

Total weight choosability of graphs

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Abstract

A graph $G = (V, E)$ is called (k, k') -total weight choosable if the following holds: For any total list assignment L which assigns to each vertex x a set $L(x)$ of k real numbers, and assigns to each edge e a set $L(e)$ of k' real numbers, there is a mapping $f : V \cup E \rightarrow \mathbb{R}$ such that $f(y) \in L(y)$ for any $y \in V \cup E$ and for any two adjacent vertices x, x' , $\sum_{e \in E(x)} f(e) + f(x) \neq \sum_{e \in E(x')} f(e) + f(x')$. We conjecture that every graph is $(2, 2)$ -total weight choosable and every graph without isolated edges is $(1, 3)$ -total weight choosable. It follows from results in [7] that complete graphs, complete bipartite graphs, trees other than K_2 are $(1, 3)$ -total weight choosable. Also a graph G obtained from an arbitrary graph H by subdividing each edge with at least three vertices is $(1, 3)$ -total weight choosable. This paper proves that complete graphs, trees, generalized theta graphs are $(2, 2)$ -total weight choosable. We also prove that for any graph H , a graph G obtained from H by subdividing each edge with at least two vertices is $(2, 2)$ -total weight choosable as well as $(1, 3)$ -total weight choosable.

1 Introduction

Suppose $G = (V, E)$ is a graph. For a vertex x of G , $E(x)$ denotes the set of edges of G incident to x and $E^*(x) = E(x) \cup \{x\}$. An *edge weighting* of G is a mapping that assigns to each edge e of G a real number $f(e)$. An edge-weighting f of G induces a vertex colouring ϕ_f of G defined as $\phi_f(x) = \sum_{e \in E(x)} f(e)$ for every vertex x . An edge weighting f is *proper* if the

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induced colouring ϕ_f is proper, i.e., for any edge xx' of G , $\phi_f(x) \neq \phi_f(x')$. The real number assigned to an edge is called the *weight* of the edge. Instead of real numbers, one can use elements of some other field as weights. For simplicity, the weights are restricted to real numbers in this paper. The study of edge weighting is initiated by Karoński, Łuczak and Thomason [8]. They proposed the following conjecture in [8]:

Conjecture 1.1. *Every connected graphs $G \neq K_2$ has a proper edge weighting f such that $f(e) \in \{1, 2, 3\}$ for every edge e .*

This conjecture is referred as the 1, 2, 3-conjecture and has received some attention. It is shown in [8] if S is a subset of \mathbb{R} of size at least 183 and is independent over the field of rational numbers, then any connected graph $G \neq K_2$ has a proper edge weighting f with $f(e) \in S$ for every edge e . Under the same restriction that S being independent over rational numbers, the bound 183 is reduced to 4 in [1]. In [3], it is shown that any connected graph $G \neq K_2$ has a proper edge weighting f with $f(e) \in \{1, 2, \dots, 30\}$ for every edge e . The result was improved in [2], where it is shown that f can be chosen so that $f(e) \in \{1, 2, \dots, 16\}$ for every edge e .

In [7], Bartnicki, Grytczuk and Niwczykthe considered the choosability version of 1, 2, 3-conjecture. A graph is said to be *k-edge-weight-choosable* if the following is true: For any list assignment L which assigns to each edge e a set $L(e)$ of k real numbers, G has a proper edge weighting f such that $f(e) \in L(e)$ for each edge e .

Conjecture 1.2. *Every graph without isolated edges is 3-edge-weight-choosable.*

Conjecture 1.2 is stronger than Conjecture 1.1. Bartnicki, Grytczuk and Niwczyk [7] verified this conjecture for complete graphs, complete bipartite graphs and some other graphs. However, it is unknown if there is a constant k such that every graph without isolated edges is *k-edge-weight-choosable*.

Przybyło and Woźniak [9, 10] studied weighting that involves both the edges and the vertices of G . Suppose $G = (V, E)$ is a graph. A mapping $f : V \cup E \rightarrow \mathbb{R}$ is called a *total weighting* of G . The *vertex-colouring* ϕ_f of G induced by f is defined as $\phi_f(x) = \sum_{y \in E^*(x)} f(y)$. A total weighting is called proper if the induced vertex colouring ϕ_f is a proper colouring of G . Przybyło and Woźniak proposed the following conjecture and named it the 1, 2-conjecture in [9]:

Conjecture 1.3. *Every simple graph G has a proper total weighting f such that $f(y) \in \{1, 2\}$ for all $y \in V \cup E$.*

Przybyło and Woźniak [9, 10] verified this conjecture for some special graphs, including complete graphs, 4-regular graphs and graphs G with $\chi(G) \leq 3$. They also proved that every simple graph G has a proper total weighting f such that $f(y) \in \{1, 2, \dots, 11\}$ for all $y \in V \cup E$.

In this paper, we consider the choosability version of total weighting. A *total list assignment* of G is a mapping $L : V \cup E \rightarrow \mathcal{P}(\mathbb{R})$ which assigns to each element $y \in V \cup E$ a set $L(y)$ of real numbers as *permissible weights*. Given a total list assignment L , a total weighting f is called an *L -total weighting* if for each $y \in V \cup E$, $f(y) \in L(y)$. We say G is *L -total weightable* if there exists a L -total weighting f of G . Given a pair (k, k') of positive integers, a total list assignment L is called a (k, k') -total list assignment if $|L(x)| = k$ for each vertex $x \in V$ and $|L(e)| = k'$ for each edge $e \in E$. We say G is (k, k') -total weight choosable ((k, k') -choosable, for short) if for any (k, k') -total list assignment L , G is L -total-weightable.

The notion of L -total weightability and the notion of (k, k') -total weight choosable is a common generalization of the above mentioned various edge weighting concepts. It also includes the ordinary vertex colouring and vertex choosability as its special cases. A total list assignment L is called a *vertex list assignment* if $L(e) = \{0\}$ for each edge $e \in E$. If L is a vertex list assignment for which $|L(x)| = k$ for each vertex x , then L is a vertex k -list assignment. A graph G is called *k -choosable* if for any k -vertex list assignment L , G is L -total weightable. The *choosability* $ch(G)$ of a graph G (also called the choice number of G or the list chromatic number of G and denoted by $\chi_l(G)$) is the minimum integer k such that G is k -choosable.

