

Chromatic Ramsey Numbers

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Abstract

Suppose G is a graph. The chromatic Ramsey number $r_c(G)$ of G is the least integer m such that there exists a graph F of chromatic number m for which the following is true: For any 2-colouring of the edges of F there is a monochromatic subgraph isomorphic to G . Let $M_n = \min\{r_c(G) : \chi(G) = n\}$. It was conjectured by S. A. Burr, P. Erdős and L. Lovász that $M_n = (n-1)^2 + 1$. This conjecture has been confirmed previously for $n \leq 4$. In this paper, we shall prove that the conjecture is true for $n = 5$. We shall also improve the upper bounds for M_6 and M_7 .

1 Introduction

Suppose F, G, H are finite graphs. We use $F \rightarrow (G, H)$ to mean the following statement:

For every colouring of the edges of F with red and blue, either the red subgraph of F contains a copy of G or the blue subgraph contains a copy of H .

A natural question is to characterize those F for which $F \rightarrow (G, H)$ for given graphs G and H . This question is in general extremely difficult. Simpler problems are to describe properties of such graphs. One such problem, discussed by S. A. Burr, P. Erdős and L. Lovász [1] is to find the smallest possible chromatic number of such a graph.

Let \mathcal{G}, \mathcal{H} be sets of graphs. We write $F \rightarrow (\mathcal{G}, \mathcal{H})$ if for every colouring of the edges of F with red and blue, there is either a red subgraph of F

isomorphic to a member of \mathcal{G} , or there is a blue subgraph isomorphic to a member of \mathcal{H} . We define the chromatic Ramsey number $r_c(\mathcal{G}, \mathcal{H})$ of \mathcal{G}, \mathcal{H} to be the least integer m such that there exists a graph F with $\chi(F) = m$ and $F \rightarrow (\mathcal{G}, \mathcal{H})$. The Ramsey number $r(\mathcal{G}, \mathcal{H})$ is the least integer n such that $K_n \rightarrow (\mathcal{G}, \mathcal{H})$. We shall write $r_c(\mathcal{G})$ for $r_c(\mathcal{G}, \mathcal{G})$ and $r(\mathcal{G})$ for $r(\mathcal{G}, \mathcal{G})$. In case $\mathcal{G} = \{G\}$ or $\mathcal{H} = \{H\}$, we write G or H as an argument. Usually, $r(K_n, K_m)$ is written as $r(n, m)$, $r(K_n, K_n)$ is written as $r(n)$.

Let \mathcal{K}_n be the class of all n -chromatic graphs. We shall be interested in the number

$$M_n = \min\{r_c(G) : G \in \mathcal{K}_n\}.$$

This number was first studied by S. A. Burr, P. Erdős and L. Lovász [1]. They proposed the following conjecture:

Conjecture 1 *For any integer n ,*

$$M_n = (n - 1)^2 + 1.$$

It was proved in [1] that Conjecture 1 is true for $n \leq 4$. We shall prove in this paper that the conjecture is true for $n = 5$. We shall give better upper bounds for M_6 and M_7 .

2 $M_5 = 17$

Suppose G, H are graphs. We call H a homomorphic image of G if there is a homomorphism (an edge preserving vertex mapping) from G to H . For any graph G , we denote by $\text{hom}(G)$ the set of all homomorphic images of G . If \mathcal{G} is a class of graphs, then $\text{hom}(\mathcal{G}) = \cup_{G \in \mathcal{G}} \text{hom}(G)$. We shall need the following result proved in [1]:

Lemma 1 *Suppose \mathcal{G}, \mathcal{H} are classes of finite graphs. Then*

$$r_c(\mathcal{G}, \mathcal{H}) = r(\text{hom}(\mathcal{G}), \text{hom}(\mathcal{H})).$$

It is easy to see that $\text{hom}(\mathcal{K}_n) = \mathcal{K}_n$. Therefore for any integers n, s we have $r_c(\mathcal{K}_n, \mathcal{K}_s) = r(\mathcal{K}_n, \mathcal{K}_s)$. It is easy to see ([1]) that $r(\mathcal{K}_n, \mathcal{K}_s) = (n - 1)(s - 1) + 1$. Therefore for any graph G of chromatic number n , $r_c(G) \geq r_c(\mathcal{K}_n) = (n - 1)^2 + 1$. Hence

$$M_n = \min\{r_c(G) : G \in \mathcal{K}_n\} \geq (n - 1)^2 + 1.$$

Conjecture 1 above asserts that the equality holds for all n . As observed in [1], this conjecture is weaker than the following conjecture of Hedetniemi [6].

