

# Sparse $H$ -colourable graphs of bounded maximum degree

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## Abstract

Let  $F$  be a graph of order at most  $k$ . We prove that for any integer  $g$  there is a graph  $G$  of girth at least  $g$  and of maximum degree at most  $5k^{13}$  such that  $G$  admits a surjective homomorphism  $c$  to  $F$ , and moreover, for any  $F$ -pointed graph  $H$  with at most  $k$  vertices, and for any homomorphism  $h$  from  $G$  to  $H$  there is a unique homomorphism  $f$  from  $F$  to  $H$  such that  $h = f \circ c$ . As a consequence, we prove that if  $H$  is a projective graph of order  $k$ , then for any finite family  $\mathcal{F}$  of prescribed mappings from a set  $X$  to  $V(H)$  (with  $|\mathcal{F}| = t$ ), there is a graph  $G$  of arbitrary large girth and of maximum degree at most  $5k^{26mt}$  (where  $m = |X|$ ) such that  $X \subseteq V(G)$  and up to an automorphism of  $H$ , there are exactly  $t$  homomorphisms from  $G$  to  $H$ , each of which is an extension of an  $f \in \mathcal{F}$ .

**Keywords:** Uniquely  $H$ -colourable graphs, girth, bounded maximum degree.

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# 1 Introduction

It was proved in 1959 [2] by Erdős that there exist graphs of arbitrary large girth and arbitrary large chromatic number. Since then, numerous generalizations of this landmark result have been published. Among these generalizations is the following result of Müller:

**Theorem 1** [4] *Let  $k, g$  be positive integers,  $X$  a finite set, and  $\mathcal{F}$  a finite family of  $k$ -colourings of  $X$ . Then there exists a graph  $G = (V, E)$  with  $X \subset V$  such that (i):  $G$  has girth  $g(G) \geq g$ ; (ii): up to a permutation of colours,  $G$  has exactly  $|\mathcal{F}|$   $k$ -colourings, each of which is an extension of a member of  $\mathcal{F}$ .*

In particular, when  $\mathcal{F}$  consists of a single colouring, the graph  $G$  is uniquely  $k$ -colourable, and hence has chromatic number  $k$ .

Suppose  $G$  and  $H$  are graphs. A *homomorphism* from  $G$  to  $H$  is a mapping  $f : V(G) \rightarrow V(H)$  such that  $f(x)f(y) \in E(H)$  whenever  $xy \in E(G)$ . We say  $G$  is  *$H$ -colourable* if there exists a homomorphism from  $G$  to  $H$ . We say  $G$  is *uniquely  $H$ -colourable* if there is a surjective homomorphism  $h$  from  $G$  to  $H$ , and for any other homomorphism  $f : G \rightarrow H$ , there is an automorphism  $c$  of  $H$  such that  $f = c \circ h$ . It is easy to see that  $G$  is  $n$ -colourable if and only if  $G$  is  $K_n$ -colourable, and  $G$  is a uniquely  $n$ -colourable if and only if  $G$  is uniquely  $K_n$ -colourable. So  $H$ -colouring is a generalization of  $n$ -colouring. Nešetřil and Zhu [6] extracted the properties needed for Müller's result to hold, and defined the concept of  $G$ -pointed graphs. A graph  $H$  is said to be  *$G$ -pointed* if for any two distinct homomorphisms  $g, g' : G \rightarrow H$  there are at least two vertices  $x, y$  such that  $g(x) \neq g'(x)$  and  $g(y) \neq g'(y)$ . In other words, if two homomorphisms  $g, g'$  from  $G$  to  $H$  are identical on every vertex except one vertex  $x$ , then they must be identical on the vertex  $x$ . The following result was proved in [6].