It follows from the definition that if a graph G is $(k, 1)$ -choosable, then G is k -choosable (as one can let $L(e) = \{0\}$ for each edge e). The converse is also true. Assume G is k -choosable, and L is a $(k, 1)$ -total list assignment with $L(e) = \{l(e)\}$ for each edge e . Let L' be a k -vertex list assignment defined as $L'(x) = \{a + \sum_{e \in E(x)} l(e) : a \in L(x)\}$. Since G is k -choosable, G has an L' -colouring f' . Then $f(x) = f'(x) - \sum_{e \in E(x)} l(e)$ and $f(e) = l(e)$ is an L -total weighting of G .

It also follows from the definition that if a graph G is $(1, k)$ -choosable, then it is k -edge-weight-choosable. The converse is not true. For example, the path $P = (v_1, v_2, v_3, v_4)$ is 2-edge-weight-choosable, but it is not $(1, 2)$ -

choosable. Let $L(v_1) = \{1\}$, $L(v_i) = \{0\}$ for $i = 2, 3, 4$, and let $L(e) = \{0, 1\}$ for each edge e . It can be verified that there is no L -total weighting. However, we shall show that those graphs shown in [7] to be k -edge-weight-choosable are also $(1, k)$ -choosable. We propose two conjectures concerning total weight choosability of graphs.

Conjecture 1.4. *There are constants k, k' such that every graph is (k, k') -choosable.*

Conjecture 1.5. *Every graph is $(2, 2)$ -choosable. Every graph with no isolated edges is $(1, 3)$ -choosable.*

Note that if a graph G is (k, k') -choosable, then it is $(k + 1, k')$ -choosable and $(k, k' + 1)$ -choosable. If Conjecture 1.5 is true, then the integer pairs are best possible, as there are graphs which are not $(1, 2)$ -choosable and for any integer k , there are graphs which are not $(k, 1)$ -choosable.

Conjecture 1.5 is stronger than Conjectures 1.4, 1.2 and 1.3. Although $(1, 3)$ -total weight choosability is stronger than the 3-edge weight choosability, we shall see that in some inductive proofs, the stronger version has its advantage. The argument in [7] actually shows that complete graphs, complete bipartite graphs and some other graphs are $(1, 3)$ -choosable. This paper shows that complete graphs, trees, cycles, generalized theta graphs, and some other graphs are $(2, 2)$ -choosable. Moreover, complete bipartite graphs $K_{2,n}$ are $(1, 2)$ -choosable and $K_{3,n}$ are $(2, 2)$ -choosable.

2 A polynomial associated with total weighting

Assume $G = (V, E)$ is a simple graph, where $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{e_1, e_2, \dots, e_m\}$. In the study of edge weight choosability of graphs, Bartnicki, Grytczuk and Niwczyk [7] considered a polynomial associated to G . Fix an arbitrary orientation D of G and denote by $E(D)$ the set of directed edges of D . Each edge e is associated with a variable x_e . Let

$$P(G) = \prod_{e=uv \in E(D)} \left(\sum_{e' \in E(v)} x_{e'} - \sum_{e' \in E(u)} x_{e'} \right).$$

It follows from the definition that an edge weight assignment $f : E \rightarrow \mathbb{R}$ is an edge weighting of G if and only if $P(G) \neq 0$ when evaluated at $x_e = f(e)$ for $e \in E(G)$.

In this paper, we consider total weighting, i.e., mappings f that assigns weights to edges as well as vertices of G . So we need a modification of the polynomial. Each vertex v of G is associated with a variable x_v , and each edge e of G is associated with a variable x_e . Denote by X the set of variables $x_{v_1}, x_{v_2}, \dots, x_{v_n}, x_{e_1}, x_{e_2}, \dots, x_{e_m}$. For each vertex v of G , let $Q_v = \sum_{e \in E(v)} x_e + x_v$. Fix an arbitrary orientation D of G . Let

$$\tilde{P}(G) = \tilde{P}(x_{v_1}, x_{v_2}, \dots, x_{v_n}, x_{e_1}, x_{e_2}, \dots, x_{e_m}) = \prod_{(u,v) \in E(D)} (Q_v - Q_u).$$

Note that for each edge (u, v) , $Q_v - Q_u$ is a polynomial of degree 1 with variable set X . Thus $\tilde{P}(G)$ is a polynomial of degree $m = |E(G)|$ with variable set X . For different orientations D of G , the polynomials defined may differ by a sign, which is irrelevant for our purpose. For convenience, we use $\tilde{P}(G)$ to denote any polynomial defined through an arbitrary orientation of G .

Suppose f is a proper total weighting of G . Recall that ϕ_f is the colouring of G induced by f . By definition, for each vertex u ,

$$\phi_f(u) = Q_u(f(v_1), f(v_2), \dots, f(v_n), f(e_1), f(e_2), \dots, f(e_m))$$

is equal to the evaluation of the polynomial Q_u with $x_v = f(v)$ for each $v \in V$ and $x_e = f(e)$ for each edge $e \in E$. Therefore f is a proper total weighting if and only if

$$\tilde{P}(f(v_1), f(v_2), \dots, f(v_n), f(e_1), f(e_2), \dots, f(e_m)) \neq 0.$$

We shall need the following theorem, called the Combinatorial Nullstellensatz [5, 6], in our proofs.

Theorem 2.1. *Let F be a field and let $p(x_1, x_2, \dots, x_n)$ be a polynomial in $F[x_1, x_2, \dots, x_n]$. Suppose the degree of p is equal to $\sum_{j=1}^n t_j$ and the coefficient of $\prod_{j=1}^n x_j^{t_j}$ in the expansion of p is nonzero. Then for any subsets S_1, S_2, \dots, S_n of F with $|S_j| = t_j + 1$, there exist $s_1 \in S_1, s_2 \in S_2, \dots, s_n \in S_n$ so that*

$$p(s_1, s_2, \dots, s_n) \neq 0.$$

It follows from Combinatorial Nullstellensatz that if $k : V \cup E \rightarrow \{0, 1, \dots\}$ is a mapping such that

- $\sum_{v \in V} k(v) + \sum_{e \in E} k(e) = m$,

- the coefficient of the monomial $\prod_{v \in V} x_v^{k(v)} \prod_{e \in E} x_e^{k(e)}$ in the expansion of $\tilde{P}(G)$ is nonzero,

then for any total list assignment L with $|L(v)| \geq k(v) + 1$ for all $v \in V$ and $|L(e)| \geq k(e) + 1$ for all $e \in E$, G has a L -total weighting.