For graphs G, H , the categorical product $G \times H$ of G and H has vertex set $V(G) \times V(H)$ and (a, b) is adjacent to (c, d) if and only if a is adjacent to c and b is adjacent to d . Hedetniemi's made the following conjecture about thirty years ago:

Conjecture 2 *For any integer n , if G, H are graphs of chromatic number at least n , then their categorical product $G \times H$ also has chromatic number at least n .*

To see that Conjecture 1 follows from Conjecture 2, we assume that Conjecture 2 is true for integer n . It is easy to see that for each 2-colouring c of the edges of $K_{(n-1)^2+1}$, there is a monochromatic subgraph of chromatic number at least n . Let G_c be a monochromatic subgraph (with respect to a 2-edge colouring c of $K_{(n-1)^2+1}$) of K_n of chromatic number n . Let c_1, c_2, \dots, c_m be all the 2 edge colourings of $K_{(n-1)^2+1}$. Let $G = G_{c_1} \times G_{c_2} \times \dots \times G_{c_m}$. Then the "projection" mappings are homomorphisms of G to the factors G_{c_i} . Therefore $\text{hom}(G) \supset \{G_{c_i} : i = 1, 2, \dots, m\}$. Hence $r_c(G) = r(\text{hom}(G)) \leq r(\{G_{c_i} : i = 1, 2, \dots, m\}) \leq (n-1)^2 + 1$. On the other hand, it follows from Hedetniemi's conjecture that the product G has chromatic number n . Therefore $M_n \leq (n-1)^2 + 1$, and hence $M_n = (n-1)^2 + 1$.

The above argument shows that if Conjecture 1 is true for an integer n , then Conjecture 2 is also true for n . Burr, Erdős and Lovász [1] confirmed Conjecture 1 for $n = 4$. Later, El-Zahar and Sauer [3] proved, with a genius argument, that Conjecture 2 is also true for $n = 4$. Conjecture 2 has attracted considerable attention [2, 3, 5, 8, 9, 12, 13]. However both Conjectures 1 and 2 remained open in general for $n \geq 5$. We shall prove that Conjecture 1 is true for $n = 5$.

Theorem 1 *There is a graph G of chromatic number 5 such that $r_c(G) = 17$. Therefore $M_5 = 17$.*

In order to construct such a graph G , we first prove two lemmas:

Lemma 2 *Suppose c is a colouring of the edges of K_{17} with two colours blue and red. If there is a monochromatic copy of K_4 , then there is a connected monochromatic subgraph A of K_{17} of chromatic number 5 such that A contains a K_4 .*

Proof. Suppose c is a colouring of the edges of K_{17} with two colours blue and red and that there is monochromatic copy of K_4 . Let R, B be the red subgraph and blue subgraph of K_{17} respectively. Without loss of generality,

suppose that there is a red copy of K_4 . If R is 4-colourable, then given a 4-colouring of R , one of the colour class contains five vertices. These five vertices induces a blue K_5 , and we may take A to be this copy of K_5 . Thus we assume that R has chromatic number at least 5. For the same reason, we may also assume that the blue subgraph B has chromatic number at least 5.

Let X be a maximal connected subgraph of R of chromatic number at least 5. If X contains a copy of K_4 then we take $A = X$ and we are done. Assume now that X contains no K_4 . Since R contains a copy of K_4 , we conclude that $V(X) \neq V(K_{17})$. By the maximality of X , we know that all the edges between $V(X)$ and $V(K_{17}) - V(X)$ are coloured blue. Therefore the blue graph B is connected. If B contains a copy of K_4 , then since $\chi(B) \geq 5$ we may take $A = B$ and we are done. Thus we may assume that B contains no K_4 .

