

# An approach to Hedetniemi's conjecture

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## Abstract

For a fixed integer  $n \in \omega$ , a graph  $G$  of chromatic number greater than  $n$  is called persistent if for all  $n + 1$ -chromatic graphs  $H$ , the products  $G \times H$  are  $n + 1$ -chromatic graphs. Whether all graphs of chromatic number greater than  $n$  are persistent is a long standing open problem due to Hedetniemi. We call a graph  $G$  strongly persistent if  $G$  is persistent and the Hajos sum of  $G$  with any other persistent graph  $H$  is still persistent. This paper extends the class of known persistent graphs by proving the following result: If  $G$  is constructed from copies of  $K_{n+1}$  by Hajos sums, adding vertices and edges and at most one contraction of non-adjacent vertices then  $G$  is strongly persistent.

## 1 Introduction

Let  $G$  be a simple graph. We denote by  $V(G)$  and  $E(G)$  the vertex set and edge set of  $G$  respectively. Instead of writing  $(x, y) \in E(G)$ , we may write  $xy \in E(G)$  or  $x \sim y$ . A proper  $n$ -coloring of  $G$  is a mapping  $\phi :$

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$V(G) \mapsto \{1, 2, \dots, n\}$  such that whenever  $xy \in E(G)$  we have  $\phi(x) \neq \phi(y)$ . The chromatic number  $\chi(G)$  of  $G$  is the least integer  $n$  such that there is a proper  $n$ -coloring of  $G$ .

If  $G$  and  $H$  are simple graphs, the product  $G \times H$  of  $G$  and  $H$  has vertex set  $V(G \times H) = \{(g, h) : g \in V(G), h \in V(H)\}$  and edge set  $E(G \times H) = \{(g, h)(g', h') : gg' \in E(G), hh' \in E(H)\}$ . Given a proper  $n$ -coloring  $\phi$  of  $G$ , there is a natural induced proper  $n$ -coloring  $\psi$  of  $G \times H$ , namely  $\psi((g, h)) = \phi(g)$ . Therefore we have  $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$ . The “product conjecture” due to Hedetniemi [12] asserts that the equality holds for all graphs  $G$  and  $H$ . Or equivalently, for all  $n$ ,  $\chi(G) = \chi(H) = n + 1$  implies that  $\chi(G \times H) = n + 1$ .

This conjecture has enjoyed considerable interest, [5, 6, 7, 9, 11, 14, 15, 16, 18, 19, 21, 22], the principal result being that it holds for  $n = 3$ , [7]. Some other special cases of the conjecture have also been proved to be true, [5, 6, 18, 19].

Among various approaches to this conjecture, D. Duffus, B. Sands and R. Woodrow [6] tried to prove it by using a theorem of Hajos. Suppose that  $G$  and  $H$  are simple graphs,  $ab \in E(G)$  and  $uv \in E(H)$ . The Hajos sum of  $G$  and  $H$  with respect to  $ab$  and  $uv$ , denoted by  $G \oplus_{ab-uv} H$ , is the graph obtained from the disjoint union of  $G$  and  $H$  by contracting  $a$  and  $u$  into a single vertex, deleting the edges  $ab$  and  $uv$  and adding the edge  $bv$ . Although the edges  $ab$  and  $uv$  are unordered pairs, the order in which we write the Hajos sum is important. For instance,  $G \oplus_{ab-uv} H$  and  $G \oplus_{ba-uv} H$  are different graphs. We will write  $G \oplus H$  instead of  $G \oplus_{ab-uv} H$  in case the edges  $ab$  and  $uv$  need not be specified.

If  $G$  and  $H$  are graphs of chromatic number greater than  $n$ , then it is easy to see that  $G \oplus H$  is also of chromatic number greater than  $n$ . Adding vertices and edges to a graph  $G$  or contracting two non-adjacent vertices of  $G$  will not decrease its chromatic number. Hajos proved that all graphs with chromatic number at least  $n + 1$  can be constructed from copies of  $K_{n+1}$  by these operations, [10].

Fix an integer  $n$ , a graph  $G$  of chromatic number greater than  $n$  is called

*persistent* if for all  $n + 1$ -chromatic graphs  $H$  we have  $\chi(G \times H) = n + 1$ . Notice that when we discuss whether a graph  $G$  is persistent, we should fix the integer  $n$  first. Throughout this paper,  $n$  is a fixed integer unless otherwise specified. Hedetniemi's conjecture, for a given fixed  $n$ , is equivalent to asserting that all graphs  $G$  with  $\chi(G) > n$  are persistent. Since every graph  $G$  with  $\chi(G) > n$  can be constructed from copies of  $K_{n+1}$  by adding vertices and edges, contracting non-adjacent vertices and performing Hajos sums, it is sufficient to prove that every such constructed graph is persistent in order to establish Hedetniemi's conjecture.

