

# Bipartite density of triangle-free subcubic graphs

Xuding Zhu \*

## Abstract

A graph is subcubic if its maximum degree is at most 3. The bipartite density of a graph  $G$  is defined as  $b(G) = \max\{|E(B)|/|E(G)| : B \text{ is a bipartite subgraph of } G\}$ . It was conjectured by Bondy and Locke that if  $G$  is a triangle-free subcubic graph, then  $b(G) \geq \frac{4}{5}$  and equality holds only if  $G$  is in a list of seven small graphs. The conjecture has been confirmed recently by Xu and Yu. This note gives a shorter proof of this result.

*Keywords:* triangle-free; subcubic; bipartite density; max-cut.

*AMS 2000 Subject Classifications:* 05C35; 05C75

## 1 Introduction

The Max-Cut problem is to find a bipartite subgraph of  $G$  with maximum number of edges [1, 6]. Given a graph  $G$  and an integer  $m$ , the problem to determine if  $G$  has a bipartite subgraph  $H$  with  $m$  edges is NP-complete even when restricted to triangle-free cubic graphs [9]. A natural question is to find lower bound for the number of edges in a maximum bipartite subgraph of  $G$ . The *bipartite density*  $b(G)$  of  $G$  is defined as

$$b(G) = \max\{|E(B)|/|E(G)| : B \text{ is a bipartite subgraph of } G\}.$$

Erdős [3] proved that if  $G$  is  $2m$ -colourable then  $b(G) \geq \frac{m}{2m-1}$ . Stanton [7] and Locke [5] proved that if  $G$  is cubic and  $G \neq K_4$ , then  $b(G) \geq \frac{7}{9}$ . Hopkins and Stanton [4] proved that if  $G$  is cubic and triangle-free then  $b(G) \geq \frac{4}{5}$ . A graph is subcubic if its maximum degree is at most 3. Bondy and Locke [2] give a polynomial time algorithm that, for a given triangle-free subcubic graph  $G$ , finds a bipartite subgraph  $H$  of  $G$  with at least  $4|E(G)|/5$  edges. They further proved that the Petersen graph and the dodecahedron are the only triangle-free cubic graphs with bipartite density  $\frac{4}{5}$ .

There are triangle-free subcubic graphs with minimum degree 2 that have bipartite density  $\frac{4}{5}$ . Graphs  $F_1, F_2, F_3, F_4, F_5$  in Figure 2 are triangle-free, subcubic, and have bipartite density  $\frac{4}{5}$ . Bondy and Locke [2] conjectured that all other triangle-free subcubic graphs have bipartite density strictly larger than  $\frac{4}{5}$ .

This conjecture was confirmed by Xu and Yu [8]. Let  $F_6, F_7$  be the Petersen graph and the dodecahedron, respectively.

**Theorem 1.1** [8] *If  $G$  is a triangle-free subcubic graph with  $b(G) = \frac{4}{5}$ , then  $G \in \{F_i : 1 \leq i \leq 7\}$ .*

---

\*Department of Applied Mathematics, National Sun Yat-sen University, and National Center for Theoretical Sciences, Taiwan. Grant number: NSC95-2115-M-110-013-MY3.

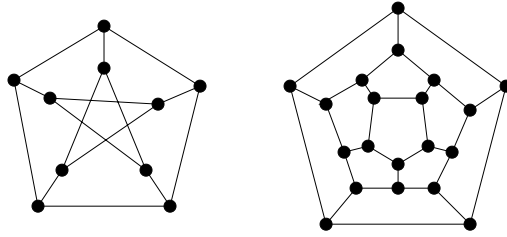


Figure 1: The Petersen graph and the dodecahedron.

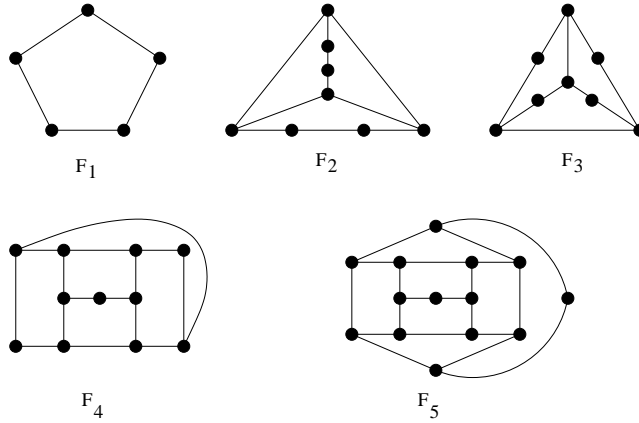


Figure 2: triangle-free subcubic graphs with bipartite density  $4/5$ .