If $|V(X)| \geq 9$, then since $r(K_4, K_3) = 9$ and since X contains no K_4 , we conclude that the blue subgraph on X , i.e., the subgraph of B induced by X , contains a copy of K_3 . Then the union of this copy of K_3 and a vertex in $V(K_{17}) - V(X)$ induces a blue copy of K_4 , contrary to our assumption. Therefore we must have $|V(X)| \leq 8$.

Since X has chromatic number 5 and contains no K_4 , we conclude that X is not complete, and hence there are two vertices $a, b \in V(X)$ such that the edge ab is coloured blue. Since $|V(K_{17}) - V(X)| \geq 9$, it follows that either there are two vertices $c, d \in V(K_{17}) - V(X)$ such that the edge cd is coloured blue, or we have red copy of K_9 . In the former case $\{a, b, c, d\}$ induces a blue copy of K_4 , contrary to our assumption, and in the latter case, we may take A to be the red K_9 . This completes the proof of Lemma 2. \blacksquare

Since $r(K_4, K_4) = 18$, there is a colouring of the edges of K_{17} such that there is no monochromatic K_4 . Actually, there is a unique such edge colouring [4]. Our next lemma explores the properties of the monochromatic subgraph of such an edge colouring of K_{17} .

For a graph G , the n -colouring graph K_n^G has vertices all the mappings $f : V(G) \rightarrow \{1, 2, \dots, n\}$, and two such mappings f, g are adjacent in K_n^G if and only if for every edge xy of G , we have $f(x) \neq g(y)$.

Lemma 3 *Suppose the edges of K_{17} are coloured by red and blue, and that there is no monochromatic copy of K_4 . Let R be the red subgraph of K_{17} . Then the 4-colouring graph K_4^R of R is 4-colourable.*

Proof. It is well-known that there is a unique edge colouring of K_{17} with two colours such that the resulting coloured graph contains no monochromatic K_4 . In such an edge colouring, the red graph R , as well as the blue graph B , is isomorphic to the graph on Z_{17} in which ij is an edge if and only if $|i - j|$ is one of the numbers 1, 2, 4, 8, 9, 13, 15, 16, [4]. For each vertex f of

the 4-colouring graph K_4^R , i.e., f is a mapping of $V(R)$ to $\{1, 2, 3, 4\}$, let i be any number in $\{1, 2, 3, 4\}$ such that $|f^{-1}(i)| \geq 5$ (because $|V(R)| = 17$, such an integer exists). We set $\Delta(f) = i$, and we shall show that Δ is a proper colouring of K_4^R .

Assume to the contrary that Δ is not a proper colouring of K_4^R . Then there are two adjacent vertices f, g of K_4^R such that $\Delta(f) = \Delta(g) = i$ for some $i \in \{1, 2, 3, 4\}$. By the definition of Δ , there exist vertices x_1, x_2, x_3, x_4, x_5 such that $f(x_1) = f(x_2) = f(x_3) = f(x_4) = f(x_5) = i$, and there exist vertices y_1, y_2, y_3, y_4, y_5 such that $g(y_1) = g(y_2) = g(y_3) = g(y_4) = g(y_5) = i$.

Since f, g are adjacent in K_4^R , we conclude that $x_i y_j \notin E(R)$ for all $i, j \in \{1, 2, \dots, 5\}$. In other words, if $x_i \neq y_j$ then the edge $x_i y_j$ is coloured blue.

Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ and let $Y = \{y_1, y_2, y_3, y_4, y_5\}$. If $|X \cap Y| \leq 1$, then we may assume that $x_i \neq y_j$ for $i, j \in \{1, 2, 3, 4\}$. Since there is no monochromatic K_4 , we conclude that there are indices $i, j \in \{1, 2, 3, 4\}$ such that the edge $x_i x_j$ is coloured blue. Similarly there are $u, v \in \{1, 2, 3, 4\}$ such that the edge $y_u y_v$ is coloured blue. Then $\{x_i, x_j, y_u, y_v\}$ induces a blue K_4 , contrary to our assumption. If $2 \leq |X \cap Y| \leq 4$, then we may assume that $x_1 = y_1, x_2 = y_2$ and $x_5 \neq y_5$. Then $\{x_1, x_2, x_5, y_5\}$ induces a blue K_4 , contrary to our assumption. If $X = Y$ then $\{x_1, x_2, x_3, x_4\}$ induces a blue K_4 , contrary to our assumption. Therefore Δ is indeed a proper 4-colouring of K_4^R . This completes the proof of Lemma 3. \blacksquare

With these two lemmas, we can construct the graph G of Theorem 1 as follows:

Let α_0 be the unique 2-colouring of the edges of K_{17} such that the resulting coloured graph contains no monochromatic K_4 . Let G_0 be the red subgraph.