D. Duffus, B. Sands and R. Woodrow [6] proved that if a graph  $G$  is constructed from copies of  $K_{n+1}$  by performing the three operations in such a way that all contractions are performed after all the Hajos sums then  $G$  is persistent.

We call a graph  $G$  *strongly persistent* if  $G$  is persistent and the Hajos sum of  $G$  with any other persistent graph  $H$  is still persistent. In this paper we prove that if  $G$  is constructed from copies of  $K_{n+1}$  by Hajos sums, adding vertices and edges and at most one contraction then  $G$  is strongly persistent. By observing that contractions of non-adjacent vertices preserve persistence, we can see that this extends the class of known persistent graphs.

A homomorphism  $\phi$  of  $G$  to  $H$  is an edge preserving mapping of  $V(G)$  to  $V(H)$ . We write  $G \rightarrow H$  if there exists a homomorphism of  $G$  to  $H$ . Otherwise we write  $G \not\rightarrow H$ . A graph  $G$  is  $n$ -colorable if and only if  $G \rightarrow K_n$ . We usually write  $G \rightarrow K_n$  instead of writing  $G$  is  $n$ -colorable. Also we do not distinguish a proper  $n$ -coloring from a homomorphism of  $G$  to  $K_n$ , and we always assume that the vertex set of the  $n$ -complete graph  $K_n$  is  $\{1, 2, \dots, n\}$ .

Given graphs  $G$  and  $H$ , the graph  $H^G$  has as vertex set  $V(H^G) = \{f : f \text{ is a mapping of } V(G) \text{ into } V(H)\}$  and edge set  $E(H^G) = \{(f, g) : \forall xy \in E(G), f(x)g(y) \in E(H)\}$ . Obviously  $H^G$  has loops if and only if  $G$  is homomorphic to  $H$ . The graph  $K_n^G$  is called the  $n$ -coloring graph of  $G$ . When  $K_n^G$  has loops, we denote by  $\mathcal{L}(K_n^G)$  the graph obtained from  $K_n^G$  by deleting the vertices of  $K_n^G$  with a loop on it.

## 2 The main theorem

In order to establish that all graphs  $G$  with  $\chi(G) > n$  are persistent, it is sufficient to prove that  $K_{n+1}$  is persistent and that the three types of operations, adding vertices and edges, taking Hajos sums and contracting non-adjacent vertices, preserve persistence. While it is easy to verify that  $K_{n+1}$  is persistent and that the operations of adding vertices and edges and contracting non-adjacent vertices preserve persistence, we do not know whether or not the operation of taking Hajos sums preserves persistence.

D. Duffus, B. Sands and R. Woodrow [6] introduced a property of graphs  $G$  with  $\chi(G) > n$  which is stronger than persistence and which is satisfied by  $K_{n+1}$  and preserved by Hajos sums. By investigating this stronger property, they were able to prove that those  $n + 1$ -chromatic graphs constructed from copies of  $K_{n+1}$  by first taking Hajos sums and then performing contractions are persistent.

In this paper we introduce some other properties of graphs  $G$  with  $\chi(G) > n$ . The discussion of these properties enables us to prove the following theorem which extends the class of known persistent graphs.

**THEOREM 1** *If  $G$  is obtained from copies of  $K_{n+1}$  by means of adding vertices and edges, taking Hajos sums and at most one contraction of non-adjacent vertices, then  $G$  is strongly persistent. Furthermore the Hajos sum of two strongly persistent graphs is a strongly persistent graph.*

The properties of graphs  $G$  with  $\chi(G) > n$  introduced in this paper are motivated by the following result of El-Zahar and N. Sauer:

**THEOREM 2** [7] *A graph  $G$  is persistent if and only if  $K_n^G \rightarrow K_n$ .*

This theorem suggests that we should look into the structure of the coloring graph  $K_n^G$  in order to prove the persistence of  $G$ .

Let  $G$  be a graph of chromatic number greater than  $n$  and let  $a$  be a vertex of  $G$ . We denote by  $G - a$  the subgraph of  $G$  induced by  $V(G) \setminus \{a\}$ . The vertex  $a$  is said to have property (\*) if

- (1)\*: The graph  $\mathcal{L}(K_n^{G-a})$  is  $n$ -colorable, i.e.,  $\mathcal{L}(K_n^{G-a}) \rightarrow K_n$ .
- (2)\*: If  $ff$  and  $fg$  are edges of  $K_n^{G-a}$  then  $f = g$ .

The graph  $G$  is said to have property (\*) if every vertex of  $G$  has property (\*).

For an edge  $ab$  of  $G$ , we denote by  $G - ab$  the subgraph of  $G$  obtained from  $G$  by removing the edge  $ab$ . An edge  $ab$  of  $G$  is said to have property (\*\*) if

- (1)\*\*: The graph  $\mathcal{L}(K_n^{G-ab})$  is  $n$ -colorable, i.e.,  $\mathcal{L}(K_n^{G-ab}) \rightarrow K_n$ .
- (2)\*\*: If  $ff$  and  $fg$  are edges of  $K_n^{G-ab}$  then  $f = g$ .