This note gives a shorter proof of Theorem 1.1, however the result in [2] on cubic graphs is used in our proof.

A degree  $i$  vertex of  $G$  is called an  $i$ -vertex of  $G$ . Let  $n_i$  be the number of  $i$ -vertices of  $G$ . For a subcubic graph  $G$ , let  $\sigma(G) = 7n_3/10 + 4n_2/5 + n_1 + n_0$ . Let  $\alpha_2(G)$  be the maximum number of vertices in an induced bipartite subgraph of  $G$ . The *heart* of a graph  $G$  is obtained from  $G$  by repeatedly deleting 1-vertices and isolated vertices. In this paper, we prove the following result.

**Theorem 1.2** *If  $G$  is a triangle-free subcubic graph, then  $\alpha_2(G) \geq \sigma(G)$  and equality holds if and only if the heart of  $G$  belong to the set  $\{F_i : 1 \leq i \leq 7\}$ .*

To see that Theorem 1.1 follows from Theorem 1.2, it suffices to observe the following: Let  $H$  be an induced bipartite subgraph of  $G$  with partite sets  $A, B$ . Partition  $V(G - H)$  into  $A', B'$  in such a way that the total number of edges in  $G[A \cup A']$  and  $G[B \cup B']$  is minimum. Then each vertex in  $A'$  has at most one neighbour in  $A \cup A'$  and each vertex in  $B'$  has at most one neighbour in  $B \cup B'$ . Therefore the bipartite subgraph  $Q$  of  $G$  with partite sets  $A \cup A'$  and  $B \cup B'$  has  $|E(Q)| \geq |E(G)| - |A' \cup B'| \geq |E(G)| - (3n_3 + 2n_2)/10 \geq 4|E(G)|/5$ , with equality holds only if  $G = F_i$  for some  $1 \leq i \leq 7$ .

## 2 Proof of Theorem 1.2

Assume Theorem 1.2 is not true and  $G$  is a minimum counterexample. I.e.,  $G$  is a minimum triangle-free subcubic graph such that either  $\alpha_2(G) < \sigma(G)$  or  $\alpha_2(G) = \sigma(G)$  and  $G \notin \{F_i : 1 \leq i \leq 7\}$ . It is obvious that  $G$  has minimum degree at least 2.

**Lemma 2.1** *Suppose  $G$  is a subcubic graph with minimum degree at least 2,  $X$  is a proper subset of  $V(G)$ . Let  $n'_i$  be the number of  $i$ -vertices of  $G$  contained in  $X$ , and  $m$  be the number of edges with exactly one end vertex in  $X$ . If  $G'$  is obtained from  $G$  by deleting  $X$  and adding  $m'$  edges and  $d_{G'}(v) \leq d_G(v)$  for each vertex  $v$ , then  $\sigma(G) - \sigma(G') \leq (7n'_3 + 8n'_2 + 2m' - m)/10$ .*

*Proof.* Let  $\phi(0) = \phi(1) = 1, \phi(2) = 8/10, \phi(3) = 7/10$ . By definition,

$$\begin{aligned}\sigma(G) &= \sum_{x \in V(G-X)} \phi(d_G(x)) + \sum_{x \in X} \phi(d_G(x)) \\ \sigma(G') &= \sum_{x \in V(G-X)} \phi(d_{G'}(x)). \\ \sigma(G) - \sigma(G') &= \sum_{x \in V(G-X)} (\phi(d_G(x)) - \phi(d_{G'}(x))) + (7n'_3 + 8n'_2)/10.\end{aligned}$$

By Assumption, for each vertex  $x \in V(G - X)$ ,  $d_G(x) \geq d_{G'}(x)$ . As  $d_G(x) \geq 2$ , it follows from the definition that

$$\phi(d_G(x)) - \phi(d_{G'}(x)) \leq (d_{G'}(x) - d_G(x))/10.$$

As

$$\sum_{x \in V(G-X)} (d_{G'}(x) - d_G(x)) = 2m' - m,$$

we have

$$\sum_{x \in V(G-X)} (\phi(d_G(x)) - \phi(d_{G'}(x))) \leq (2m' - m)/10.$$

Therefore  $\sigma(G) - \sigma(G') \leq (7n'_3 + 8n'_2 + 2m' - m)/10$ .  $\blacksquare$

For a graph  $Q$ , denote by  $\mathcal{B}(Q)$  the family of all maximum induced bipartite subgraphs of  $Q$ . For an induced subgraph  $H$  of  $G$ , and for  $X \subseteq V(G)$ , we shall denote by  $H + X$  (respectively,  $H - X$ ) the subgraph of  $G$  induced by  $V(H) \cup X$  (respectively,  $V(H) - X$ ). We denote by  $G + uv$  the graph obtained from  $G$  by adding the edge  $uv$ .

