Oscillation for Nonlinear Neutral Difference Equations*

Guang Zhang†‡

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Abstract

In this note, some oscillatory results for neutral difference equations are obtained.

Recently, Tang and Liu in [1] considered delay difference equations of the form
\[ \Delta x_n + q_n x_n^{\alpha_{n-k}} = 0, \quad n = 0, 1, 2, \ldots, \quad (1) \]
where \( \{q_n\} \) is a sequence of nonnegative numbers, \( k \) is a positive integer, and \( \alpha \in (0, \infty) \) is a quotient of odd positive integers. They obtained the following theorems.

THEOREM 1. Assume that \( 0 < \alpha < 1 \). Then every solution of (1) oscillates if, and only if,
\[ \sum_{n=0}^{\infty} q_n = \infty. \quad (2) \]

THEOREM 2. Assume that \( \alpha > 1 \). Suppose further that there exists a \( \lambda > k^{-1} \ln \alpha \) such that
\[ \liminf_{n \to \infty} q_n \exp (-e^{\lambda n}) > 0, \]
then every solution of (1) oscillates.

In this note, we will consider the neutral difference equations
\[ \Delta (x_n - px_{n-l}) + q_n x_n^{\alpha_{n-k}} = 0, \quad n = 0, 1, 2, \ldots, \quad (3) \]
and
\[ \Delta^2 (x_n + px_{n-1}) + q_n x_n^{\alpha_{n-k}} = 0, \quad n = 0, 1, 2, \ldots, \quad (4) \]
where \( \alpha, k \) and \( \{q_n\} \) are defined as before, \( l \) is a positive integer, and \( 0 \leq p < 1 \). It is clear that equation (1) is a particular case of (3). We will discuss the oscillation of (3) and (4) in two cases where \( \alpha < 1 \) and \( \alpha > 1 \). In the sequel, for convenience, when we write a functional inequality without specifying its domain of validity, we assume that it holds for all sufficiently large \( n \).

THEOREM 3. Assume that \( 0 \leq p < 1 \) and \( 0 < \alpha < 1 \). Then every solution of (3) oscillates if, and only if, (2) holds.

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†Department of Mathematics, Yanbei Normal College, Datong, Shanxi 037000, P. R. China
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PROOF. The fact that oscillation of (3) implies (2) can be found in [2]. Next, assume that \( \{x_n\} \) is an eventually positive solution of (3). In view of Lemma 1 in [3], we get that \( z_n = x_n - px_{n-l} \) is eventually positive. Thus,

\[
x_{n-k} = z_{n-k} + px_{n-l-k} \geq z_{n-k} + pz_{n-l-k} \geq (1 + p)z_n.
\]

Substituting it into (3), we have

\[
\Delta z_n + q_n (1 + p)^\alpha z_n^\alpha \leq 0.
\]

Thus

\[
z_n^{-\alpha} \Delta z_n + q_n (1 + p)^\alpha \leq 0. \quad (5)
\]

We define \( r(t) = z_m + (t - m)\Delta z_m, m \leq t \leq m + 1 \). Since \( \Delta z_m \leq 0 \), then \( z_{m+1} \leq r(t) \leq z_m \) and

\[
\frac{r'(t)}{r^\alpha(t)} \leq \frac{\Delta z_m}{z_m^\alpha}. \quad (6)
\]

In view of (5), (6) and (2), we obtain

\[
\int_{r(N)}^{r(\infty)} \frac{dr}{r^\alpha} = -\infty,
\]

which contradicts the fact \( \alpha \in (0, 1) \). The proof is complete.

If \( \alpha > 1 \), \( 0 \leq p < 1 \) and \( \{x_n\} \) is an eventually positive solution of (3), then we have

\[
x_{n-k} = z_{n-k} + px_{n-l-k} = z_{n-k} + pz_{n-l-k} + p^2 x_{n-l-2k} \\
= z_{n-k} + pz_{n-l-k} + \ldots + p^m z_{n-l-mk} + p^{m+1} x_{n-l-(m+1)k} \\
\leq p^m z_{n-l-mk}.
\]

By (3), we have

\[
\Delta z_n + q_n p^m z_{n-l-mk} \leq 0
\]

for any \( m \geq 0 \). Note that when \( p > 0 \),

\[
\liminf_{n \to \infty} q_n \exp(-e^{\lambda n}) > 0 \iff \liminf_{n \to \infty} p^m q_n \exp(-e^{\lambda n}) > 0.
\]

In view of Theorem 2, we have the following result.

THEOREM 4. Assume that \( \alpha > 1 \) and \( 0 \leq p < 1 \). Suppose further that for some nonnegative integer \( m \) there exists a \( \lambda > (l + mk)^{-1} \ln \alpha \) (where \( m = 0 \) if \( p = 0 \)) such that

\[
\liminf_{n \to \infty} q_n \exp(-e^{\lambda n}) > 0,
\]

then every solution of (3) oscillates.

To obtain oscillatory results of equation (4), we need the following lemma which can be found in [4].

LEMMA 1. An eventually concave sequence \( \{x_n\} \) (i.e. \( \Delta^2 x_n \leq 0 \) for all large \( n \)) is of constant sign eventually. If \( x_n > 0 \) and \( \Delta^2 x_n \leq 0 \) eventually and \( \{\Delta^2 x_n\} \) has
a negative subsequence, then \( \{\Delta x_n\} \) is eventually positive. Furthermore, there is a number \( \theta \in (0, 1) \) such that \( x_n \geq \theta n \Delta x_n \) for all large \( n \).

Assume that \( \{x_n\} \) is an eventually positive solution of equation (4). Clearly, we have \( y_n = x_n + px_{n-l} > \theta n \Delta y_n > 0 \) and \( \Delta^2 y_n \leq 0 \). Thus,

\[
x_{n-k} = y_{n-k} - p y_{n-k-l} + p^2 x_{n-l-2k} \geq (1 - p) y_{n-k-l} \\
\geq (1 - p) \theta (n - k - l) \Delta y_{n-k-l}.
\]

Substituting it into (4), we have

\[
\Delta^2 y_n + q_n (1 - p) \theta^{\alpha} (n - k - l)^\alpha (\Delta y_{n-k-l})^\alpha \leq 0.
\]

In view of Theorem 1, we have the following theorem.

**THEOREM 5.** Assume that \( \alpha \in (0, 1) \) and \( p \in [0, 1) \). Suppose further that

\[
\sum_{n=0}^{\infty} q_n (n - k - l)^\alpha = \infty,
\]

then every solution of (4) oscillates. While if \( \alpha > 1, p \in [0, 1) \), and there exists \( \lambda > (k + l)^{-1} \ln \alpha \) such that

\[
\liminf_{n \to \infty} q_n (n - k - l)^\alpha \exp (-e^{\lambda n}) > 0,
\]

then all solutions of (4) oscillate.

**References**