Let $\alpha_1, \alpha_2, \dots, \alpha_m$ be all the 2-colourings of the edges of K_{17} such that the resulting coloured graph contains a monochromatic K_4 . For each $1 \leq i \leq m$, let G_i be a connected monochromatic subgraph (with respect to the colouring α_i) such that $\chi(G_i) \geq 5$ and G_i contains a copy of K_4 . (By Lemma 2, such graphs G_i exist.) Finally let $G = G_0 \times G_1 \times G_2 \times \dots \times G_m$.

We shall show that $r_c(G) \leq 17$. It suffices to show that $r(\text{hom}(G)) \leq 17$. Since it is obvious that $G_i \in \text{hom}(G)$ for $i = 0, 1, \dots, m$, it suffices to show that $r(\{G_0, G_1, \dots, G_m\}) \leq 17$. However this follows trivially from the definition of G_i (i.e., for each 2-colouring of the edges of K_{17} there is a monochromatic subgraph isomorphic to one of the graphs G_i).

To complete the proof of Theorem 1, it suffices to show that G has chromatic number at least 5. This follows easily from the following two lemmas, which are proved in [2] and [3], respectively:

Lemma 4 *Suppose A, B are connected graphs of chromatic number at least n and each of A, B contains a K_{n-1} . Then $A \times B$ has chromatic number at least n .*

Lemma 5 *Suppose A, B are graphs of chromatic number at least n . If the $(n-1)$ -colouring graph K_{n-1}^A of A is $(n-1)$ -colourable, then the product $A \times B$ has chromatic number at least n .*

Indeed, by inductively applying Lemma 4, we conclude that $G_1 \times \cdots \times G_m$ has chromatic number at least 5. Since G_0 has chromatic number 5 and $K_4^{G_0}$ is 4-colourable, by Lemma 5, $G_0 \times G_1 \times \cdots \times G_m$ has chromatic number 5. This completes the proof of Theorem 1.

3 Better upper bounds for M_6 and M_7

It follows from the definition that $M_n \leq r(K_n, K_n)$. Therefore any upper bound for $r(K_n, K_n)$ is an upper bound for M_n . However this upper bound is much too big. In this section, we shall give better upper bounds for M_6 and M_7 .

Theorem 2 *There is a graph G of chromatic number 6 such that $r_c(G) \leq 41$. Therefore $26 \leq M_6 \leq 41$.*

We shall first prove the following lemma:

Lemma 6 *For any 2-colouring of the edges of K_{41} , either there is a monochromatic connected subgraph A which has chromatic number 6 and which contains a K_5 , or there is a monochromatic subgraph A' which has chromatic number 6 and $K_5^{A'}$ is 5-colourable.*

Proof. Let c be a 2-colouring of the edges of K_{41} by red and blue, and let R, B be the red and blue subgraphs of K_{41} respectively. If R is 5-colourable, then for a 5-colouring of R , there is a colour class of order at least 6. Therefore B contains a K_6 , and we are done. Thus we may assume that $\chi(R) \geq 6$. Similarly, we may assume that $\chi(B) \geq 6$. We now consider two cases.

Case 1. There is a monochromatic K_5 .

Without loss of generality, we may assume that R contains a K_5 . Let H be a maximal connected subgraph of R which has chromatic number at least 6. If H contains a K_5 , then we may choose $A = H$ and we are done. If H does not contain a K_5 , then $|V(H)| \leq 36$ and all the edges between

$V(H)$ and $V(K_{41}) - V(H)$ are coloured blue. Therefore B is connected. If B contains a K_5 , then we may choose $A = B$ and we are done. Thus we assume that B contains no K_5 .

If $|V(H)| = 36$, we shall show that K_5^H is 5-colourable, and hence we may choose $A' = H$.