**Observation 2.2** *Each  $H \in \mathcal{B}(F_1)$  equals  $F_1 - v$  for some vertex  $v$ . For any two non-adjacent 2-vertices  $x, y$ ,  $F_2 - \{x, y\} \in \mathcal{B}(F_2)$ . For any two 2-vertices  $x, y$  of distance 3,  $F_3 - \{x, y\} \in \mathcal{B}(F_3)$ . For  $i = 4, 5$ , there is an  $H \in \mathcal{B}(F_i)$  such that  $H$  contains no 2-vertices.*  $\blacksquare$

**Lemma 2.3** *The graph  $G$  is 2-connected.*

*Proof.* Otherwise  $G$  has a cut-edge  $uv$ . Let  $G_1, G_2$  be two components of  $G - uv$ . Let  $H_i \in \mathcal{B}(G_i)$  for  $i = 1, 2$ . Then  $H_1 \cup H_2 \in \mathcal{B}(G)$ . Hence  $\alpha_2(G) \geq \alpha_2(G_1) + \alpha_2(G_2) \geq \sigma(G_1) + \sigma(G_2) > \sigma(G)$ .  $\blacksquare$

**Lemma 2.4** *No 3-vertex is adjacent to two 2-vertices.*

*Proof.* Assume a 3-vertex  $x$  has two 2-vertices  $a, b$  as its neighbours. Let  $a', b', x'$  be the other neighbour of  $a, b, x$ , respectively. Let  $G' = G - \{x, a, b\}$ .

Then  $\sigma(G) - \sigma(G') \leq (7 + 16 - 3)/10 = 2$ , and equality holds only if none of  $a', b', x'$  is a 2-vertex in  $G$ , which implies that the heart of  $G'$  is itself. For any  $H \in \mathcal{B}(G')$ ,  $H + \{a, b\}$  is a bipartite subgraph of  $G$ . So  $\alpha_2(G) \geq \alpha_2(G') + 2 \geq \sigma(G)$ . If equality holds, then  $\alpha_2(G') = \sigma(G')$  and hence  $G' = F_i$  for some  $i \in \{1, 2, 3, 4\}$  (observe that  $G'$  has at least three 2-vertices). If  $G' = F_i$  for

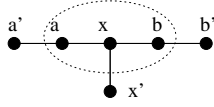


Figure 3: A 3-vertex adjacent to two 2-vertices.

some  $i \in \{2, 3, 4\}$ , then by Observation 2.2, there is an  $H \in \mathcal{B}(G')$  such that  $H$  contains at most one of the vertices  $a', b', x'$ . Then  $H + \{a, x, b\} \in \mathcal{B}(G)$ , implying that  $\alpha_2(G) > \sigma(G)$ . If  $G' = F_1$  then  $x'$  is adjacent to at least one of  $a', b'$ , for otherwise  $G = F_3$ . Assume  $x'b'$  is an edge, then  $H = G' - \{a'\} \in \mathcal{B}(G')$ , and  $H + \{a, x, b\} \in \mathcal{B}(G)$ , implying that  $\alpha_2(G) > \sigma(G)$ . ■

**Lemma 2.5** *No 4-cycle contains a 2-vertex.*

*Proof.* Assume  $(a, b, c, d)$  is a 4-cycle and  $a$  is a 2-vertex. By Lemma 2.4,  $c$  is a 3-vertex. Let  $G' = G - \{a, b, c, d\}$ . Then  $\sigma(G) - \sigma(G') \leq (14 + 16 - 2)/10 < 3$  (as  $G$  is 2-connected, at least two of  $b, c, d$  are 3-vertices). If one of  $b, d$  is a 2-vertex, say  $b$  is a 2-vertex, then for  $H \in \mathcal{B}(G')$ ,  $H + \{a, b, c\}$  is a bipartite subgraph of  $G$ . Hence  $\alpha_2(G) > \sigma(G)$ . Assume  $b, d$  are 3-vertices, and  $b', c', d'$  are the other neighbour of  $b, c, d$ , respectively. Suppose  $H \in \mathcal{B}(G')$ . If one of  $b', c', d' \notin V(H)$ , say  $b' \notin V(H)$ , then  $H + \{a, b, c\}$  is a bipartite subgraph of  $G$ . Assume  $b', c', d' \in V(H)$ . If  $b', d'$  are in the same partite set of  $H$ , then  $H + \{d, a, b\}$  is a bipartite subgraph of  $G$ . Otherwise without loss of generality, we may assume  $c'$  and  $d'$  are in different partite sets. Then  $H + \{a, c, d\}$  is a bipartite subgraph of  $G$ . In any case  $\alpha_2(G) \geq \alpha_2(G') + 3$ , hence  $\alpha_2(G) > \sigma(G)$ . ■