To calculate the coefficient of the term $\prod_{v \in V} x_v^{k(v)} \prod_{e \in E} x_e^{k(e)}$, we need to calculate the permanent of some matrices.

For an $m \times m$ square matrix $A = (a_{ij})$, the *permanent* of A is defined by

$$\text{per}(A) = \sum_{\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{m\sigma(m)}$$

where σ is taken over all the permutations of $\{1, 2, \dots, m\}$.

Let A be a matrix with m rows and s columns A_1, A_2, \dots, A_s . Let $K = (k_1, k_2, \dots, k_s)$ be a sequence of nonnegative integers with $k_1 + k_2 + \dots + k_s = m$. Let $A(K)$ be the square matrix in which the j th column A_j is repeated k_j times (in case $k_j = 0$ then the column A_j does not appear). The following lemma can be verified easily and is observed in [4].

Lemma 2.2. *Let A be an $m \times s$ matrix. For the polynomial $P(x_1, x_2, \dots, x_s) = \prod_{i=1}^m \sum_{j=1}^s a_{ij} x_j$, the coefficient of $x_1^{k_1} x_2^{k_2} \cdots x_s^{k_s}$ is equal to $\frac{\text{per}(A(K))}{k_1! \cdots k_s!}$.*

For a graph G with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set $E = e_1, e_2, \dots, e_m$ and with a fixed orientation, let A_G be the $m \times (n + m)$ matrix with rows indexed by the edges of G and columns indexed by vertices and edges of G such that

$$\tilde{P}(G) = \prod_{e \in E} \left(\sum_{v \in V} a_{ev} x_v + \sum_{e' \in E} a_{ee'} x_{e'} \right).$$

For each vertex $v \in V$ (respectively, for each edge $e \in E$), we denote by A_v (respectively, by A_e) the column of A_G indexed by v (respectively, indexed by e). So

$$A_G = [A_{v_1}, A_{v_2}, \dots, A_{v_n}, A_{e_1}, A_{e_2}, \dots, A_{e_m}].$$

The columns A_v for $v \in V$ are called the vertex columns, and columns A_e for $e \in E$ are called the edge columns. We denote by B_G the submatrix of A_G consisting the edge columns of A_G .

Given a matrix A , the *perrank* of A , denoted by $\text{perrank}(A)$, is the largest integer k such that A has a $k \times k$ submatrix with nonzero permanent. If A is an $m \times s$ -matrix, then $\text{perrank}(A) \leq m$. Let $A^{(k)} = [A, A, \dots, A]$ be the

matrix formed of k copies of A . The *permanent index* of an $m \times s$ matrix A , denoted by $\text{pind}(A)$, is the minimum k for which $\text{perrank}(A^{(k)}) = m$. If there is no such k , then let $\text{pind}(A) = \infty$.

The following lemma follows directly from Lemma 2.2 and Theorem 2.1.

Lemma 2.3. *Given two positive integers k, k' , let $A_G(k, k')$ be the matrix formed by the columns of A_G in such a way that each vertex column appears k times and each edge column appears k' times. If the perrank of $A_G(k, k')$ is equal to $|E(G)|$, then G is $(k + 1, k' + 1)$ -choosable. In particular, if $\text{pind}(A_G) = 1$, then G is $(2, 2)$ -choosable. If $\text{pind}(B_G) = 1$, then G is $(1, 2)$ -choosable. If $\text{pind}(B_G) \leq 2$, then G is $(1, 3)$ -choosable.*

It was conjectured in [7] that for any graph with no isolated edges, $\text{pind}(B_G) \leq 2$. This conjecture implies that every graph with no isolated edges is $(1, 3)$ -choosable. The following two conjectures are stronger than Conjecture 1.4 and the first half 1.5.

Conjecture 2.4. *There is a constant k such that every graph G has $\text{pind}(A_G) \leq k$.*

Conjecture 2.5. *Every graph G has $\text{pind}(A_G) = 1$.*

3 Complete graphs

In this section, we prove that complete graphs K_n are $(2, 2)$ -choosable, by showing that $\text{pind}(A_{K_n}) = 1$.

For a complete graph K_n , the vertices are v_1, v_2, \dots, v_n , and the edges are $e_{ij} = v_i v_j$ for $n \geq i > j \geq 1$. For $i > j$, we orient the edge e_{ij} from v_i to v_j . Let $\psi(k)$ be the function defined as $\psi(1) = 1$ and $\psi(k) = \psi(k - 1)k!$ for $k \geq 2$.

Lemma 3.1. *Let Q_{K_n} be obtained from A_{K_n} by deleting the following columns*

$$A_{v_1}, A_{e_{21}}, A_{e_{32}}, \dots, A_{e_{i(i-1)}}, \dots, A_{e_{n(n-1)}}.$$

Then $|\text{per}(Q_{K_n})| = \psi(n - 1)$.

As a consequence of this lemma and Lemmas 2.2 and 2.3, we have the following theorem, which is a main result of this paper.

Theorem 3.2. *Complete graphs are $(2, 2)$ -choosable.*

We shall prove Lemma 3.1 by induction on $n \geq 2$. The case $n = 2$ is trivial. For K_3 ,

$$A_G = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 & -1 & 0 \end{bmatrix}.$$

Remove the columns $A_{v_1}, A_{e_{21}}, A_{e_{32}}$ from A_{K_3} , we obtain the matrix

$$Q_{K_3} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix}.$$

It is easy to verify that $|\text{per}(Q_{K_3})| = 2 = \psi(2)$. (The case $n = 3$ need not be considered separately. We do it explicitly to illustrating the lemma). It remains to show that if $n \geq 4$ and Lemma 3.1 holds for $n - 1$, it also holds for n . Before the proof of Lemma 3.1, we shall prove some other lemmas.