**Theorem 2** [6] *For every graph  $F$  and every choice of positive integers  $k$  and  $l$  there exists a graph  $G$  together with a surjective homomorphism  $c : G \rightarrow F$  with the following properties:*

- i.  $g(G) > l$ ;*

ii. For every graph  $H$  with at most  $k$  vertices, there exists a homomorphism  $g : G \rightarrow H$  if and only if there exists a homomorphism  $f : F \rightarrow H$ .

iii. For every  $F$ -pointed graph  $H$  with at most  $k$  vertices and for every homomorphism  $g : G \rightarrow H$  there exists a unique homomorphism  $f : F \rightarrow H$  such that  $g = f \circ c$ .

For a graph  $H = (V, E)$ , the power  $H^k$  is a graph with vertex set  $V^k$  and in which  $(x_1, x_2, \dots, x_k)$  is adjacent to  $(y_1, y_2, \dots, y_k)$  if and only if  $x_i$  is adjacent to  $y_i$  for every  $i \in \{1, 2, \dots, k\}$ . For each index  $i$ , the projection:  $\pi_i : H^k \rightarrow H$  defined as  $\pi_i(x_1, x_2, \dots, x_k) = x_i$  is a homomorphism from  $H^k$  to  $H$ . A graph  $H$  is called *projective* if, up to an automorphism of  $H$ , the only homomorphisms from  $H^k$  to  $H$  are the projections. Theorem 2 implies the following:

**Theorem 3** [6] *Suppose  $H$  is a projective graph,  $g$  is a positive integer and  $X$  is a finite set. Then for any finite family  $\mathcal{F}$  of mappings from  $X$  to  $V(H)$ , there exists a graph  $G = (V(G), E(G))$  with  $X \subset V(G)$  such that (i):  $G$  has girth at least  $g$ ; (ii): up to an automorphism of  $H$ ,  $G$  has exactly  $|\mathcal{F}|$   $H$ -colourings, each of which is an extension of a member of  $\mathcal{F}$ .*

As the complete graphs  $K_n$  are projective, Theorem 3 is a generalization of Theorem 1. If  $k \geq 2d$  are positive integers such that  $(k, d) = 1$ , then the *circular complete graph*  $K_{k/d}$  is the graph with vertex set  $\{0, 1, \dots, k-1\}$  in which  $i \sim j$  if and only if  $d \leq |i-j| \leq k-d$ . A graph  $G$  is said to be  $(k, d)$ -colourable if  $G$  admits a homomorphism to  $K_{k/d}$ . The *circular chromatic number*  $\chi_c(G)$  of a graph  $G$  is the infimum of those ratios  $k/d$  for which  $G$  is  $(k, d)$ -colourable. It is known [7, 8] that for every graph  $G$ ,  $\chi(G) = \lceil \chi_c(G) \rceil$ . So  $\chi_c(G)$  is a refinement of  $\chi(G)$ . As the graphs  $K_{k/d}$  are projective [3, 5, 6], Theorem 3 applies to the circular chromatic number of graphs as well.

Recently, it is proved in [1] that there exist uniquely  $k$ -colourable graphs  $G$  of large girth with maximum degree  $\Delta(G) \leq 5k^{13}$ . We note that uniquely  $k$ -colourable graphs of large girth were constructed in [4]. However, for the graphs constructed in [4], the maximum degree goes to infinity along with the number of vertices.

In this note, we show that the same result is true for  $H$ -colouring problems. We shall prove the following:

**Theorem 4** *Let  $k$  and  $l$  be positive integers. For every graph  $F$  on at most  $k$  vertices there exists a graph  $G$  together with a surjective homomorphism  $c : G \rightarrow F$  with the following properties:*

*i.  $g(G) > l$  and  $\Delta(G) \leq 5k^{13}$ ;*

*ii. For every graph  $H$  with at most  $k$  vertices, there exists a homomorphism  $g : G \rightarrow H$  if and only if there exists a homomorphism  $f : F \rightarrow H$ .*