**Lemma 2.6** *No two 2-vertices of  $G$  are adjacent.*

*Proof.* Assume to the contrary that  $x, y$  are two adjacent 2-vertices of  $G$ . Let  $a, b$  be the other neighbour of  $x, y$ , respectively. Let  $G' = (G - \{x, y\}) + ab$ . Then  $\sigma(G) - \sigma(G') = 16/10 < 2$  and for any  $H \in \mathcal{B}(G')$ ,  $H + \{x, y\}$  is a bipartite subgraph of  $G$ . If  $G'$  is not cubic, then by induction hypothesis,  $\alpha_2(G') \geq \sigma(G')$ . If  $G'$  is cubic, then by a result in [2],  $G'$  has bipartite density at least  $4/5$ , implying that  $\alpha_2(G') \geq \sigma(G')$ . Hence  $\alpha_2(G) \geq \alpha(G') + 2 > \sigma(G)$ . ■

In Lemmas 2.7, 2.8, 2.9, we assume  $x$  is a 2-vertex and  $u, v$  are the neighbours of  $x$ . By Lemma 2.6, both  $u$  and  $v$  are 3-vertices. Assume  $N(u) = \{x, a, b\}$ . By Lemma 2.5,  $N(u) \cap N(v) = \{x\}$ . Assume  $N(v) = \{x, c, d\}$ .

**Lemma 2.7** *Vertices  $a, v$  have a common neighbour and vertices  $b, v$  have a common neighbour.*

*Proof.* Assume  $b, v$  have no common neighbour. Let  $a', a''$  be the other two neighbours of  $a$  (by Lemma 2.4,  $a$  is a 3-vertex). Let  $G' = (G - \{a, u, x\}) + bv$ .

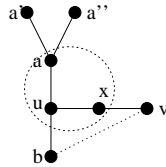


Figure 4: Vertices  $x, u, v$  and some of their neighbours

Then  $\sigma(G) - \sigma(G') \leq (14 + 8 - 2)/10 = 2$  and equality holds only if none of  $a', a''$  is a 2-vertex in  $G$ , which implies that the heart of  $G'$  is itself. For  $H \in \mathcal{B}(G')$ ,  $H + \{u, x\}$  is a bipartite subgraph

of  $G$ . So  $\alpha_2(G) \geq \alpha_2(G') + 2 \geq \sigma(G') + 2 \geq \sigma(G)$ . If equality holds, then  $\alpha_2(G') = \sigma(G')$ . Since  $G'$  is not a counterexample to Theorem 1.2, and  $G'$  has at least two non-adjacent 2-vertices, namely  $a', a''$ , at least two adjacent 3-vertices, namely  $b, v$ , we conclude that  $G' = F_i$  for some  $i = 2, 3, 4, 5$ . By Observation 2.2, if  $i \neq 3$  or  $a', a''$  have no common neighbour in  $G'$ , then there is an  $H \in \mathcal{B}(G')$  such that  $a', a'' \notin V(H)$ . Then  $H + \{a, u, x\} \in \mathcal{B}(G)$ , implying that  $\alpha_2(G) > \sigma(G)$ . In case  $G' = F_3$  and  $a', a''$  have a common neighbour in  $G'$ , it is easy to find an  $H \in \mathcal{B}(G')$  in which  $a', a''$  are in the same partite set and  $b$  is in the other partite set. Again,  $H + \{a, u, x\} \in \mathcal{B}(G)$ , implying that  $\alpha_2(G) > \sigma(G)$ .  $\blacksquare$

Assume  $c$  is a common neighbour of  $a, v$ .

**Lemma 2.8** *Vertices  $b$  and  $c$  are not adjacent.*

*Proof.* Assume  $bc$  is an edge. If none of  $ad, bd$  is an edge, then let  $G' = (G - \{a, c, u, x, v, b\})$ .