Lemma 3.3. *If $e' = uv$ is an edge of G , then $A_{e'} = A_u + A_v$.*

Proof. It suffices to show that for any edge e , $a_{ee'} = a_{eu} + a_{ev}$. This follows from the following facts:

$$a_{eu} = \begin{cases} 1, & \text{if } u \text{ is the head of } e, \\ -1, & \text{if } u \text{ is the tail of } e, \\ 0, & \text{otherwise.} \end{cases}$$

$$a_{ee'} = \begin{cases} 1, & \text{if } e' \neq e \text{ is incident to the head of } e, \\ -1, & \text{if } e' \neq e \text{ is incident to the tail of } e, \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, if e is not incident to any of u or v , then $a_{ee'} = 0, a_{eu} = a_{ev} = 0$. If $e \neq e'$ and u or v is the head of e , then $a_{ee'} = 1$ and $a_{eu} + a_{ev} = 1$. If $e \neq e'$ and u or v is the tail of e , then $a_{ee'} = -1$ and $a_{eu} + a_{ev} = -1$. If $e = e'$ then $a_{ee'} = 0$. Assume e is oriented from u to v . Then $a_{eu} = -1, a_{ev} = 1$, so $a_{eu} + a_{ev} = 0$. \square

When we write the matrix A_{K_n} , we order the edges of G as

$$e_{21}, e_{31}, e_{32}, e_{41}, e_{42}, e_{43}, \dots, e_{n1}, e_{n2}, \dots, e_{n(n-1)}.$$

Let D be any square matrix obtained from Q_{K_n} as follows: First delete the columns

$$A_{e_{n1}}, A_{e_{n2}}, \dots, A_{e_{n(n-2)}}.$$

Then for some integer $0 \leq s \leq n - 3$, add s times the column A_{v_n} and choose $n - 2 - s$ distinct columns from $A_{v_1}, A_{v_2}, \dots, A_{v_{n-2}}$ and add them to D . Note that each of the columns $A_{v_2}, A_{v_3}, \dots, A_{v_{n-2}}$ appeared in Q_{K_n} once already. So if any of these columns is added to D , then it will appear in D twice. However, A_{v_1} is not contained in Q_{K_n} . So A_{v_1} appears in D at most once. Also observe that $A_{e_{n(n-1)}}$ is not contained in Q_{K_n} . So for any edge e incident to v_n , D does not contain the column A_e .

Lemma 3.4. *The matrix D constructed above has zero permanent.*

Proof. By Lemma 2.2, the permanent of D is the coefficient of the term

$$\prod_{v \in V} x_v^{k(v)} \prod_{e \in E} x_e^{k(e)}$$

where for each vertex v (respectively, for each edge e), $k(v)$ (respectively, $k(e)$) is the number of times column A_v (respectively, A_e) appears in D . By the construction of D , we know that

- $k(v_n) = s + 1$.
- $k(e) = 0$ for $e \in \{e_{21}, e_{32}, \dots, e_{n(n-1)}, e_{n1}, e_{n2}, \dots, e_{n(n-2)}\}$.
- $k(e) = 1$ for every other edge e .
- If $j \in \{2, 3, \dots, n-2\}$ and A_{v_j} is added to D in the construction above (so A_{v_j} appeared twice in D), then $k(v_j) = 2$.
- If A_{v_1} is added to D in the construction, then $k(v_1) = 1$. Otherwise $k(v_1) = 0$.
- $k(v) = 1$ for every other vertex v .

If D has non-zero permanent, then by Theorem 2.1, K_n has a L -total weighting for any total list assignment L for which $|L(v)| = k(v) + 1$ for each vertex v and $|L(e)| = k(e) + 1$ for each edge e . Let L be the total list assignment defined as $L(v) = \{0, 1, \dots, k(v)\}$ for each vertex v and $L(e) = \{0, 1, \dots, k(e)\}$ for each edge e .

Let f be an L -total weighting of K_n . Consider the induced colouring ϕ_f of K_n . If $v = v_1$, or $v = v_{n-1}$, then v is incident to two edges whose weights are 0: For v_1 , the two edges are e_{21}, e_{n1} . For v_{n-1} , the two edges are $e_{(n-1)(n-2)}, e_{n(n-1)}$. Each other edge incident to v has weight 0 or 1, and vertex v itself has weight 0 or 1. Therefore $\phi_f(v) \in \{0, 1, \dots, n-2\}$. If $v = v_j$ for some $2 \leq j \leq n-2$, then v is incident to three edges $v_{j(j-1)}, v_{(j+1)j}, v_{nj}$, of weight 0. The other edges have weights 0 or 1, and v_j itself has weight 0, 1 or 2. So again $\phi_f(v) \in \{0, 1, \dots, n-2\}$. All the edges incident to v_n have weight 0, and v_n itself has weight in $\{0, 1, \dots, s+1\}$. As $s \leq n-3$, so $\phi_f(v_n) \in \{0, 1, \dots, n-2\}$. Therefore ϕ_f is not a proper colouring of K_n , in contrary to the assumption that f is a L -total weighting of K_n . \square

The following lemma follows from the definition of permanent.

Lemma 3.5. *Assume $C = [A_1 + B_1, A_2, \dots, A_m]$ is a square matrix, where A_i, B_1 are column vectors of dimension m . Let $A = [A_1, A_2, \dots, A_m]$ and $B = [B_1, A_2, \dots, A_m]$. Then*

$$\text{per}(C) = \text{per}(A) + \text{per}(B).$$

Now we are ready to prove Lemma 3.1. For convenience, we move the column A_{v_n} from its current position to the right most position (i.e., after the move, A_{v_n} is the last column of Q_{K_n} . The order is not important, the moving is just for convenience). Let $m = n(n-1)/2$ and $m' = (n-1)(n-2)/2 = m - n + 1$. Then Q_{K_n} is an $m \times m$ matrix, and after moving A_{v_n} to the last column, the square submatrix of Q_{K_n} induced by the first m' row and first m' columns is equal to $Q_{K_{n-1}}$. The last $n-1$ columns of Q_{K_n} (after moving A_{v_n} to the last column) are $A_{e_{n1}}, A_{e_{n2}}, \dots, A_{e_{n(n-2)}}, A_{v_n}$. By Lemma 3.3, $A_{e_{nj}} = A_{v_n} + A_{v_j}$. We write each of the $n-2$ columns $A_{e_{n1}}, A_{e_{n2}}, \dots, A_{e_{n(n-2)}}$ of Q_{K_n} as the sum of two vertex columns.

By Lemma 3.5, $\text{per}(Q_{K_n})$ is the summation of a set of permanents $\text{per}(D)$, where each D is obtained from Q_{K_n} by choosing, for each of the $n-2$ columns that are of the form $A_{v_n} + A_{v_j}$, one of the two columns A_{v_n}, A_{v_j} . Altogether there are 2^{n-2} such D 's. One of the choice is that all the last $n-1$ columns are A_{v_n} . For this choice, the last $n-1$ columns has an all 1 square submatrix of order $(n-1) \times (n-1)$ at the bottom, and all the other entries are 0. Therefore for this choice, the permanent D is equal to $\text{per}(Q_{K_{n-1}})(n-1)!$. For all the other choices, it follows from Lemma 3.4 that

the permanent of D is 0. Therefore $|\text{per}(Q_{K_n})| = \psi(n-1)$. This completes the proof of Lemma 3.1.

