))^{(3)} = \lim_{n} \left( \sum_n f(x_n)T(1_{X_n}) \right)^{(3)} = \lim_{n} \sum f(x_n)^{(3)}(T(1_{X_n}))^{(3)}$$

$$= \lim_{n} \sum f(x_n)^{(3)}T(1_{X_n}) = T(f^{(3)}),$$

for all $f$ in $C(X)$. This completes the proof. 

References