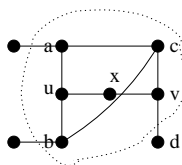


Figure 5:  $v$  is a 3-vertex and none of  $ad, bd$  is an edge.

Then  $\sigma(G) - \sigma(G') \leq (35 + 8 - 3)/10 = 4$  and equality holds only if none of  $a', b', d$  is a 2-vertex in  $G$  (note that by Lemma 2.4,  $a, b$  are 3-vertices). For  $H \in \mathcal{B}(G')$ ,  $H + \{a, c, u, x\}$  is a bipartite subgraph of  $G$ . Hence  $\alpha_2(G) \geq \alpha_2(G') + 4 \geq \sigma(G') + 4 \geq \sigma(G)$ . If equality holds, then  $G' = F_i$  for some  $1 \leq i \leq 4$  (as  $G'$  has at least three 2-vertices  $d$  and  $a', b'$ , which are the other neighbours of  $a, b$ , respectively). By Observation 2.2, there is an  $H \in \mathcal{B}(G')$  that does not contain  $b'$ . So  $H + \{a, c, u, x, b\}$  is a bipartite subgraph, and hence  $\alpha_2(G) > \sigma(G)$ .

If  $ad$  is an edge, then let  $G' = (G - \{a, c, u, x, v, b, d\})$ .

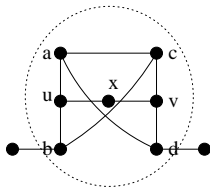


Figure 6:  $ad$  is an edge.

Then  $\sigma(G) - \sigma(G') \leq (42 + 8 - 2)/10 = 24/5$ . For  $H \in \mathcal{B}(G')$ ,  $H + \{a, c, u, x, b\}$  is a bipartite subgraph of  $G$ . Hence  $\alpha_2(G) \geq \alpha_2(G') + 4 \geq \sigma(G') + 4 > \sigma(G)$ .  $\blacksquare$

Since  $bc$  is not an edge, and  $b, v$  have a common neighbour, we conclude that  $ac, bd$  are edges of  $G$ , and  $ad, bc$  are not edges of  $G$ . The graph induced by  $\{a, b, u, x, v, c, d\}$  is depicted in Figure 7 (i).

Let  $a', b', c', d'$  be the other neighbour of  $a, b, c, d$ , respectively.

**Lemma 2.9** *Vertices  $a', b'$  are distinct and adjacent, and vertices  $c', d'$  are distinct and adjacent.*

*Proof.* If  $a' = b'$  or  $a' \neq b'$  and  $a'b'$  is not an edge, then let  $G' = (G - \{u, v, x, b, c, d\}) + ab'$ . Then  $\sigma(G) - \sigma(G') \leq (35 + 8 - 3)/10 = 4$  and equality holds only if none of  $c', d'$  is a 2-vertex. For

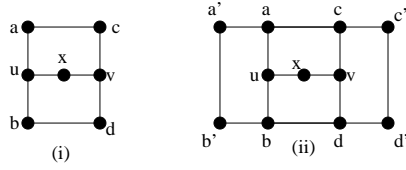


Figure 7: The graphs induced by  $\{a, b, u, x, v, c, d\}$  and by  $\{a, b, u, x, v, c, d, a', b', c', d'\}$ .

$H \in \mathcal{B}(G')$ ,  $H + \{u, x, v, b\}$  is a bipartite subgraph of  $G$ . So  $\alpha_2(G) \geq \alpha_2(G') + 4 \geq \sigma(G') + 4 \geq \sigma(G)$ . If equality holds, then  $\alpha_2(G') = \sigma(G')$  and  $G' \in \{F_i : i = 2, 3, 4\}$ . By Observation 2.2, there is an  $H \in \mathcal{B}(G')$  such that  $a, c' \notin V(H)$ . Then  $H + \{c, v, d, x, u\}$  is a bipartite subgraph of  $G$ , showing that  $\alpha_2(G) > \sigma(G)$ . ■

Let  $X = \{a', b', c', d', a, b, c, d, u, x, v\}$ . The subgraph of  $G$  induced by  $X$  is depicted in Figure 7 (ii). Observe that this is a subgraph of  $F_4$  and  $F_5$ .