Corollary 3.6. *Suppose G is obtained from K_n by deleting a set X of edges that can be extended to a Hamilton path of K_n . Then $\text{pind}(A_G) = 1$.*

Proof. Since X is a set of edges that can be extended to a Hamilton path of K_n , we may assume, without loss of generality, that $X \subseteq \{e_{21}, e_{32}, \dots, e_{n(n-1)}\}$. Observe that A_G is obtained from A_{K_n} by removing those columns and row corresponding to edges in X . Let D be the matrix formed by those columns of A_G corresponding to the columns of Q_{K_n} . Then D can be obtained from Q_{K_n} by removing the rows corresponding to edges in X . Assume $|X| = s$, and let $m = \frac{n(n-1)}{2} - s$ be the number of edges of G . Denote by Q_X the s row matrix formed by the s rows of Q_{K_n} corresponding to the edges in X . The permanent of Q_{K_n} can be calculated as the sum $\sum \text{per}(S)\text{per}(R)$, where S ranges over all matrices formed by s columns of Q_X , and R is the complement matrix of S (i.e., R is obtained from Q_{K_n} by deleting the columns and row of S). Since $\text{per}(Q_{K_n})$ is nonzero, one of the matrices R has nonzero permanent. That R is a required $m \times m$ square submatrix of A_G . So $\text{pind}(A_G) = 1$. \square

4 Bipartite graphs

Suppose $G = (X \cup Y, E)$ is a bipartite graph. In writing the polynomial A_G , we assume that the edges of G are oriented from Y to X . We denote by R_e the row of B_G corresponding to edge e . The entry at row R_e and column $A_{e'}$ is denoted by $a_{e,e'}$. So $a_{e\tilde{e}} = 1$ if e, \tilde{e} have a common end vertex in X , $a_{e\tilde{e}} = -1$ if e, \tilde{e} have a common end vertex in Y , and $a_{e\tilde{e}} = 0$ otherwise.

The matrix B_G defines a bipartite graph H with bipartite sets S, T as follows: $S = \{s_e : e \in E(G)\}$ and $T = \{t_e : e \in E(G)\}$. If $a_{e\tilde{e}} \neq 0$ then there is an edge between s_e and $t_{\tilde{e}}$. For a perfect matching M of H , let $\sigma(M) = \prod_{s_e t_{\tilde{e}} \in M} a_{e\tilde{e}}$. By definition of permanent, $\text{per}(B_G) = \sum \sigma(M)$, where the summation ranges over all perfect matchings of H .

Identify s_e with t_e in H , we obtain the line graph $L(G)$ of G (parallel edges are replaced by a single edge). We call a spanning subgraph F of $L(G)$ a semi-2-factor of $L(G)$ if each component of F is either a cycle or an edge. If each cycle in F is assigned a direction to form a directed cycle, then F is called a directed semi-2-factor. A perfect matching M in H

corresponds to a directed semi-2-factor in $L(G)$. As each cycle in a semi-2-factor can be assigned two direction, if F is semi-2-factor with q cycles then it corresponds to 2^q perfect matchings in H . For each semi-2-factor F of $L(G)$, let $m(F)$ be the number of cycles in F and let $\sigma(F) = \prod_{e\tilde{e}\in F} a_{e\tilde{e}}$. Then $\text{per}(B_G) = \sum \sigma(F)2^{m(F)}$.

If $G = K_{n,m}$ is a complete bipartite graph, then $L(G)$ is the Cartesian product $K_n \square K_m$ of K_n and K_m . We draw the graph $K_n \square K_m$ in the shape of rectangle, with m horizontal copies of K_n , and n vertical copies of K_m . So each edge of $K_n \square K_m$ is either a horizontal edge or a vertical edge. For a semi-2-factor F of $L(G) = K_n \square K_m$, $\sigma(F)$ is equal to 1 or -1 , depending on the number of horizontal edges is even or odd.

Lemma 4.1. *For any positive integer m , $\text{per}(B_{K_{2,m}}) \neq 0$. As a consequence, $K_{2,m}$ is $(1,2)$ -choosable.*

Proof. If $G = K_{2,m}$, the m horizontal edge of $K_2 \square K_m$ form a cut of $K_2 \square K_m$. By the discussion above, $\text{per}(B_G) = \sum \sigma(F)2^{m(F)}$. Since each cycle of a semi-2-factor contains an even number of edges of any cut, so F contains an even number of horizontal edges of $K_2 \square K_m$. Therefore $\sigma(F) = 1$ for all semi-2-factor F of $L(K_{2,m})$, and hence $\text{per}(B_{K_{2,m}}) \neq 0$. \square

Lemma 4.2. *Assume G is a graph, x_1, x_2, \dots, x_k is a sequence of vertices of G and $y_i \in N_G[x_i]$ for $i = 1, 2, \dots, k$. Assume the following hold:*

- *For $i = 2, 3, \dots, k$, $N_G[x_i] \cap (\cup_{j=1}^{i-1} N_G[x_j]) \subseteq \{y_1, y_2, \dots, y_i\}$.*
- *For any $1 \leq i \leq k$, if a vertex z is contained in t of the sets $N_G[x_1], N_G[x_2], \dots, N_G[x_i]$, then there are at least $t - 1$ indices $j \leq i$ such that $y_j = z$.*
- *$B_{G-\{x_1, x_2, \dots, x_k\}}$ has nonzero permanent.*

Then $\text{pind}(A_G) = 1$.

Proof. For $i = 0, 1, \dots, k$, let $G_i = G - \{x_{i+1}, x_{i+2}, \dots, x_k\}$. We shall prove by induction on i that there is a square matrix B_i with nonzero permanent formed by some edge columns of A_{G_i} and some vertex columns (each edge column and each vertex column appears in B_i at most once), and the vertex columns are contained in the set $\{A_v : v \in \cup_{j=1}^i N_G[x_j]\}$. Moreover, if z is contained in t of the sets $N_G[x_i]$ and there are at least t indices $j \leq i$ with $y_j = z$, then A_z is not a column of B_i .