*iii. For every  $F$ -pointed graph  $H$  with at most  $k$  vertices and for every homomorphism  $g : G \rightarrow H$  there exists a unique homomorphism  $f : F \rightarrow H$  such that  $g = f \circ c$ .*

The following result is a consequence of Theorem 4:

**Corollary 5** *Suppose  $H$  is a projective graph on  $k$  vertices,  $g$  is a positive integer,  $X$  is a set of size  $m > 0$ , and  $\mathcal{F}$  is a family of mappings from  $X$  to  $V(H)$  with  $|\mathcal{F}| = t > 0$ . Then there exists a graph  $G = (V, E)$  with  $X \subset V$  such that (i):  $G$  has girth at least  $g$  and maximum degree  $\Delta(G) \leq 5k^{26mt}$ ; (ii): up to an automorphism of  $H$ ,  $G$  has exactly  $t$   $H$ -colourings, each of which is an extension of a member of  $\mathcal{F}$ .*

## 2 Proof of Theorem 4

The proof is by the probabilistic method. The construction of the random graph and the calculations are similar to that of [1]. Suppose  $F$  is a graph with  $k$  vertices,  $V(F) = [k] = \{1, \dots, k\}$ . Given a positive integer  $n$ , let  $G(n, F) = F[\overline{K}_n]$  be the lexicographic product of  $F$  and  $\overline{K}_n$ . In other words,  $G(n, F)$  has vertex set  $V_1 \cup V_2 \cup \dots \cup V_k$ , where  $|V_i| = n$  and  $x \in V_i$  is adjacent to  $y \in V_j$  if and only if  $ij$  is an edge of  $F$ .

**Lemma 6** *If  $n$  is sufficiently large, then there exists a subgraph  $G$  of  $G(n, F)$  with the following properties:*

1. *For any edge  $ij$  of  $F$  and any  $U \subset V_i, W \subset V_j$ , of size  $|U| = \lceil \frac{k-3n}{2} \rceil$  and  $|W| = \lceil \frac{(k-1)n}{k} \rceil$ , there are at least  $\frac{k^7 n}{4}$  edges between  $U$  and  $W$  in  $G$ .*

2. For any edge  $ij$  of  $F$  and any  $U \subset V_i, W \subset V_j$ , of size  $|U| = |W| = \lceil \frac{n}{40k} \rceil$ , there is at least one edge between  $U$  and  $W$  in  $G$ .
3. For any edge  $ij$  of  $F$  and any  $U \subset V_i, W \subset V_j$ , of size  $k \leq |W| = k|U| \leq \frac{n}{40}$ , there are less than  $\frac{|U|k^{10}}{2}$  edges between  $U$  and  $W$  in  $G$ .
4. Let  $g := \lfloor \frac{1}{11} \frac{\log n}{\log k} \rfloor$  and define  $C := \{v : v \text{ is a vertex contained in a cycle in } G \text{ of length at most } g - 1\}$ . Then  $|C| \leq \frac{n}{4k}$ .
5. Let  $Y := \{v : v \text{ has degree in } G \text{ larger than } 5k^{13}\}$ . Then  $|Y| \leq \frac{n}{4k} - 1$ .

To prove Lemma 6, we construct a random subgraph  $G$  of  $G(n, F)$  as follows: Let  $p = k^{10}n^{-1}$ . Put each edge of  $G(n, F)$  into  $G$  independently with probability  $p$ . We shall prove that with a positive probability, a random graph  $G$  so constructed has all the properties listed in Lemma 6. The calculation of the probability are standard and similar to the proof in [1].

First we prove that if  $n$  is sufficiently large, then with probability at least  $9/10$ , a random graph  $G$  constructed above has property (3).