Let $f \in V(K_5^H)$ be a mapping of $V(H) \rightarrow \{1, 2, \dots, 5\}$. Then there is an index i such that $|f^{-1}(i)| \geq 8$. We colour f with colour $\Delta(f) = i$. We now prove that Δ is a proper colouring of K_5^H .

Assume to the contrary that there are two adjacent vertices f, g of $V(K_5^H)$ such that $\Delta(f) = \Delta(g) = i$. By the definition of Δ , there exist vertices x_1, x_2, \dots, x_8 such that $f(x_1) = f(x_2) = \dots = f(x_8) = i$, and there exist vertices y_1, y_2, \dots, y_8 such that $g(y_1) = g(y_2) = \dots = g(y_8) = i$.

Let $X = \{x_1, x_2, \dots, x_8\}$ and let $Y = \{y_1, y_2, \dots, y_8\}$. Since f, g are adjacent, it follows that if $x_i \neq y_j$, the $x_i y_j$ is not an edge of H . Hence the edge $x_i y_j$ is coloured blue.

If $|X \cap Y| \leq 3$, then we may assume that $\{x_1, x_2, \dots, x_5\} \cap \{y_1, y_2, \dots, y_5\} = \emptyset$. Since H contains no K_5 , there are indices $i, j \in \{1, 2, \dots, 5\}$ such that $x_i x_j$ is coloured blue, and there are indices $u, v \in \{1, 2, \dots, 5\}$ such that $y_u y_v$ is coloured blue. Let z be a vertex in $V(K_{41}) - V(H)$, the five vertices $\{z, x_i, x_j, y_u, y_v\}$ induces a blue K_5 , contrary to our assumption.

If $|X \cap Y| \geq 4$, then we may assume that $x_1 = y_1, x_2 = y_2, x_3 = y_3, x_4 = y_4$. Similarly, let z be a vertex in $V(K_{41}) - V(H)$, the five vertices $\{z, x_1, x_2, x_3, x_4\}$ induces a blue K_5 , contrary to our assumption.

Thus we may assume that $|V(H)| \leq 35$. It follows that there are two vertices $x, y \in V(K_{41}) - V(H)$ such that xy is a blue edge. (For otherwise, we would have a red K_6 , and we may take A to be the red K_6 and we are done.) If there is a blue K_3 induced by three vertices of H , then this K_3 and the blue edge in $V(K_{41}) - V(H)$ would form a blue K_5 , contrary to our assumption. Therefore we may assume that the coloured graph restricted to $V(H)$ does not contain a blue K_3 . Since $r(K_5, K_3) = 14$, and H contains no K_5 , i.e., the coloured graph restricted to $V(H)$ contains no red K_5 , and no blue K_3 . Therefore $|V(H)| \leq 13$. It follows that $|V(K_{41}) - V(H)| \geq 28$.

If there is red K_6 contained in $V(K_{41}) - V(H)$, we can choose A to be this red K_6 and we are done. Thus we may assume that the coloured graph restricted to $V(K_{41}) - V(H)$ contains no red K_6 . Since $r(K_6, K_3) = 18$, we conclude that there is a blue K_3 contained in $V(K_{41}) - V(H)$. Take a blue edge from $V(H)$ (which obviously exists), we obtain a blue K_5 , contrary to our assumption. This completes the proof of Case 1.

Case 2. There is no monochromatic K_5 .

Similarly to the argument in the previous case, we have $\chi(R) \geq 6$ and $\chi(B) \geq 6$. First we consider the 5-colouring graph of the red subgraph.

Let f be a vertex of K_5^R , i.e., f is a mapping of $V(R) \rightarrow \{1, 2, \dots, 5\}$. Since $|V(R)| = 41$, there is an integer i such that $|f^{-1}(i)| \geq 9$. Let $\Delta(f) = i$. If Δ is a proper 5-colouring of K_5^R , then we may take $A' = R$ and we are done. Thus we assume that Δ is not a proper colouring of K_5^R . Therefore there are two adjacent vertices f, g of K_5^R such that $\Delta(f) = \Delta(g)$.

