1. Let $A = (a_{ij})_{i,j=1}^n$ denote a $n \times n$ positive definite matrix.

(a) Write the definition of the positive definite matrix. (5%)

(b) Find a simple method different from (a) to determine if $A$ is a positive definite matrix. (10%)

(c) What can you say about the eigenvalues and eigenvectors of $A$, $\det A$ and $A^{-1}$? (10%)

(d) How to verify that the point $(2, 1)$ is a local minimum of $f(x, y) = 2x^3 - 24xy + 16y^3$? (5%)

解答：

(a) $x^T Ax > 0$ for all $x \neq 0 \in \mathbb{R}^n$.

(b) $A$ is a symmetry matrix and $\det A_k > 0$ for all $k = 1, \ldots, n$ where $A_k = (a_{ij})_{i,j=1}^k$.

(c) 
- Eigenvalues are all positive.
- There are $n$ real eigenvectors and orthogonal to each other.
- $\det A > 0$
- $A^{-1}$ is also a positive matrix.

(d) 
- $f_x = 6(x^2 - 4y)$ and $f_y = -24(x - 2y^2)$
- The point $(2, 1)$ is a critical point since $f_x(2, 1) = f_y(2, 1) = 0$
- Hessian matrix $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 12x & -24 \\ -24 & 96y \end{pmatrix}$
- $H(2, 1) = \begin{pmatrix} 24 & -24 \\ -24 & 96 \end{pmatrix}$ is a positive matrix since $24$ and $24 \times 94 - 24 \times 24 > 0$.
- Therefore the point $(2, 1)$ is a local minimum.

2. Let $X$ be a $N(\mu, \sigma^2)$ random variable. Find the probability density function of $X^2$. (10%)

解答：

- $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}, x \geq 0$

3. Find the moment generating function of a Poisson$(\lambda)$ random variable. (10%)

解答：

- $M(t) = \sum_{n=0}^{\infty} e^{tn} e^{-\lambda} \lambda^n / n! = e^{\lambda(e^t-1)}$

4. Assume $X_1, X_2, \ldots, X_n$ are i.i.d. samples from exponential distribution with the density function $f(x) = (1/\lambda)e^{-x/\lambda}, x > 0, \lambda > 0$. Find the maximum likelihood estimator (mle) of $\lambda$. Does the mle of $\lambda$ attain the Cramer-Rao lower bound? (15%)

解答：

- $\hat{\lambda} = \bar{X}$.
Since $\text{Var}(\overline{X}) = \frac{\lambda^2}{n}$ and $-nE(\frac{\partial^2}{\partial \lambda^2} \ln f(x)) = \frac{n}{\lambda^2}$, $\hat{\lambda}$ attains the Cramer-Rao lower bound $\text{Var}(\overline{X}) = \frac{1}{-nE(\frac{\partial^2}{\partial \lambda^2} \ln f(x))}.$

5. Let

$$f(x, y) = \begin{cases} \frac{1+xy}{4} & \text{if } |x| \leq 1, |y| \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal density functions of $X$ and $Y$. (7%)

(b) Find $E[X|Y = y]$ and $E[Y|X = x]$. (7%)

(c) Find $\text{Cov}(X, Y)$. (7%)

(d) Show that $X^2$ and $Y^2$ are independent. (7%)

(e) Find $P\{X > Y^2\}$. (7%)

解答：

(a) $f_X(x) = \int_{-1}^{1} f(x, y)dy = \int_{-1}^{1} \frac{1+xy}{4} dy = \frac{y}{4} + \frac{x}{8}y^2|_{y=-1}^{y=1} = \frac{1}{2}, x \in [-1, 1]$

$f_Y(y) = \frac{1}{2}, y \in [-1, 1]$

(b) $E[X|Y = y] = \int_{-1}^{1} x f_X(x|y)dx = \int_{-1}^{1} x \frac{1+xy}{8} dx = \frac{x^2}{16} + \frac{x^3y}{24}|_{x=-1}^{x=1} = \frac{y}{12}, y \in [-1, 1]$

$E[Y|X = x] = \frac{x}{12}, x \in [-1, 1]$

(c) $\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$

$E[X] = E[Y] = 0$

$E[XY] = \int_{-1}^{1} \int_{-1}^{1} xy \frac{1+xy}{4} dxdy = \frac{1}{9}$

(d) Let $U = X^2$ and $V = Y^2$. To show that

$$P\{U \leq u^2, V \leq v^2\} = P\{U \leq u^2\}P\{V \leq v^2\}$$

$$- P\{U \leq u^2, V \leq v^2\} = P\{-u \leq X \leq u\}P\{-v \leq Y \leq v\} = \int_{-v}^{v} \int_{-u}^{u} \frac{1+xy}{4} dxdy = \frac{uv}{4}$$

$$- P\{U \leq u^2\} = P\{-u \leq X \leq u\} = u$$

$$- P\{V \leq v^2\} = P\{-v \leq Y \leq v\} = v$$

$$- P\{U \leq u^2, V \leq v^2\} = P\{U \leq u^2\}P\{V \leq v^2\}$$

(e) $P\{X > Y^2\} = \int_{-1}^{1} \int_{-1}^{1} \frac{1+xy}{4} dxdy = \frac{1}{3}$

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