Let  $U$  and  $W$  be as in the statement of property (3) and let  $q := |W| = k|U|$ . For any set  $S$  of  $\frac{qk^9}{2}$  edges between  $U$  and  $W$ , the probability that  $S \subset E(G)$  is  $p^{\frac{qk^9}{2}}$ . There are  $\binom{\frac{q^2}{k}}{\frac{qk^9}{2}}$  choices of the set  $S$ . So the probability that there are at least  $\frac{|U|k^{10}}{2} = \frac{qk^9}{2}$  edges between  $U$  and  $W$  is at most:

$$\binom{\frac{q^2}{k}}{\frac{qk^9}{2}} p^{\frac{qk^9}{2}} \leq \left( \frac{2eq^2}{qk^{10}} \cdot \frac{k^{10}}{n} \right)^{\frac{qk^9}{2}} = \left( \frac{2eq}{n} \right)^{\frac{qk^9}{2}}.$$

Summing over all possible choices for  $U$  and  $W$ , we conclude that the probability that  $G$  does not fulfill the statement of property 3 is at most:

$$\begin{aligned} \sum_{q=k}^{\lfloor \frac{n}{40} \rfloor} k^2 \binom{n}{q} \binom{n}{\frac{q}{k}} \left( \frac{2eq}{n} \right)^{\frac{qk^9}{2}} &\leq \sum_{q=k}^{\lfloor \frac{n}{40} \rfloor} \left( \frac{ekn}{q} \right)^{2q} \left( \frac{q}{n} \right)^{\frac{qk^9}{4}} \left( \frac{4e^2q}{n} \right)^{\frac{qk^9}{4}} \\ &\leq \sum_{q=k}^{\lfloor \frac{n}{40} \rfloor} (ek)^{2q} \left( \frac{4e^2q}{n} \right)^{\frac{qk^9}{4}}, \end{aligned}$$

where we used the fact that  $k^9 \geq 8$ . Furthermore, by assuming that  $k \geq 3$  (as the case that  $k = 2$  is trivial) and using the fact that  $(ek)^{\frac{8}{k^9}} \leq (e3)^{\frac{8}{3^9}} \leq \frac{9}{e^2}$ , we can bound

the last sum by:

$$\sum_{q=k}^{\lfloor \frac{n}{40} \rfloor} \left( \frac{(ek)^{\frac{8}{k^9}} 4e^2 q}{n} \right)^{\frac{qk^9}{4}} \leq \sum_{q=k}^{\lfloor \frac{n}{40} \rfloor} \left( \frac{36q}{n} \right)^{2q}.$$

Because  $f(q) = \left(\frac{36q}{n}\right)^{2q}$  is convex in  $q$ ,  $f(q)$  is bounded by  $\max\{f(1), f(\lfloor \frac{n}{40} \rfloor)\}$  in the interval  $[1, \lfloor \frac{n}{40} \rfloor]$ . Therefore the sum is at most:

$$\frac{n}{40} \max\left\{\left(\frac{36}{n}\right)^2, \left(\frac{36}{40}\right)^{\frac{n}{20}}\right\}.$$

It is obvious that  $\frac{n}{40} \max\left\{\left(\frac{36}{n}\right)^2, \left(\frac{36}{40}\right)^{\frac{n}{20}}\right\}$  approaches 0 as  $n$  goes to infinity. Therefore, if  $n$  is large enough, the maximum is at most  $\frac{1}{10}$ . So with probability at least 9/10,  $G$  has property (3).

Similar argument shows that, with probability at least 9/10,  $G$  has each of the properties (1), (2), (4), (5). (See [1] for detailed calculations for the proofs of similar properties.) Therefore with probability at least 1/2, a random graph  $G$  constructed above has all the properties (1)-(5). This completes the proof of Lemma 6.