**Lemma 2.10** *There is no 2-vertex.*

*Proof.* Assume  $x$  is a 2-vertex of  $G$ . Then  $G$  contains the graph depicted in Figure 7 (ii) as a subgraph. If  $a'd'$  is an edge, then since  $G \neq F_4$  and  $\alpha_2(G) \leq \sigma(G)$ ,  $G - X$  is nonempty. Let  $G' = G - X$ . Then  $\sigma(G) - \sigma(G') \leq (70 + 8 - 2)/10 = 38/5$ . For  $H \in \mathcal{B}(G')$ ,  $H + \{a', a, c, v, x, b, d, d'\}$  is a bipartite subgraph of  $G$ . Hence  $\alpha_2(G) > \sigma(G)$ .

If  $a', c'$  have no common neighbour, then let  $G' = (G - \{a, c, u, x, v, b, d\}) + a'c'$ . Then  $\sigma(G) - \sigma(G') \leq (42 + 8 - 2)/10 = 24/5$ . For  $H \in \mathcal{B}(G')$ ,  $H + \{a, c, x, v, b\}$  is a bipartite subgraph of  $G$ . Hence  $\alpha_2(G) > \sigma(G)$ .

Thus we assume that  $a', c'$  have a common neighbour  $s$ . Similarly,  $b', d'$  have a common neighbour  $t$ . If  $s, t$  have a common neighbour  $w$ , then since  $G \neq F_5$ , we conclude that  $G$  has a cut edge  $ww'$ . But then  $\alpha_2(G - ww') = \alpha_2(G) \geq \sigma(G - ww') > \sigma(G)$ . So  $s, t$  have no common neighbour. If  $G$  has no other vertex, then  $\alpha_2(G) = 10 > \sigma(G) = 46/5$ . Otherwise, let  $G' = G - (X \cup \{s, t\})$ . Then  $\sigma(G) - \sigma(G') \leq (84 + 8 - 2)/10 = 9$ . For  $H \in \mathcal{B}(G')$ ,  $H + \{s, c', c, a, v, x, d, b, b', t\}$  is a bipartite subgraph of  $G$ . So  $\alpha_2(G) \geq \alpha_2(G') + 9 \geq \sigma(G') + 9 \geq \sigma(G)$ . If equality holds, then  $\alpha_2(G') = \sigma(G')$  and hence  $G' \in \{F_i : i = 1, 2, 3, 4, 5\}$ . By Observation 2.2, there is an  $H \in \mathcal{B}(G')$  such that  $s' \notin V(H)$ , where  $s'$  is the other neighbour of  $s$  in  $G$ . Then  $H + \{s, c', c, a, v, x, d, b, b', t\}$  is a bipartite subgraph of  $G$ , showing that  $\alpha_2(G) > \sigma(G)$ . ■

By Lemma 2.10,  $G$  is a cubic graph. It is proved in [2] that if  $G$  is cubic and triangle-free, then  $b(G) \geq 4/5$  and equality holds if and only if  $G = F_6, F_7$ . Therefore  $G$  has an induced bipartite graph with at least  $\sigma(G) = 7|V(G)|/10$  vertices, and equality holds if and only if  $G = F_6, F_7$ . This completes the proof of Theorem 1.2.

## References

- [1] B. Bollobás and A. D. Scott, Problems and results on judicious partitions, *Random Structure and Algorithm*, **21**(2002)414-430.
- [2] J. A. Bondy and S. C. Locke, Largest bipartite subgraphs in triangle-free graphs with maximum degree three, *J. Graph Theory* **10** (1986) 477-504.
- [3] P. Erdős, Problems and results in graph theory and combinatorial analysis, *Graph Theory and Related Topics* (Proceedings of Waterloo Conference, dedicated to W. T. Tutte on his sixties birthday, Waterloo, 1977), Academic, New York (1979) 153-163.

- [4] G. Hopkins and W. Staton, Extremal bipartite subgraphs of cubic triangle-free graphs, *J. Graph Theory* **6** (1982) 115–121.
- [5] S. C. Locke, Maximum  $k$ -colourable subgraphs, *J. Graph Theory* **6** (1982) 123–132.
- [6] G. I. Orlova and Y. G. Dorfman, Finding the maximal cut in a graph, *Eng. Cyber.* **10** (1972) 502–506.
- [7] W. Stanton, Edge deletions and the chromatic number, *Ars Combinatoria* **10** (1980) 103–106.
- [8] B. Xu and X. Yu, Triangle-free subcubic graphs with minimum bipartite density, *to appear in Journal of Combinatorial Theory Ser. B*, 98(3) (2008).
- [9] M. Yannakakis, Node- and Edge-Deletion NP-Complete Problems, *STOC* (1978)253-264.