The level of nonmultiplicativity of graphs

Claude Tardif
Department of Mathematics
Universität Bielefeld
33501, Bielefeld, Germany

Xuding Zhu*
Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung, Taiwan, 80424

June 4, 1999

Abstract

We introduce a parameter called the *level of nonmultiplicativity* of a graph, which is related to Hedetniemi's conjecture. We show that this parameter is equal to the number of factors in a factorization of the graph into a product of multiplicative graphs. Apart from the known multiplicative graphs, no graph is known to have a finite level of nonmultiplicativity. We show that the countably infinite complete graph $K_{\aleph_0}$ has an infinite level of nonmultiplicativity and that there exist Kneser graphs with arbitrarily high levels of nonmultiplicativity.

1 Introduction

Given graphs G and H, the *categorical product* $G \times H$ of G and H has vertex set $V(G \times H) = \{(g,h) : g \in V(G) \text{ and } h \in V(H)\}$ and edge set $E(G \times H) =$

*Supported in part by the National Science Council of R. O. C. under grant NSC88-2115-M-110-001.

$\{(g,h)(g',h') : gg' \in E(G)$ and $hh' \in E(H)\}$. This product is also called the *tensor product*. If c is an n-coloring of G, then it is straightforward to verify that the mapping c' defined by $c'(g,h) = c(g)$ is an n-coloring of $G \times H$. Therefore, $\chi(G \times H) \leq \chi(G)$. Similarly, we have $\chi(G \times H) \leq \chi(H)$, and hence

$$\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}. \tag{1}$$

Hedetniemi's conjecture [7] asserts that the equality holds in (1) for all graphs G and H.

Conjecture 1 (Hedetniemi [7]) *For any finite simple graphs G and H,*

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}.$$

This conjecture remains open, although it has enjoyed considerable attention. The many different approaches to this conjecture suggested other interesting questions and concepts. Amongst them we find the "multiplicativity" of graphs, which is the central theme of our paper.

For two graphs G and H, a mapping $f : V(G) \to V(H)$ is called a *homomorphism* from G to H if for every edge xy of G, $f(x)f(y)$ is an edge of H. We write $G \to H$ if G admits a homomorphism to H and $G \not\to H$ otherwise. Two graphs G and H are said to be *homomorphically equivalent* if both $G \to H$ and $H \to G$ hold; we then write $G \sim H$. It is obvious that a homomorphism of G to K_n is simply an n-coloring of the vertices of G. Indeed, many results suggest that homomorphisms provide a natural way of viewing coloring problems. Using the language of homomorphism, Hedetniemi's conjecture asserts that if neither G nor H admit a homomorphism to K_n, then the product $G \times H$ does not admit a homomorphism to K_n. The concept of a "multiplicative graph" is derived from this interpretation (see [5, 12]).

Definition 2 A graph K is called *multiplicative* if for any two graphs G and H such that $G \not\to K$ and $H \not\to K$, we have $G \times H \not\to K$. Otherwise K is called *nonmultiplicative*.

Hedetniemi's conjecture is equivalent to the statement that all complete graphs are multiplicative. However, it is a difficult task to prove that any given graph K is multiplicative, since this property depends on the behaviour of the class of all finite graphs. The complete graphs K_1, K_2, K_3 and the odd cycles are multiplicative (see [5]), but up to homomorphic equivalence, these are the only examples known so far. In fact, it is not even certain that other (finite) multiplicative graphs necessarily exist.

On the other hand, the categorical product provides a natural way of constructing nonmultiplicative graphs. Let G and H be graphs such that $G \not\to H$ and $H \not\to G$. Then $K = G \times H$ is nonmultiplicative, because $G \not\to K$ and $H \not\to K$, but $G \times H \to K$. As a matter of fact, up to homomorphic equivalence, all nonmultiplicative graphs can be written as a product in this fashion. For suppose that K is nonmultiplicative, and let G, H be graphs such that $G, H \not\to K$ and $G \times H \to K$. Since the product $\times$ distributes over the disjoint union $\cup$, we then have

$$K \sim (G \cup K) \times (H \cup K),$$

where $G \cup K \not\to K$ and $H \cup K \not\to K$.

