In the following, let $I = \{ x \in \mathbb{R} : 0 \leq x \leq 1 \}$.

1. Let $f$ be the real valued function defined on the interval $I$ by
   $$f(x) = \begin{cases} \frac{1}{n} & \text{whenever } n > 0 \text{ is an integer and } 2^{-n} < x \leq 2^{-n+1}, \\ 0 & \text{whenever } x = 0. \end{cases}$$
   (i) Prove that $f$ is integrable on $I$.
   (ii) Evaluate $\int_0^1 f(x) \, dx$.

2. For every integer $n > 0$, let $f_n(x) = \frac{x}{1 + n^4x^2}$ for $x \in I$.
   (i) Prove that for every $x \in I$ the series $\sum_{n=1}^{\infty} f_n(x)$ converges.
   (ii) Let $f(x) = \sum_{n=1}^{\infty} f_n(x)$ for $x \in I$. Prove that $f$ continuous on $I$.

3. Let $K$ be a nonempty compact subset of the Euclidean plane $\mathbb{R}^2$, and let $f : K \to \mathbb{R}$ be a function. Assume that for every $r \in \mathbb{R}$ there is an open subset $U(r)$ of $\mathbb{R}^2$ such that $\{(x, y) \in K : f(x, y) < r\} = K \cap U(r)$.
   (i) Prove that there is a $p \in \mathbb{R}$ such that $f(x, y) \leq p$ for all $(x, y) \in K$.
   (ii) Prove that there is a point $(a, b) \in K$ such that $f(x, y) \leq f(a, b)$ for all $(x, y) \in K$.

4. Let $V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : 2x_1 - 3x_2 + 4x_3 - 5x_4 = 0\}$.
   (i) Prove that $V$ is a vector subspace of $\mathbb{R}^4$.
   (ii) Find the dimension of $V$, and prove your answer.

5. Prove that the matrix $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$ is diagonalizable, and find an invertible $3 \times 3$ matrix $B$ such that $B^{-1}AB$ is diagonal.