If $i = 0$, then this follows from our assumption that B_{G_0} has nonzero permanent. Assume $i \geq 1$ and the statement holds for $i - 1$. The graph G_i , which is obtained from G_{i-1} by adding vertex x_i and edges incident to x_i in G_i . Assume the edges in G_i incident to x_i are $e_1 = x_i u_1, e_2 = x_i u_2, \dots, e_s = x_i u_s$.

Since the statement holds for $i - 1$, there is a square matrix B_{i-1} formed by some edge columns and vertex columns of $A_{G_{i-1}}$. Moreover, the assumption implies that the vertex columns $A_{u_1}, A_{u_2}, \dots, A_{u_s}$ are not contained in B_{i-1} , except that in case $y_i = u_j$, then A_{u_j} may be contained in B_{i-1} . The vertex columns in B_{i-1} do not contain the columns corresponding to the neighbours of x_i in G_{i-1} . Let D be the submatrix of A_{G_i} formed by the columns corresponding to those columns in B_{i-1} . The matrix D can also be viewed as obtained from B_{i-1} by adding s rows (at the bottom) corresponding to those edges in G_i incident to x_i .

By Lemma 3.5, for $j = 1, 2, \dots, s$, $A_{e_j} - A_{u_j} = A_{x_i}$. Let D' be the matrix obtained from D by adding s copies of the columns A_{x_i} . Since A_{x_i} has 1's in the last s rows and 0's in the other rows, we conclude that $\text{per}(D') = s! \text{per}(B_{i-1}) \neq 0$. If $y_i = x_i$, then we replace the s copies of A_{x_i} by the columns $A_{e_j} - A_{u_j}$ for $j = 1, 2, \dots, s$. If $y_i = u_t$, then we replace $s - 1$ copies of A_{x_i} by the columns $A_{e_j} - A_{u_j}$ for $j = 1, 2, \dots, s, j \neq t$. Let $Z = \{e_j : j = 1, 2, \dots, s\} \cup \{u_j : j = 1, 2, \dots, s\} - \{y_i\}$. By Lemma 3.5, there is a matrix B_i with nonzero permanent which is obtained from D by adding s distinct columns from $\{A_y : y \in Z\}$. This completes the proof of the lemma. \square

Corollary 4.3. *If G is a graph for which B_G has nonzero permanent, and G' is obtained from G by adding a set $\{x_1, x_2, \dots, x_k\}$ of vertices and some edges so that $N(x_i) \cap N(x_j) = \emptyset$ if $i \neq j$, and for each $i \geq 2$, there is at most one index $j < i$ such that x_i is adjacent to x_j , then $\text{pind}(A_{G'}) = 1$.*

Proof. Arbitrarily choose y_1 . For $i \geq 2$, if there is an index j such that x_i is adjacent to x_j , then let $y_i = x_j$. Otherwise, arbitrarily choose y_i . Then the conclusion follows from Lemma 4.2. \square

Since $B_{K_{2,m}}$ has nonzero permanent, any graph G obtained from $K_{2,m}$ by adding one vertex and some edges has $\text{pind}(A_G) = 1$. In particular, we have the following corollary.

Corollary 4.4. *For any positive integer m , $\text{pind}(A_{K_{3,m}}) = 1$. Hence $K_{3,m}$ is $(2, 2)$ -choosable.*

The question whether every complete bipartite graph is $(2, 2)$ -choosable remains open.

5 Construction of $(2, 2)$ -choosable graphs

This section proves that some other graphs, including trees, cycles, generalized theta graphs are $(2, 2)$ -choosable.

Lemma 5.1. *Suppose G is obtained from G' by adding one vertex v and one edge $e^* = uv$, where u is a vertex of G' . If $\text{pind}(A_{G'}) = 1$, then $\text{pind}(A_G) = 1$. If G' is $(2, 2)$ -choosable, then G is $(2, 2)$ -choosable.*

Proof. Assume $A_{G'}$ is an $m \times (m + n)$ matrix. Then A_G is obtained from $A_{G'}$ by adding one row and two columns. Let B' be an $m \times m$ submatrix of $A_{G'}$ with nonzero permanent. Let B be the submatrix of A_G obtained from B' by adding the last row (the row corresponding to the edge e^*) and the column A_v . Note that the column A_v has 1 in the last row and 0 in other rows. Therefore $\text{per}(B) = \text{per}(B') \neq 0$.

Assume G' is $(2, 2)$ -choosable and L is a $(2, 2)$ -total-list assignment of G . Let L' be the $(2, 2)$ -total list assignment of G' defined as $L'(y) = L(y)$ if $y \in V(G') \cup E(G')$ and $y \neq u$. Arbitrarily choose $w \in L(e^*)$ and let $L'(u) = w + L(u)$. By assumption, G' has a proper L' -total weighting f . By let $f(e^*) = w$ and by choosing $f(v) \in L(v)$ so that $\phi_f(v) \neq \phi_f(u)$, we obtain a proper L -total weighting of G . \square

A *thread* in a graph G is a path $P = (v_1, v_2, \dots, v_k)$ in G such that $d_G(v_i) = 2$ for $i = 2, 3, \dots, k - 1$. The vertices v_1, v_k need not be distinct. If we need to specify the two end vertices of a thread, then we say P is a v_1 - v_k -thread. By deleting the thread $P = (v_1, v_2, \dots, v_k)$ from G , we mean delete the vertices v_2, v_3, \dots, v_{k-1} (and hence edges incident to them).

Lemma 5.2. *Assume G is a graph and $P_1 = (x, u, y), P_2 = (x, v, y)$ are two x - y -threads. Let G' be obtained from G by deleting the two threads P_1, P_2 . If $\text{pind}(A_{G'}) = 1$, then $\text{pind}(A_G) = 1$.*

Proof. Assume $A_{G'}$ is an $m \times (m + n)$ matrix. Let B' be an $m \times m$ submatrix of $A_{G'}$ with $\text{per}(B') \neq 0$. Let $e_1 = xu, e_2 = xv, e_3 = yu, e_4 = yv$.

Orient the edges e_1, e_2, e_3, e_4 from u, v to x, y . Let B'' be obtained from B' by adding the 4 rows corresponding to added edges and the following 4 columns: $A_u, A_v, A_{e_1} - A_{e_2}, A_{e_3} - A_{e_4}$. The added 4 columns have zero entries everywhere except those entries at the added 4 rows. Let C be the 4×4 submatrix of B'' consisting of the added rows and columns. Then $\text{per}(B'') = \text{per}(B') \times \text{per}(C)$.