Hence, the nonmultiplicative graphs are somehow "composite" with respect to the categorical product. In this frame of mind, it is then natural to wonder whether these "composite" graphs admit a factorization into "primes". These "primes" are the multiplicative graphs; thus a positive answer would at least establish the existence of a large class of multiplicative graphs. Bačík [1] presents many interesting consequences of such factorization of graphs into a product of finitely many multiplicative graphs. The prospect of bounding the number of factors in such a factorization leads us to the idea of quantifying the nonmultiplicativity of a graph. A priori, there appear to be many natural measures of nonmultiplicativity. One nice surprise is that these different measures give rise to the same parameter, and we are able to provide a unified account of the "level of nonmultiplicativity" of a graph in the next section.

Our working hypothesis is that the level of nonmultiplicativity of any finite graph is finite. However, efficient ways of bounding this parameter from above have yet to be devised, and we are only able to provide negative results. We will show that the infinite complete graph $K_{\aleph_0}$ has an infinite level of nonmultiplicativity, and that there exist Kneser graphs with arbitrarily high levels of nonmultiplicativity. On the positive side, the presentation of these results gives us the opportunity to survey other "natural colorings" of a categorical product of graphs, apart from the canonical colorings underlying the inequality (1) and Hedetniemi's conjecture. If the latter turns out to be false, these alternative colorings of a categorical product of graphs also need to be investigated.

2 Measures of nonmultiplicativity

Definition 3 A graph K is called *n-composite* if there exist graphs $H_1, \ldots, H_n$ such that $\prod_{i=1}^n H_i \to K$ and $\prod_{i\in I} H_i \not\to K$ for every proper subset I of $\{1, \ldots, n\}$. The *level of nonmultiplicativity* of K is the largest integer n such that K is n-composite, if such an integer n exists. Otherwise, K is said to have an infinite level of nonmultiplicativity.

Our main concern is the distinction between graphs with a finite level of nonmultiplicativity and graphs with an infinite level of nonmultiplicativity. One tentative construction for a graph with a given finite level of nonmultiplicativity is presented in the next result.

Lemma 4 *Suppose that $M_1, \ldots, M_n$ are multiplicative graphs such that $M_i \not\to M_j$ for $i \neq j$. Then the graph $K = \prod_{i=1}^n M_i$ has a level of nonmultiplicativity of n.*

Proof. Let I be a proper subset of $\{1, \ldots, n\}$. Then for $j \notin I$, we have $M_i \not\to M_j$ for all $i \in I$. Thus $\prod_{i\in I} M_i \not\to M_j$ since M_j is multiplicative. This implies $\prod_{i\in I} M_i \not\to K$; whence K is n-composite.

Now let $H_1, \ldots, H_{n+1}$ be graphs such that $\prod_{i=1}^{n+1} H_j \to K$. Then for $i = 1, \ldots, n$, we have $\prod_{i=1}^{n+1} H_j \to M_i$, hence there exists an index $f(i) \in \{1, \ldots, n+1\}$ such that $H_{f(i)} \to M_i$. The range I of the function f is a proper subset of $\{1, \ldots, n+1\}$, and

$$\prod_{j\in I} H_j \sim \prod_{i=1}^n H_{f(i)} \to \prod_{i=1}^n M_i = K.$$

This shows that the level of nonmultiplicativity of K is at most n.

■

However, no examples of multiplicative graphs M_1, M_2 satisfying $M_1 \not\to M_2$ and $M_2 \not\to M_1$ are known for the moment. The multiplicative graphs have a level of nonmultiplicativity of 1, and it is not yet established that other finite values belong to the range of the level of nonmultiplicativity.

The situation is different in the case of directed graphs. Homomorphism and products of directed graphs are defined in the same way, and the multiplicavity of directed graphs has been studied quite extensively [5, 8, 18, 19, 20][1]. There are more directed graphs which are known to be multiplicative, and also more directed graphs which are known to be nonmultiplicative. In particular, it is known [5, 20] that each directed cycle of prime power length is multiplicative, and all other directed cycles are nonmultiplicative. It is easy to verify that if $\gcd(p, q) = 1$, then $\vec{C}_p \times \vec{C}_q = \vec{C}_{pq}$, where $\vec{C}_n$ denotes the directed cycle of length n. Moreover, $C_m \to C_n$ if and only if m is a multiple of n. Therefore we have the following result:

Theorem 5 *Suppose n is a positive integer which has k distinct prime factors. Then the level of nonmultiplicativity of $\vec{C}_n$ is equal to k.*

Corollary 6 *For every integer k, there is a directed graph $\vec{G}$ whose level of nonmultiplicativity is equal to k.*

The remainder of this section is devoted to the converse of Lemma 4: any graph with a finite level of nonmultiplicativity is homomorphically equivalent to a product of multiplicative graphs. We begin with an alternative characterization of the level of nonmultiplicativity.