Let $X = \{x_1, x_2, \dots, x_9\}$ be the 9-set such that $f(x_j) = \Delta(f)$ and let $Y = \{y_1, y_2, \dots, y_9\}$ be the 9-set such that $g(y_j) = \Delta(g)$. If $X \cap Y \neq \emptyset$, then we may assume that $x_1 = y_1$. There are two vertices $x_i, x_j \in X - \{x_1\}$ such that the edge $x_i x_j$ is coloured blue, for otherwise we would have a red K_8 , and we may take $A = K_8$ and we are done. For the same reason, there are also vertices $y_u, y_v \in Y - \{y_1, x_i, x_j\}$ such that the edge $y_u y_v$ is coloured blue. Now $\{x_1, x_i, x_j, y_u, y_v\}$ is a blue K_5 , contrary to our assumption.

Thus we may assume that $X \cap Y = \emptyset$. Consider the blue graph induced by X . If the blue graph has chromatic number 2, then there would be a red K_5 , contrary to our assumption. Therefore there is an odd blue cycle, say X' , contained in X . Similarly there is an odd blue cycle, say Y' , contained in Y . Let H be the blue graph induced by $X' \cup Y'$. We shall show that K_5^H is 5-colourable.

Suppose f is a vertex of K_5^H , i.e., f is a mapping of $V(H)$ to $\{1, 2, 3, 4, 5\}$. If there are vertices $x \in X', y \in Y'$ such that $f(x) = f(y) = i$, then we let $\Delta(f) = i$. Otherwise $f(X') \cap f(Y') = \emptyset$. In case $|f(X')| \leq |f(Y')|$, then $|f(X')| \leq 2$. Since X' contains an odd cycle, there is an edge xx' of H such that $f(x) = f(x')$. In this case we let $\Delta(f) = f(x) = f(x')$. Otherwise $|f(X')| > |f(Y')|$, and $|f(Y')| \leq 2$. Similarly there is an edge yy' of H such that $f(y) = f(y')$. In this case we let $\Delta(f) = f(y) = f(y')$. In the following we shall prove that Δ is a proper colouring of K_5^H .

Assume to the contrary that there are two adjacent vertices f, g of K_5^H such that $\Delta(f) = \Delta(g) = i$. Since for all $x \in X'$ and $y \in Y'$ and f, g are adjacent in K_5^H , we conclude that for all $x \in X'$ and $y \in Y'$ we have $f(x) \neq g(y)$ and $g(x) \neq f(y)$. In other words, $f(X') \cap g(Y') = \emptyset$ and $f(Y') \cap g(X') = \emptyset$. If $f(X') \cap f(Y') \neq \emptyset$ then by the definition of Δ , we have $\Delta(f) \in f(X') \cap f(Y')$. Hence $\Delta(f) \notin g(X') \cup g(Y')$. However, it follows from the definition that $\Delta(g) \in g(X') \cup g(Y')$, contrary to the assumption that $\Delta(g) = \Delta(f)$. Thus we assume that $f(X') \cap f(Y') = \emptyset$. Similarly we may assume that $g(X') \cap g(Y') = \emptyset$. This implies that $(f(X') \cup g(X')) \cap (f(Y') \cup g(Y')) = \emptyset$.

First we consider the case that $|f(X') \cup g(X')| \leq |f(Y') \cup g(Y')|$. Then $|f(X') \cup g(X')| \leq 2$. Since f is adjacent to g in K_5^H , we know that $|f(X') \cup g(X')| \neq 1$. Therefore $|f(X') \cup g(X')| = 2$. Without loss of generality, we assume that $f(X') \cup g(X') = \{1, 2\}$. Suppose $X' = \{x_1, x_2, \dots, x_{2k+1}\}$

and $x_i x_{i+1}$ is an edge of H for $i = 1, 2, \dots, 2k + 1$ (where $2k + 2 = 1$). Without loss of generality, we assume that $f(x_1) = 1$. Then $g(x_2) = 2$, because f, g are adjacent vertices in K_5^H . Inductively we can show that $f(x_3) = 1, g(x_4) = 2, \dots, f(x_{2k+1}) = 1, g(x_1) = 2, \dots, g(x_{2k+1}) = 2$. In other words, we have $f(X') = \{1\}$ and $g(X') = \{2\}$. By the definition of Δ , we should have $\Delta(f) = 1$ and $\Delta(g) = 2$, contrary to the assumption that $\Delta(f) = \Delta(g)$. The case that $|f(X') \cup g(X')| > |f(Y') \cup g(Y')|$ can be treated similarly (although not exactly the same), and we omit the details. This completes the proof of Lemma 6 \blacksquare

Theorem 2 follows easily from Lemma 6 and Lemmas 4 and 5. The argument is the same as that in the proof of Theorem 1, and we omit the details.

