1. Rachel and Robert run on a circular track. Rachel runs counterclockwise and completes a lap every 90 seconds, and Robert runs clockwise and completes a lap every 80 seconds. Both start from the start line at the same time. At some random time between 10 minutes and 11 minutes after they begin to run, a photographer standing inside the track takes a picture that shows one-fourth of the track, centered on the starting line. What is the probability that both Rachel and Robert are in the picture?

解答：After 10 minutes (600 seconds), Rachel will have completed 6 laps and be 30 seconds from completing her seventh lap. Because Rachel runs one-fourth of a lap in 22.5 seconds, she will be in the picture between 18.75 seconds and 41.25 seconds of the tenth minute. After 10 minutes Robert will have completed 7 laps and will be 40 seconds past the starting line. Because Robert runs one-fourth of a lap in 20 seconds, he will be in the picture between 30 and 50 seconds of the tenth minute. Hence both Rachel and Robert will be in the picture if it is taken between 30 and 41.25 seconds of the tenth minute. So the probability that both runners are in the picture is \( \frac{41.25 - 30}{60} = \frac{3}{16} \). □

2. If \( X \) is an exponential random variable with parameter \( \lambda = 1 \), compute the probability density function of the random variable \( Y \) defined by \( Y = \log X \). 

\[
e^{-y} - e^{-y}, \quad y \in \mathbb{R}
\]
解答:

\[
F_Y(y) = P(\log X \leq y) = P(X \leq e^y) = F_X(e^y)
\]

\[
f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(e^y) \frac{dx}{dy} = f_X(e^y)e^y = e^{y-x}, \quad y \in \mathbb{R}
\]

3. Let \(X\) be a discrete random variable taking values in positive integers. Assume that \(P(X = k) \geq P(X = k + 1)\) for all positive integer \(k\). Show that for any positive integer \(k\),

\[
P(X = k) \leq \frac{2E[X]}{k^2}.
\]

解答: Since \(P(X = i) \geq 0\), it follows that for any \(k \in \mathbb{N}\),

\[
E[X] = \sum_{i=1}^{\infty} i P(X = i) \geq \sum_{i=1}^{k} i P(X = i).
\]

Moreover, since \(P(X = k) \geq P(X = k + 1)\) for all \(k \in \mathbb{N}\), we have \(P(X = k) \geq P(X = i)\) for all positive integer \(i \leq k\) and so for any \(k \in \mathbb{N}\),

\[
\sum_{i=1}^{k} i P(X = i) \geq \sum_{i=1}^{k} i P(X = k) = \frac{k(k + 1)}{2} P(X = k) \geq \frac{k^2}{2} P(X = k).
\]

Hence \(E[X] \geq k^2 P(X = k)/2\) for any positive integer \(k\) and the proof is complete. \(\square\)

4. Let \(X\) and \(Y\) be independent standard normal random variables. Show that \(X^2/(X^2 + Y^2)\) and \(X^2 + Y^2\) are independent.

解答: Note that \(X^2\) and \(Y^2\) are independent and have the chi-square distribution \(\chi^2_1\). Hence their joint probability density function is

\[
f_{X^2, Y^2}(x, y) = \frac{1}{2\pi} (xy)^{-1/2} e^{-(x+y)/2}, \quad x, y > 0.
\]

Let \(U = X^2 + Y^2\) and \(V = X^2/U\). Then \(X^2 = UV\) and \(Y^2 = U - UV\). The joint probability density function for \((U, V)\) is

\[
\frac{1}{2\pi \sqrt{v(1-v)}} \frac{e^{-u/2}}{u} = \frac{1}{2\pi} e^{-u/2} \frac{1}{\sqrt{v(1-v)}}, \quad u > 0, \ 0 < v < 1.
\]

Hence \(U\) and \(V\) are independent. \(\square\)
5. Let $X_1, X_2, \ldots, X_n$ be iid $N(\theta, 1)$ random variables. Assume the parameter $\theta \in [0, \infty)$. Find the maximum likelihood estimator of $\theta$. 

$$\hat{\theta} = \max \{ \bar{X}_n, 0 \}$$

**Answer:** Since the likelihood

$$L(\theta|x) = (2\pi)^{-\frac{n}{2}} \exp \left\{ -\frac{n}{2} (\theta - \bar{x}_n)^2 - \frac{1}{2} \sum_{i=1}^{n} (x_i - \bar{x}_n)^2 \right\}.$$ 

If $\bar{x}_n < 0$, then $L(\theta|x)$ is a decreasing function of $\theta$ for $\theta \geq 0$. Thus the maximum of $L(\theta|x)$ occurs at $\theta = 0$. If $\bar{x}_n > 0$, then the maximum of $L(\theta|x)$ occurs at $\bar{x}_n$. Consequently, the mle of $\theta$ is $\hat{\theta} = \max \{ \bar{X}_n, 0 \}$. □

6. Let $X_1, X_2, \ldots, X_n$ be iid Bernoulli($\theta^2$), $0 < \theta < 1$. Find the CR lower bound of the variance of unbiased estimator of $\theta^2$.

**Answer:** Note that the pdf of $X_1$ is

$$f(x|\theta) = (\theta^2)^x (1 - \theta^2)^{1-x}, \quad x = 0, 1.$$ 

Thus

$$\left( \frac{\partial}{\partial \theta} \log f(x|\theta) \right)^2 = \left( \frac{2x}{\theta} - \frac{2\theta(1-x)}{1-\theta^2} \right)^2,$$

and the Fisher information

$$I(\theta) = E \left[ \frac{\partial}{\partial \theta} \log f(x|\theta) \right]^2 = \frac{4}{1-\theta^2}.$$ 

The CRLB is

$$\frac{\theta^2(1-\theta^2)}{4n/(1-\theta^2)} = \frac{\theta^2(1-\theta^2)}{n}. \quad \Box$$