Proposition 7 *A graph K has a level of nonmultiplicativity of n if and only if n is the largest integer such that there exist graphs $G_1, \ldots, G_n$ satisfying $G_i \not\to K, i = 1, \ldots n$ and $G_i \times G_j \to K$ whenever $i \neq j$.*

Proof. The equivalence is clear if $n = 1$. Let n be greater than 1 and suppose that $G_1, \ldots, G_n$ are such that $G_i \not\to K, i = 1, \ldots n$ and $G_i \times G_j \to K$ whenever $i \neq j$. Put

$$H_i = \bigcup \{G_j : j \neq i\} \text{ for } i = 1, \ldots, n.$$

[1]Indeed, the results of this section also hold for directed graphs. However, the level of nonmultiplicativity is more relevant to the context of undirected graphs, where the results on multiplicativity are so scarce. Moreover, the concept of multiplicativity is "category sensitive": The complete graph K_3 is multiplicative in the category of undirected graphs [3], but viewed as a symmetric directed graph, it is nonmultiplicative in the category of directed graphs [13].

Then for any proper subset I of $\{1, \ldots, n\}$ and $k \notin I$, we have $G_k \to H_i$ for all $i \in I$. This implies that $G_k \to \prod_{i \in I} H_i$ and therefore $\prod_{i \in I} H_i \not\to K$. However, since $H_i = \bigcup\{G_j : j \neq i\}$ for $i = 1, \ldots, n$, $\prod_{i=1}^n H_i$ can be written as the union of all the terms $T_f = \prod_{i=1}^n G_{f(i)}$, where f ranges over the set F of all functions $f : \{1, \ldots, n\} \mapsto \{1, \ldots, n\}$ such that $f(i) \neq i$ for $i = 1, \ldots, n$. Hence any term T_f has at least two distinct factors, say G_i and G_j, whence $T_f \to G_i \times G_j \to K$. Therefore,

$$\prod_{i=1}^n H_i = \bigcup_{f \in F} T_f \to K.$$

This shows the level of nonmultiplicativity of K is at least n.

Conversely, if $H_1, \ldots, H_n$ are such that $\prod_{i=1}^n H_i \to K$ and $\prod_{i \in I} H_i \not\to K$ for every proper subset I of $\{1, \ldots, n\}$. Then the family

$$G_i = \prod\{H_j : j \neq i\}, i = 1, \ldots, n$$

is easily seen to satisfy $G_i \not\to K, i = 1, \ldots n$ and $G_i \times G_j \to K$ whenever $i \neq j$.

■

Definition 3 and Proposition 7 respectively present "coatomic" and "atomic" measures of nonmultiplicativity. The fact that these measures coincide agrees with the work of Duffus and Sauer [2] on the boolean lattices generated by graph exponentiation, that is, the following operation.

Definition 8 *For graphs K, G, the* exponential graph K^G *is the graph with vertex set all the mappings from $V(G)$ to $V(K)$, where two mappings f, g are adjacent if $f(u)$ and $g(v)$ are adjacent in K whenever u and v are adjacent in G.*

This definition encodes the following equivalence.

Lemma 9 ([2, 3, 5, 10]) *For graphs K, G, H, we have $G \times H \to K$ if and only if $H \to K^G$.*

In particular, K is multiplicative if and only if $K^G \to K$ whenever $G \not\to K$. This characterization of multiplicativity has been used successfully in [3, 5, 14]. Duffus and Sauer [2] investigated the order-theoretic properties of objects of the type $K^{\mathcal{G}}$, where K is a fixed graph and $\mathcal{G}$ is the class of all finite graphs. It turns out that these properties are closely related to the level of nonmultiplicativity of K and the factorization of K into multiplicative graphs. However, we will choose to avoid this setting and rest our final proofs on the power of a finite numerical parameter alone.