Given a graph  $G$  with properties listed in Lemma 6, we now describe a method (given in [1]) which constructs a graph  $G^*$  satisfying the condition of Theorem 4, by deleting some vertices from  $G$ . Let  $C := \{v : v \text{ is a vertex contained in a cycle in } G \text{ of length at most } g - 1\}$ . Let  $Y := \{v : v \text{ has degree in } G \text{ larger than } 5k^{13}\}$ . Let  $s := \max\{|V_i \cap (C \cup Y)| : i \in [k]\}$ . First of all we remove from each  $V_i$  exactly  $s$  vertices in such a way that all vertices from  $C \cup Y$  are removed. Call the remaining sets  $W_i$ . The resulting graph  $G'$  now has girth at least  $g$ , and has maximum degree at most  $5k^{13}$ . We further modify the graph  $G'$  by the following WHILE-loop:

WHILE there is a vertex  $x \in W_i$  such that  $x$  has less than  $\frac{k^{10}}{2}$  neighbors in  $W_j$  and  $ij$  is an edge of  $F$ , then

DO the following: delete  $x$  from  $W_i$ , and remove for every  $r \neq i$  an arbitrary vertex from  $W_r$ .

Note that in the WHILE-loop, each time we search in the new graph (not in the original graph  $G$ ) for the “bad” vertex  $x$ . So a vertex  $x$  may be good at the beginning, and become bad later. However, it is proved in [1] that the WHILE-loop will eventually stop.

**Lemma 7** [1] *The WHILE-loop is executed at most  $\frac{n}{2k}$  times.*

**Proof.** Assume to the contrary that the WHILE-loop is executed more than  $\frac{n}{2k}$  times. Let  $W'_i$ ,  $i \in [k]$ , be the remaining parts of the sets  $W_i$  after  $\lceil \frac{n}{2k} \rceil$  executions of the WHILE-loop. Then  $|W'_i| \geq \frac{(k-1)n}{k}$  as  $s \leq \frac{n}{2k} - 1$ .

For each execution of the WHILE-loop, an ordered pair  $(i, j)$  is selected so that  $ij$  is an edge of  $F$  and  $W_i$  contains a vertex which has less than  $\frac{k^{10}}{2}$  neighbors in  $W'_j$ . Since there are only  $k^2$  ordered pairs, there exists an ordered pair  $(i, j)$  which was selected more than  $\lceil \frac{n}{2k^3} \rceil$  times. Therefore, there exists a set  $U \subset V_i$  of size  $\lceil \frac{n}{2k^3} \rceil$  such that all vertices from  $U$  have less than  $\frac{k^{10}}{2}$  neighbors in  $W'_j$ , and  $ij$  is an edge of  $F$ . Let  $W$  be a subset of  $W'_j$  of size  $\lceil \frac{(k-1)n}{k} \rceil$ . Then there are less than  $\frac{k^7 n}{4}$  edges between  $U$  and  $W$ , contradicting property 1.  $\blacksquare$

After the WHILE-loop is completed, we obtain a subgraph  $G^*$  of  $G(n, F)$ , whose  $k$  partite sets are  $W'_1, W'_2, \dots, W'_k$ , with  $|W'_1| = |W'_2| = \dots = |W'_k| \geq \frac{(k-1)n}{k}$ . With  $V_i$  replaced by  $W'_i$ , the graph  $G^*$  has properties (1)-(5). Moreover,  $G^*$  has girth at least  $g$ , maximum degree at most  $5k^{13}$ , and for each edge  $ij$  of  $F$ , each vertex  $x$  of  $W'_i$  has at least  $\frac{k^{10}}{2}$  neighbors in  $W'_j$ . Now we shall prove that  $G^*$  has the properties of Theorem 4.