Since

$$C = \begin{bmatrix} -1 & 0 & -1 & -1 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ 0 & -1 & 1 & 1 \end{bmatrix},$$

which has nonzero permanent, we conclude that $\text{per}(B'') \neq 0$. By Lemma 3.3, $\text{per}(B'')$ is a linear combination of the permanents of matrices obtained from B'' by replacing the row $A_{e_1} - A_{e_2}$ with either A_{e_1} or A_{e_2} , and replacing the row $A_{e_3} - A_{e_4}$ with either A_{e_3} or A_{e_4} . Since the linear combination is nonzero, there exists $i \in \{1, 2\}$ and $j \in \{3, 4\}$ such that the matrix obtained from B'' by replacing $A_{e_1} - A_{e_2}$ with A_{e_i} and replacing $A_{e_3} - A_{e_4}$ with A_{e_j} has nonzero permanent. Such a matrix B is a submatrix of A_G . So $\text{pind}(A_G) = 1$. \square

It is unknown if the following is true: If G' is obtained from G by deleting a pair of x - y -threads of length 2, and G' is $(2, 2)$ -choosable, then G is also $(2, 2)$ -choosable.

Given a total list assignment L of $G = (V, E)$, to find a proper L -total weighting f of G , we need to determine $f(y)$ for all $y \in V \cup E$. In the following, we shall determine the weighting on edges first. Assume $f(e)$ are determined for all edges e . For each vertex v , let $L_f(v) = \{w + \sum_{e \in E(v)} f(e) : w \in L(v)\}$. Then L_f is a vertex list assignment. It is easy to see that f can be extended to a L -total weighting of G (by determining $f(x)$ for each vertex $x \in V(G)$) if and only if there is an L_f -colouring of G .

Suppose s is a real number and A is a set of real numbers. Then the set $s + A$ is defined as $s + A = \{s + a : a \in A\}$. If A, B are sets of real numbers, then $A + B = \{a + b : a \in A, b \in B\}$.

Lemma 5.3. *Suppose G' is obtained from a graph G by deleting a thread $P = (v_0, v_1, v_2, v_3)$ of length 3. If G' is $(2, 2)$ -choosable, then G is $(2, 2)$ -choosable.*

Proof. Suppose L is a $(2, 2)$ -total-weight-list assignment of G . Let $e_0 = v_0v_1, e_1 = v_1v_2, e_2 = v_2v_3$. We need to find an L -total weighting of G . Let $e_0 = v_0v_1, e_1 = v_1v_2, e_2 = v_2v_3$. Choose $w_0 \in L(e_0), w_2 \in L(e_2)$ so that $w_0 + L(v_1) \neq w_2 + L(v_2)$. (This can be done as there are two different choices for each of w_0, w_2).

Let L' be the $(2, 2)$ -total-list assignment of G' defined as follows: $L'(y) = L(y)$ if $y \neq v_0, v_3$, and let $L'(v_0) = w_0 + L(v_0)$ and $L'(v_3) = w_2 + L(v_3)$ (in case $v_0 = v_3$, then let $L'(v_0) = w_0 + w_2 + L(v_0)$). By assumption, G' has a proper L' -total-weighting f . Let ϕ_f be the induced colouring of G' .

Assume $L(e_1) = \{a, b\}$ and $a < b$. Let f_1 be the extension of f to the edges e_0, e_1, e_2 so that $f(e_0) = w_0, f(e_2) = w_2$ and $f(e_1) = a$. Let f_2 be the extension of f to the edges e_0, e_1, e_2 so that $f(e_0) = w_0, f(e_2) = w_2$ and $f(e_1) = b$. We shall show that one of f_1, f_2 can be extended to a proper L -total weighting of G . This is equivalent to say that either ϕ_f can be extended to an L_{f_1} -colouring of G or an L_{f_2} -colouring of G .

For $i = 1, 2$, assume $L_{f_i}(v_i) = \{a_i, b_i\}$, where $a_i < b_i$. Let $t = b - a > 0$. Then $L_{f_2}(v_i) = \{a_i + t, b_i + t\}$. If $\phi_f(v_0) \notin L_{f_i}(v_1)$ for some $i = 1, 2$, then ϕ_f can be easily extended to an L_{f_i} -colouring of G : first choose $\phi_f(v_2) \in L_{f_i}(v_2)$ so that $\phi_f(v_2) \neq \phi_f(v_3)$ (as $|L_{f_i}(v_2)| = 2$, there is such a choice). Then choose $\phi_f(v_1) \in L_{f_i}(v_1)$ so that $\phi_f(v_1) \neq \phi_f(v_2)$. Since $\phi_f(v_0) \notin L_{f_i}(v_1)$, we automatically have $\phi_f(v_0) \neq \phi_f(v_1)$. Thus we assume that $\phi_f(v_0) \in L_{f_i}(v_1)$ for $i = 1, 2$. Similarly we assume that $\phi_f(v_3) \in L_{f_i}(v_2)$ for $i = 1, 2$. Thus $\phi_f(v_0) \in \{a_1, b_1\} \cap \{a_1 + t, b_1 + t\}$ and $\phi_f(v_3) \in \{a_2, b_2\} \cap \{a_2 + t, b_2 + t\}$. Since $a_i < b_i$, it must be the case that $\phi_f(v_0) = a_1 + t = b_1$ and $\phi_f(v_3) = a_2 + t = b_2$. Since $\{a_1, b_1\} \neq \{a_2, b_2\}$, we conclude that $a_1 \neq a_2$. Now ϕ_f can be extended to an L_{f_1} -colouring of G by letting $\phi_f(v_1) = a_1$ and $\phi_f(v_2) = a_2$. \square

It is unknown if the following is true: Assume G' is obtained from G by deleting a thread of length 3. If $\text{pind}(A_{G'}) = 1$, then $\text{pind}(A_G) = 1$. However, if G' is obtained from G by deleting a thread of length at least 4 and $\text{pind}(A_{G'}) = 1$, then it can be proved that $\text{pind}(A_G) = 1$.