Proposition 10 *Suppose that the level of nonmultiplicativity of K is n and $G_1, \ldots, G_n$ are graphs such that $G_i \not\to K, i = 1, \ldots, n$ and $G_i \times G_j \to K$ whenever $i \neq j$. Then the graphs $K^{G_1}, \ldots, K^{G_n}$ are multiplicative and $K \sim \prod_{i=1}^n K^{G_i}$.*

Proof. We first show that K^{G_1} is multiplicative. Suppose that there exists graphs A, B such that $A, B \not\to K^{G_1}$ and $A \times B \to K^{G_1}$. Then by Lemma 9, $G_1 \times A \not\to K$ and $G_1 \times B \not\to K$. We then have

$$(G_1 \times A) \times (G_1 \times B) \to G_1 \times (A \times B) \to G_1 \times K^{G_1} \to K,$$

and for $i = 2, \ldots, n$, we also have

$$(G_1 \times A) \times G_i \to G_1 \times G_i \to K \text{ and } (G_1 \times B) \times G_i \to G_1 \times G_i \to K.$$

By Proposition 7, this contradicts the fact that the level of nonmultiplicativity of K is n.

It remains to show that $K \sim \prod_{i=1}^n K^{G_i}$. For any graph H, there is a natural homomorphism ϕ from K to K^H which maps any vertex u of K to the constant map $f_u : V(H) \mapsto V(K)$ defined by $f_u(v) = u$ for all $v \in V(H)$. In particular, we have $K \to \prod_{i=1}^n K^{G_i}$. However, we also have $G_j \times \prod_{i=1}^n K^{G_i} \to G_j \times K^{G_j} \to K$ for $j = 1, \ldots, n$. Since the family $G_1, \ldots, G_n, \prod_{i=1}^n K^{G_i}$ has one more member than the level of nonmultiplicativity of K, this shows that $\prod_{i=1}^n K^{G_i} \to K$. Hence $K \sim \prod_{i=1}^n K^{G_i}$.

■

Combining Lemma 4 and Proposition 10, we get:

Theorem 11 *A graph has a finite level of nonmultiplicativity if and only if it is homomorphically equivalent to a categorical product of finitely many multiplicative graphs.*

Hence the finiteness of the level of nonmultiplicativity of a graph turns out to have nontrivial consequences. For instance, suppose that the level of nonmultiplicativity of the complete graph K_4 is finite. We then can write

$$K_4 \sim M_1 \times \ldots \times M_n,$$

where $M_1, \ldots, M_n$ are multiplicative. Put $m = \chi(M_1)$. Then for G, H such that $\chi(G), \chi(H) > m$, we have $G, H \not\to M_1$, whence $G \times H \not\to M_1$ and $G \times H \not\to K_4$. This implies the validity of the following:

> There exists a value m such that $\chi(G) > m$ and $\chi(H) > m$ implies $\chi(G \times H) > 4$.

For the moment, even this simple statement is not known to be true (see [1, 2, 13, 21]).

3 The level of nonmultiplicativity of $K_{\aleph_0}$

In a paper entitled "The chromatic number of the product of two $\aleph_1$-chromatic graphs can be countable", Hajnal [6] showed that Hedetniemi's conjecture fails for infinite chromatic numbers. We will show that $K_{\aleph_0}$ has an infinite level of nonmultiplicativity. The presentation of this result gives us an opportunity to provide a reference to the Galvin-Laver-Kurepa result on which Hajnal's construction is based.

The cardinal $\aleph_1$ is the set of all countable ordinals. Injective (i.e., one-to-one) functions $f : \alpha \mapsto \aleph_0$ with $\alpha \in \aleph_1$ are endowed with a natural *extension* ordering: For functions $f : \alpha \mapsto \aleph_0$ and $g : \beta \mapsto \aleph_0$, we write $f < g$ if $\alpha \subset \beta$ and f is the restriction of g to α. If S is any subset of $\aleph_1$, we define the graph G_S as follows:

$$\begin{aligned} V(G_S) &= \{f : \alpha \mapsto \aleph_0 : \alpha \in S \text{ and } f \text{ is injective } \} \\ E(G_S) &= \{fg : f < g \text{ or } g < f\}. \end{aligned}$$

The key point in Hajnal's argumentation is the following:

Lemma 12 *If S and T are disjoint subsets of $\aleph_0$, then $\chi(G_S \times G_T) \leq \aleph_0$*

Proof. Put $A = \{(f, g) \in V(G_S \times G_T) : \mathrm{Dom}(f) < \mathrm{Dom}(g)\}$ (where $\mathrm{Dom}(h)$ denotes the domain of the function h). We get a proper $\aleph_0$-colouring ϕ of A by putting $\phi(f, g) = g(\mathrm{Dom}(f))$. This shows that $\chi(A) \leq \aleph_0$. Similarly, $\chi(G_S \times G_T - A) \leq \aleph_0$ hence $\chi(G_S \times G_T) \leq \aleph_0$.