It follows from the construction that  $G^*$  admits a surjective homomorphism to  $F$ , with  $W'_i$  mapped to  $i$ . Assume that  $H$  has at most  $k$  vertices and there exists a homomorphism  $g : G^* \rightarrow H$ . We need to prove that there exists a homomorphism  $f : F \rightarrow H$ . For each  $i \in V(F)$ , let  $v \in V(H)$  be a vertex such that  $|g^{-1}(v) \cap W'_i| \geq \frac{(k-1)n}{k^2}$ . Let  $f(i) = v$ . We claim that  $f$  is a homomorphism from  $F$  to  $H$ . Otherwise, there is an edge  $ij$  of  $F$  for which  $f(i)f(j) = uv$  is not an edge of  $H$ . This implies that in  $G^*$  there is no edge between  $g^{-1}(u) \cap W'_i$  and  $g^{-1}(v) \cap W'_j$ . However,  $g^{-1}(u) \cap W'_i$  and  $g^{-1}(v) \cap W'_j$  are subsets of  $W'_i$  and  $W'_j$  of size at least  $\frac{(k-1)n}{k^2} > \frac{n}{40k}$ . This is in contrary to property (2).

Finally, assume that  $H$  is an  $F$ -pointed graph with at most  $k$  vertices. We need to prove that for every homomorphism  $g : G \rightarrow H$  there exists a unique homomorphism  $f : F \rightarrow H$  such that  $g = f \circ c$ , where  $c$  is the homomorphism from  $G$  to  $F$  with  $c(W'_i) = i$ . Let  $f$  be the homomorphism from  $F$  to  $H$  defined as in the previous paragraph. We shall prove that  $g = f \circ c$ . Assume to the contrary that  $g \neq f \circ c$ . So for some  $i$ ,  $W'_i - g^{-1}(f(i)) \neq \emptyset$ .

First we observe that for any  $i, j$  such that  $f(i) \neq f(j)$ ,  $|W'_i \cap g^{-1}(f(j))| < \frac{n}{40k}$ . Otherwise, let  $f'$  be the mapping from  $V(F)$  to  $V(H)$  be defined as  $f'(a) = f(a)$  if  $a \neq i$ , and  $f'(i) = f(j)$ . The argument in the previous paragraph shows that  $f'$  is also a homomorphism from  $F$  to  $H$ , contrary to the assumption that  $H$  is  $F$ -pointed.

Let  $i_0$  be an index such that  $|W'_{i_0} - g^{-1}(f(i_0))| = q$  is maximum among  $|W'_i - g^{-1}(f(i))|$ . Let  $j_0$  be an index such that  $f(i_0) \neq f(j_0)$  and  $|W'_{i_0} \cap g^{-1}(f(j_0))| = q'$  is maximum among  $|W'_{i_0} \cap g^{-1}(f(j))|$  where  $f(i_0) \neq f(j)$ . Then  $q' \geq q/k$ . By the previous paragraph,  $q' < \frac{n}{40k}$ .

First we show that there is an index  $j$  such that  $j$  is adjacent to  $i_0$  in  $F$  but  $f(j)$  is not adjacent to  $f(j_0)$  in  $H$  (the vertex  $j$  could be  $j_0$  itself). For otherwise, the mapping  $f' : V(F) \rightarrow V(H)$  defined as  $f'(a) = f(a)$  for  $a \neq i_0$  and  $f'(i_0) = f(j_0)$  would be a homomorphism from  $F$  to  $H$ , contrary to the assumption that  $H$  is  $F$ -pointed.

Let  $U = W'_{i_0} \cap g^{-1}(f(j_0))$  and  $W = W'_j - g^{-1}(f(j))$ . By the choice of  $i_0$ , we have  $|W| \leq |W'_{i_0} - g^{-1}(f(i_0))| \leq k|U|$ .

By property (3), there are less than  $\frac{|U|k^{10}}{2}$  edges between  $U$  and  $W$  in  $G^*$ . However, each vertex of  $U$  is adjacent to at least  $\frac{k^{10}}{2}$  neighbors in  $W'_j$ . So there is an edge, say  $xy$ , between  $U$  and  $W'_j \cap g^{-1}(f(j))$ . However  $g(x) = f(j_0)$  and  $g(y) = f(j)$  and  $f(j_0)$  is not adjacent to  $f(j)$ , contrary to the assumption that  $g$  is a homomorphism. This completes the proof of Theorem 4.