Theorem 3 *There is a graph G of chromatic number 7 such that $r_c(G) \leq 102$. Therefore $M_7 = 102$.*

Proof. It suffices to show that for any 2-colouring of the edges of K_{102} , there is a monochromatic subgraph H which is connected and which contains a copy of K_6 .

Let c be a 2-colouring of the edges of K_{102} and let R, B be the red subgraph and the blue subgraph, respectively. Since $r(K_6, K_6) \leq 102$ ([4]), there is a monochromatic copy of K_6 . Without loss of generality, we may assume that R contains a copy of K_6 . Similarly to the arguments in the previous proofs, we can assume that $\chi(R) \geq 7$ and $\chi(B) \geq 7$. Let H be a maximal connected subgraph of R with $\chi(H) \geq 7$. If H contains a copy of K_6 then we are done. Thus we may assume that H contains no K_6 . Since R contains a K_6 , we conclude that $|V(H)| \leq 102 - 6 = 96$. It follows that B is connected. If B contains a copy of K_6 then we are done. Thus we may assume that B contains no K_6 . In following we shall use V to denote the vertex set of H , and denote by \overline{V} the set $V(K_{102}) - V$.

If $|V| \geq 94$ then V contains a blue K_5 (i.e., the restriction of B to V contains a copy of K_5), because $r(K_6, K_5) \leq 94$ and V contains no red K_6 . Now take another vertex from \overline{V} , we get a blue K_6 , contrary to our assumption. Thus $|V| < 94$, and $|\overline{V}| \geq 9$. If \overline{V} contains no blue edge, then we are done, as it is a red K_9 . Assume that \overline{V} contains a blue edge. If V contains a blue K_4 , then the union of this blue K_4 and the blue edge in \overline{V} is a blue K_6 , contrary to our assumption. Therefore V contains no blue K_4 . This implies that $|V| \leq 43$, because $r(K_6, K_4) = 44$. Hence $|\overline{V}| \geq 59$.

We now consider the restriction of R to \overline{V} . We denote this graph by R' . If R' is 6-colourable, then one of the colour classes has cardinality at least 10, which implies that there is blue copy of K_{10} , contrary to our assumption. Thus we may assume that $\chi(R') \geq 7$. Let H' be a maximal connected

subgraph of R' with $\chi(H') \geq 7$. If H' contains a copy of K_6 then we are done. If H' does not contain a copy K_6 , then since R' contains a copy of K_6 , we have $|V(H')| \leq |V(R')| - 6$. Let U be the vertex of H' and let $\overline{U} = \overline{V} - U$. Then the 102 vertices of K_{102} is partitioned into three sets V, U, \overline{U} . All the edges between the three parts are blue edges. Moreover the restriction of the red subgraph R to V, U are connected, contains no K_6 and has chromatic numbers at least 7. Hence each of the sets U, V contains a blue edge. If the set \overline{U} also contain a blue edge, then the union of the vertices of the three blue edges induces a blue K_6 , contrary to our assumption. Assume that the set \overline{U} contains no blue edge. Then it is complete red graph. If $|\overline{U}| \geq 7$, then we are done. If $|\overline{U}| = 6$, then $|U| \geq 53$ and hence U contains a blue K_3 , because $r(K_6, K_3) = 18$ and U contains no red K_6 . Now the union of the vertices of this blue K_3 , the vertices of the blue edge in V and any vertex in \overline{U} induces a blue K_6 , contrary to our assumption.

Thus we have proved that for any 2-colouring of the edges of K_{102} , there is a monochromatic subgraph H which is connected and which contains a copy of K_6 . Now Theorem 3 follows easily from Lemma 5 (by using the corresponding arguments in the proof of Theorem 1). ■

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