■

The above colouring uses an interplay between the structure of the factors rather than colourings of the factors. Colouring G_S amount to covering a poset with antichains, which is a fruitful topic in set theory. The following is known.

Theorem 13 (Kurepa (see [17]), Galvin-Laver) *If S is a stationary subset of $\aleph_1$, then $\chi(G_S) = \aleph_1$.*

Hajnal [6] credits Theorem 13 to Galvin and Laver as an unpublished result. These authors later found out about the work of Kurepa, who proved that $\chi(G_{\aleph_1}) = \aleph_1$, and decided not to publish their work. We refer the reader to [17] for the proof of Kurepa's result, and to [4] for combinatorial set theoretic concepts such as stationary sets. For our purposes, it is sufficient to consider the following result.

Theorem 14 (Solovay (see [4])) *$\aleph_1$ contains infinitely many disjoint stationary sets.*

For any family $S_\alpha, \alpha \in \aleph_0$ of pairwise disjoint stationary sets, we then have $G_{S_\alpha} \not\to K_{\aleph_0}$ for all $\alpha \in \aleph_0$, and $G_{S_\alpha} \times G_{S_\beta} \to K_{\aleph_0}$ whenever $\alpha \neq \beta$. By Proposition 7, we then have the following:

Corollary 15 *The level of nonmultiplicativity of $K_{\aleph_0}$ is infinite.*

4 Multicolorings and Kneser graphs

One variation of graph colorings devised by Stahl [15] is particularily well suited to our discussion. A *n-tuple coloring* of a graph G is a map which assigns to each vertex of G a n-subset of some set of colors, in such a way that adjacent vertices are mapped to disjoint sets. The *n-chromatic number* $\chi_n(G)$ of G is the least number of colors required for a n-tuple coloring of G.

In the spirit of our investigations, it is natural to look at n-tuple colorings of products of graphs. We note the following.

Remark 16 *For any finite simple graphs G and H,*

$$\chi_n(G \times H) \leq \min\{\chi_k(G) + \chi_{n-k}(H) : k = 0, \dots, n\}. \tag{2}$$

Proof. Given a k-tuple coloring $f : V(G) \mapsto \mathcal{P}_k(A)$ of G and a $n-k$-tuple coloring $g : V(H) \mapsto \mathcal{P}_{n-k}(B)$ of H, where A and B are disjoint sets, we can define a n-tuple coloring h of $G \times H$ by $h(u,v) = f(u) \cup g(v)$. This shows that $\chi_n(G \times H) \leq \chi_k(G) + \chi_{n-k}(H)$.

■

In particular, $\chi_n(G \times H) \leq \min\{\chi_n(G), \chi_n(H)\}$. This is similar to the bound (1) for the ordinary chromatic number. The bound (2) uses "hybrid" colorings of products of graphs, where the structure of both factors is put to contribution. This is a different situation from the case of Hedetniemi's conjecture, where the proposed bound implies that one of the factors should always act as a dummy for coloring purposes. Hence the bound (2) is not related to multiplicativity as closely as Hedetniemi's conjecture. However, it does reveal the nonmultiplicativity of a large class of graphs.

For integers m, n, the *Kneser graph* $K(m,n)$ is the graph whose vertices are the n-subsets of $\{1, \dots, m\}$, where two vertices are joined by an edge if and only if they are disjoint. Thus, a n-tuple coloring of a graph G with m colors can be viewed as a homomorphism $\phi : G \mapsto K(m,n)$, and $\chi_n(G) = \min\{m : G \to K(m,n)\}$. Stahl [15, 16] mostly investigated multicolorings *of* Kneser graphs, that is, homomorphisms between Kneser graphs. This is a generalization of Kneser's original problem on the chromatic number of Kneser graphs. In all cases, it is the lower bounds that are hard to establish. Kneser's conjecture can be formulated without the language of graph theory: *If the n-subsets of a m-set are partitioned into $m - 2n + 1$ classes, then one of the classes contains two disjoint subsets.* This was proved by Lovász [11].