To prove Corollary 5, assume that  $\mathcal{F} = \{f_1, f_2, \dots, f_t\}$ . Let  $F = H^t \times K_N$  where  $K_N$  is the complete graph with  $N$  vertices,  $N = \max\{k^t, m\}$ . Then  $|F| \leq k^{2mt}$ . We apply Theorem 4 for the graphs  $F$  and  $H$ . Thus there exists a graph  $G$  and a surjective homomorphism  $c : G \rightarrow F$  such that for any homomorphism  $g : G \rightarrow H$  there exists a homomorphism  $f : F \rightarrow H$  such that  $g = f \circ c$ .

Since  $N \geq |H^t| = k^t$ , there is a homomorphism  $\phi : H^t \rightarrow K_N$ . For any homomorphism  $f : F \rightarrow H$ , the restriction of  $f$  to  $H^t \times \phi(H^t)$  defines a homomorphism  $f'$  from  $H^t \rightarrow H$  as follows: for  $x \in H^t$ , let  $f'(x) = f(x, \phi(x))$ . As  $H$  is projective, we conclude that, up to an automorphism of  $H$ , all the homomorphisms  $F \rightarrow H$  are induced by the  $t$  projections  $\pi_1, \pi_2, \dots, \pi_t : H^t \rightarrow H$ . In other words, every homomorphism  $f : F \rightarrow H$  for which  $f(x, \dots, x, a) = x$  is of the form  $f(x, a) = \pi_i(x)$  for every vertex  $(x, a)$  of  $F$  and some  $i, 1 \leq i \leq t$ , here  $x \in H^t$ . Hence, up to an

automorphism of  $H$ , there are exactly  $t$  homomorphisms from  $G$  to  $H$ :  $\pi_i \circ c$ ,  $i = 1, 2, \dots, t$ . Now consider mappings  $f_1, f_2, \dots, f_t$  together with an injective mapping  $f_0 : X \rightarrow V(K_N)$ . Then the corresponding mapping  $\phi = (f_1, f_2, \dots, f_t, f_0) : X \rightarrow V(F)$  is injective. Thus we can identify  $X$  with its image  $\phi(X)$  and also replace some vertices of the graph  $G$  with elements of  $\phi(X)$ . Call the resulting graph again  $G$ . Then each of the  $t$  homomorphisms  $\pi_i \circ c$  is an extension of the mapping  $f_i$ . Clearly all homomorphisms  $f : G \rightarrow H$  coincide on  $X = \phi(X)$  with one of the maps  $f_i$ ,  $i = 1, 2, \dots, t$ .

A graph  $H$  is a *core* if  $H$  does not admit a homomorphism to any of its proper subgraphs.

**Corollary 8** *If  $H$  is a core on  $k$  vertices, then for any integer  $g$ , there is a uniquely  $H$ -colourable graph  $G$  of girth at least  $g$  and with maximum degree at most  $5k^{13}$ .*

This corollary follows easily from Theorem 4 and the fact that if  $H$  is a core then it is  $H$ -pointed.

The question that for which graph  $G$  we have  $\chi_c(G) = \chi(G)$  has been studied extensively in the literature. For all such graphs  $G$  constructed before, if  $G$  is  $\chi_c$ -critical (i.e., deleting any edge will decrease its circular chromatic number), then its maximum degree increases to infinity along with the size of  $G$ . As uniquely  $K_{k/d}$ -colourable graphs have circular chromatic number  $k/d$  [8], it follows that there are graph  $G$  of arbitrary large girth and of bounded maximum degree which have circular chromatic number  $k/d$ . As any such graph contains a  $\chi_c$ -critical subgraph, it follows that for any  $k/d$ , we can find a  $\chi_c$ -critical graph  $G$  with  $\chi_c(G) = k/d$  with maximum degree bounded by a constant which is independent of the size of  $G$ .

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