Corollary 5.4. *Assume G' is obtained from G by repeatedly deleting pairs of x - y -threads of length 2, threads of lengths at least 3, and leaves. If $\text{pind}(A_{G'}) = 1$, then G is $(2, 2)$ -choosable. In particular, if G has only one cycle, or G has at most two vertices of degree greater than 2, or G is obtained from another graph H by subdividing each edge with two vertices, then G is $(2, 2)$ -choosable.*

6 (1, 3)-choosable graphs

By Lemma 2.3, if $\text{pind}(B_G) \leq k$, then G is $(1, k+1)$ -choosable. It was shown in [7] that if $\text{pind}(B_{G'}) \leq 2$, then $\text{pind}(B_G) \leq 2$, if one of the following holds:

- G is obtained from G' by adding two vertices x, y , and $N_G(x) - \{y\} = N_G(y) - \{x\}$.
- G is obtained from G' by adding two vertices x, y and two edges xy, yz , where $z \in V(G')$.
- G is obtained from G' by adding three vertices x, y, z and edges ux, xy, yz, zv , where u, v are (not necessarily distinct) vertices of G' .

It follows from this result that complete graphs, complete bipartite graphs and trees (other than K_2) have $\text{pind}(B_G) \leq 2$ (and hence are $(1, 3)$ -choosable).

The following simple result shows that there are some advantages by considering $(1, 3)$ -total weight choosability instead of 3-edge weight choosability.

Lemma 6.1. *Suppose G' is obtained from G by deleting a thread of length 3. If G' is $(1, 3)$ -choosable, then G is $(1, 3)$ -choosable.*

Proof. Assume $P = (v_0, v_1, v_2, v_3)$ is a thread in G and G' is obtained from G by deleting this thread. Let $e_i = v_i v_{i+1}$ for $i = 0, 1, 2$ be the added edges. Let L be a $(1, 3)$ -total list assignment of G . We shall find an L -total weighting of G . For each vertex v of G , let $f(v)$ be the unique permissible weight of v , i.e., $L(v) = \{f(v)\}$.

Arbitrarily choose a weight $f(e_0) \in L(e_0)$ and a weight $f(e_2) \in L(e_2)$ so that $f(e_0) + f(v_1) \neq f(e_2) + f(v_2)$ (such a choice is certainly possible).

Let \tilde{L} be the $(1, 3)$ -total list assignment of G' defined as follows: $\tilde{L}(y) = L(y)$ if $y \in E(G) \cup V(G)$ and $y \neq v_0, v_3$. Let $\tilde{L}(v_0) = \{f(v_0) + f(e_0)\}$ and $\tilde{L}(v_3) = \{f(v_3) + f(e_2)\}$. If $v_0 = v_3$, then $\tilde{L}(v_0) = \{f(v_0) + f(e_0) + f(e_2)\}$. Since G' is $(1, 3)$ -choosable, there is an \tilde{L} -weighting f of G' . Assume $\phi_f(v_0) = \alpha$ and $\phi_f(v_3) = \beta$.

To extend f to a proper total weighting of G , it remains to determine $f(e_1)$ (as each vertex has only one permissible weight, so we do not need to choose the weight for vertices). For the extend f to be a weighting of G , it suffices to choose $f(e_1)$ so that $\phi_f(v_1) \neq \alpha$ and $\phi_f(v_2) \neq \beta$.

Now the edge e_1 has three permissible weights, say $L(e_1) = \{a, b, c\}$ where $a < b < c$. By letting $f(e_1) = a, b, c$, respectively, $\phi_f(v_1)$ takes values $\alpha_1 < \alpha_2 < \alpha_3$, respectively, and $\phi_f(v_2)$ takes values $\beta_1 < \beta_2 < \beta_3$, respectively. At most one of the three choices makes $\phi_f(v_1) = \alpha$ and at most one of the three choices makes $\phi_f(v_2) = \beta$. Therefore there is a choice of $f(e_1)$ so that $\phi_f(v_1) \neq \alpha$ and $\phi_f(v_2) \neq \beta$. \square

The argument above can be easily modified to show that if G' is $(1, 3)$ -choosable and G is obtained from G' by adding a path of length at least 3 connecting two vertices of G' , then G is $(1, 3)$ -choosable.

A *hanging edge* of a graph G is an edge uv of G such that $d_G(v) = 1$ and $d_G(u) = 2$ or 3 . A *pair of leaves* is a pair of vertices u, v such that $d_G(u) = d_G(v) = 1$ and u, v have the same neighbour.

Lemma 6.2. *Assume G' is obtained from G by deleting the vertices of a hanging edge. If $\text{pind}(B_{G'}) \leq 2$, then $\text{pind}(B_G) \leq 2$.*

Proof. Assume $e = uv$ is a hanging edge of G (with $d_G(v) = 1$ and $d_G(u) = 2$ or 3) and $G' = G - \{u, v\}$. Let B' be a square formed by columns of $B_{G'}$ with $\text{per}(B') \neq 0$. If $d_G(u) = 2$, then let w be the other neighbour of u and let $e' = uw$. Let B be the square matrix formed by columns of B_G which is obtained from B' by adding the two rows corresponding to the added edges e', e , and two columns A_e and $A_{e'}$. It can be easily verified that $|\text{per}(B)| = |\text{per}(B')| \neq 0$. If $d_G(u) = 3$, then let w, w' be the two other neighbours of u and let $e_1 = uw, e_2 = uw'$. Let B be obtained from B' by adding three rows corresponding to the three added edges, and three columns A_e, A_{e_1}, A_{e_2} . It can be easily checked that $|\text{per}(B)| = 2|\text{per}(B')|$. \square

It follows from the above mentioned results in [7] that if G' is obtained from G by deleting a pair of leaves or by deleting a pair of x - y -threads of length 2, and $\text{pind}(B_{G'}) \leq 2$, then $\text{pind}(B_G) \leq 2$. Thus we have the following corollary.

Corollary 6.3. *Assume G' is obtained from G by repeatedly deleting pairs of x - y -threads of length 2, threads of lengths at least 3, vertices of hanging edges and pairs of leaves. If $\text{pind}(B_{G'}) \leq 2$, then G is $(1, 3)$ -choosable. In particular, if G has only one cycle, or G has at most two vertices of degree greater than 2, or G is obtained from another graph H by subdividing each edge with two vertices, then G is $(1, 3)$ -choosable.*

Proof. If G has only one cycle, or has at most two vertices of degree greater than 2, or is obtained from another graph H by subdividing each edge with two vertices, then by repeatedly deleting pairs of x - y -threads of length 2, threads of lengths at least 3, vertices of hanging edges and pairs of leaves, we can get a graph G' which is either a tree with at least two edges, or a K_3 , or a K_3 plus a leaf vertex, or a C_4 , or a K_4 minus one edge. Each of these graphs G' has $\text{pind}(B_G) \leq 2$. Hence G is $(1, 3)$ -choosable. \square

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