The question of existence of homomorphisms between Kneser graphs $K(m,n)$ and $K(m',n')$ is easily settled in the case where the ratios $\frac{m}{n}$ and $\frac{m'}{n'}$ are equal.

Lemma 17 ([15]) *Let m, n, m', n' be integers such that $\frac{m}{n} = \frac{m'}{n'}$. Then there exists a homomorphism from $K(m,n)$ to $K(m',n')$ if and only if*[2] *n' is a multiple of n.*

[2]The "only if" part is false if $m \leq 2n$, but the corresponding Kneser graphs are just perfect matchings or isolated vertices. ¿From now on, we only consider Kneser graphs $K(m,n)$ with $m > 2n$

For instance, Lemma 17 implies both $K(6,2) \not\to K(15,5)$ and $K(9,3) \not\to K(15,5)$. However,

$$\chi_5(K(6,2) \times K(9,3)) \le \chi_2(K(6,2)) + \chi_3(K(9,3)) \le 6 + 9 = 15,$$

whence $K(6,2) \times K(9,3) \to K(15,5)$. This shows that $K(15,5)$ is nonmultiplicative. We generalize this observation as follows:

Proposition 18 *There exist Kneser graphs with arbitrarily high levels of nonmultiplicativity.*

Our proof uses the following number-theoretic argument.

Fact. Given an integer n, there exists integers $\mu_1, \dots, \mu_n$ and ν such that

(i) ν is a multiple of $\mu_i + \mu_j$ whenever $i \ne j$,

(ii) ν is not a multiple of μ_i, $i = 1, \dots, n$.

Proof of Fact. The sequence $\mu_1, \dots, \mu_n$ consists of primes selected recursively as follows: Put $\mu_1 = 3$. If $\mu_1, \dots, \mu_{k-1}$ are already defined, then by Dirichlet's theorem, we can find a prime μ_k such that

$$\mu_k \equiv 1 \pmod{\prod_{i=1}^{k-1} \mu_i}.$$

Put $\nu = \prod\{\mu_i + \mu_j : 1 \le i < j \le n\}$. Clearly, condition (i) above is satisfied. Now, suppose that ν is a multiple of some μ_i. Then μ_i divides $\mu_j + \mu_k$ for some $j, k \ne i$. However, we can successively rule out the case $i < j, k$ because $\mu_j + \mu_k \equiv 2 \pmod{\mu_i}$, the case $j < i < k$ because $\mu_j + \mu_k \equiv \mu_j + 1 \pmod{\mu_i}$ and the case $j, k < i$ because $\mu_i > \mu_j + \mu_k$. This concludes the proof.

Proof of Proposition 18. Let $\mu_1, \dots, \mu_n$ and ν be as above. Then $K(3\mu_i, \mu_i) \not\to K(3\nu, \nu)$ for $i = 1, \dots, n$ by Lemma 17. For $i \ne j$, we have

$$K(3\mu_i, \mu_i) \times K(3\mu_j, \mu_j) \to K(3\mu_i + 3\mu_j, \mu_i + \mu_j) \to K(3\nu, \nu)$$

by Remark 16 and Lemma 17. This shows that the level of nonmultiplicativity of $K(3\nu, \nu)$ is at least n by Proposition 7.

■

The complete graphs are the Kneser graphs $K(m,1)$. These are conjectured to be multiplicative, while Proposition 18 shows that other Kneser graphs have high levels of nonmultiplicativity. The question as to precisely which Kneser graphs are multiplicative is related to a central conjecture on multicolorings.

Conjecture 19 (Stahl [15, 16]) *$K(m,n) \to K(m',n')$ if and only if $m' \geq am - 2b$, where $n' = an - b, a \geq 1$ and $0 \leq b < n$.*

Remark 20 *If Stahl's conjecture is true, then the only Kneser graphs that are possibly multiplicative are the following:*

- $K(2n+1,n), K(2n+2,n), n \geq 1$,
- $K(n,1), n \geq 3$ *and* $K(n,2), n \geq 5$,
- $K(9,3)$, $K(10,3)$, $K(12,3)$ *and* $K(12,4)$.

Proof. Put $m = \omega n + t$, where $0 \leq t < n$. Then, ω is the clique number of $K(m,n)$. We have

$$K(\omega+1,1) \times K(m-\omega-1, n-1) \to K(m,n)$$

by Remark 16 while $K(\omega+1,1) \not\to K(m,n)$. Therefore, $K(m,n)$ is nonmultiplicative or

$K(m-\omega-1,n-1) \to K(m,n)$. According to Stahl's conjecture, the latter holds if and only if $(\omega-2)(n-2)+t \leq 2$. The Kneser graphs satisfying this property are precisely those listed above.

■

This contrasts with a result of the first author and B. Larose (see [9]) stating that when a Kneser graph is homomorphically equivalent to a product of *connected* graphs, then it is equivalent to one of the factors. Thus most Kneser graphs are "composite" in the sense of our discussion, but "prime" in the sense of factorization by connected graphs.

Acknowledgements. We thank Professors F. Galvin and R. Laver for the story of and reference to Theorem 13.

References

[1] R. Bačík, *Structure of graph homomorphisms*, Ph. D thesis, Simon Fraser University, 1997.

[2] D. Duffus, N. Sauer, Lattices arising in categorial investigations of Hedetniemi's conjecture, *Discrete Math.* **152** (1996), 125–139.

[3] H. El-Zahar and N. Sauer, The chromatic number of the product of two 4-chromatic graphs is 4, *Combinatorica* **5** (1985), 121-126.

[4] P. Erdős, A. Hajnal, A. Máté, R. Rado, Combinatorial Set Theory: Partition Relations for Cardinals, Studies in Logic and in the Foundations of Mathematics 106, North-Holland Publishing Co., Amsterdam-New York, 1984, 347 pp.

[5] R. Häggkvist, P. Hell, D.J. Miller and V. Neumann-Lara, *On multiplicative graphs and the product conjecture,* Combinatorica 8 (1988), 71-81.

[6] A. Hajnal, The chromatic number of the product of two $\aleph_1$-chromatic graphs can be countable, Combinatorica **5** (1985), 137–139.

[7] S. H. Hedetniemi, Homomorphisms of graphs and automata, University of Michigan Technical Report 03105-44-T, 1966.

[8] P. Hell, H. Zhou and X. Zhu, Multiplicativity of oriented cycles, *Combinatorica* **13** (1993), 421-433.

[9] B. Larose, C. Tardif, Projective graphs and multiplicative graphs, in preparation.

[10] L. Lovász, Operations with structures, *Acta Math. Acad. Sci. Hungar.* **18** (1967), 321–328.

[11] L. Lovász, Kneser's conjecture, chromatic number, and homotopy, *J. Combinatorial Theory Ser. A* **25** (1978), 319–324.

[12] J. Nešetřil, A. Pultr, On classes of relations and graphs determined by subobjects and factorobjects, *Discrete Math.* **22** (1978), 287–300.

[13] S. Poljak, V. Rödl, On the arc-chromatic number of a digraph. *J. Combin. Theory Ser. B* **31** (1981), 190–198.

[14] N. Sauer and X. Zhu, An approach to Hedetniemi's conjecture. *Journal of Graph Theory* **16** (1992), 423–436.

[15] S. Stahl, n-tuple colorings and associated graphs, *J. Combinatorial Theory Ser. B* **20** (1976), 185–203.

[16] S. Stahl, The multichromatic number of some Kneser graphs, *Discrete Math.* **185** (1998), 287–291.

[17] S. Todorčević, Trees and linearly ordered sets, Handbook of set-theoretic topology, 235–293, North-Holland, Amsterdam-New York, 1984.

[18] H. Zhou, On the non-multiplicativity of oriented cycles, *SIAM J. on Discrete Mathematics* **5** (1992), 207-218.

[19] H. Zhou and X. Zhu, Multiplicativity of acyclic local tournaments, *Combinatorica* **17** (1997), 135-145.

[20] X. Zhu, A simple proof of the multiplicativity of directed cycles of prime power length, *Discrete Applied Mathematics* **36** (1992), 313-315.

[21] X. Zhu, A survey on Hedetniemi's conjecture, *Taiwanese Journal of Mathematics*, **2** (1998), 1–24.