The graph  $G$  is said to have property (\*\*) if every edge of  $G$  has property (\*\*).

Now we proceed to prove Theorem 1 by first establishing a sequence of lemmas.

**LEMMA 3** *If  $G$  is a graph of chromatic number greater than  $n$  and  $a$  is a vertex of  $G$  with property (\*), then for any other vertex  $b$  of  $G$  with  $ab \in E(G)$ , the edge  $ab$  has property (\*\*).*

**Proof.** First we prove that  $\mathcal{L}(K_n^{G-ab}) \rightarrow K_n$ . Observe that if  $(f, g)$  is an edge of  $K_n^{G-ab}$  then  $(f|G-a, g|G-a)$  is an edge of  $K_n^{G-a}$ , where  $f|G-a$  is the restriction of  $f$  to  $G-a$ .

Let  $S$  be the subgraph of  $\mathcal{L}(K_n^{G-ab})$  induced by the set  $\{f \in \mathcal{L}(K_n^{G-ab}) : f|G-a \in \mathcal{L}(K_n^{G-a})\}$ . Then the mapping which takes  $f$  to  $f|G-a$  is a homomorphism of  $S$  to  $\mathcal{L}(K_n^{G-a})$ . Therefore  $S \rightarrow K_n$  since  $\mathcal{L}(K_n^{G-a}) \rightarrow K_n$ .

We claim that  $\mathcal{L}(K_n^{G-ab}) \setminus S$  is a set of isolated vertices, which would then imply that  $\mathcal{L}(K_n^{G-ab}) \rightarrow K_n$ .

First we show that there is no edge between elements of  $\mathcal{L}(K_n^{G-ab}) \setminus S$ . Otherwise suppose that  $f, g \in \mathcal{L}(K_n^{G-ab}) \setminus S$  and that  $f \sim g$ . Then  $f|G-a \sim g|G-a$  and  $f|G-a \sim f|G-a$  in  $K_n^{G-a}$ . Therefore  $f|G-a = g|G-a$  since  $G-a$  satisfies (2)\*. Because  $f$  is not a homomorphism of  $G-ab$  to  $K_n$ , there is a neighbor  $c$  of  $a$  in  $G-ab$  such that  $f(a) = f(c)$ . Thus  $f(a) = f(c) = g(c)$ , in contradiction to  $f \sim g$ .

A similar argument shows that there is no edge between  $S$  and  $\mathcal{L}(K_n^{G-ab}) \setminus S$ . Therefore the claim is proved and  $G-ab$  satisfies (1)\*\*.

To prove that  $G-ab$  satisfies (2)\*\*, we let  $f, g$  be vertices of  $K_n^{G-ab}$  such that  $f \sim f$  and  $f \sim g$ . Then  $f|G-a \sim f|G-a$  and  $f|G-a \sim g|G-a$  in  $K_n^{G-a}$ . Therefore  $f|G-a = g|G-a$  because  $G-a$  satisfies (2)\*. Since  $\chi(G) > n$  and  $f \sim f$  in  $K_n^{G-ab}$ , we have  $f(a) = f(b)$  for otherwise  $f$  would be a proper  $n$ -coloring of  $G$ . Also  $f \sim g$  in  $K_n^{G-ab}$  implies that  $f(a) = g(a)$  for otherwise we would have a proper  $n$ -coloring  $\phi$  of  $G$  defined as  $\phi(a) = g(a)$  and  $\phi(t) = f(t)$  for  $t \neq a$ . Therefore  $f = g$  and hence  $G-ab$  satisfies (2)\*\*.

**LEMMA 4** *The complete graph  $K_{n+1}$  has property (\*).*

**Proof.** For any vertex  $a$  of  $K_{n+1}$ ,  $K_{n+1} - a$  is a copy of  $K_n$ . We first prove that  $\mathcal{L}(K_n^{K_n}) \rightarrow K_n$  by explicitly giving a homomorphism  $\phi$  of  $\mathcal{L}(K_n^{K_n})$  to  $K_n$ .

For each  $f \in V(\mathcal{L}(K_n^{K_n}))$ , since  $f$  is not a homomorphism, there exist  $x, y \in K_n$  such that  $f(x) = f(y)$ . We set  $\phi(f) = f(x) = f(y)$ . If  $\phi(f) = \phi(g) = f(x) = f(y) = g(x') = g(y')$ , then  $x \sim x'$  or  $x \sim y'$ . Hence  $(f, g) \notin E(\mathcal{L}(K_n^{K_n}))$ . Therefore  $\phi$  is a homomorphism.

Now we prove that  $K_{n+1} - a$  satisfies (2)\*. Suppose that  $f \sim f$  and  $f \sim g$

in  $K_n^{K_n}$ . Since  $f$  is a homomorphism of  $K_n$  to  $K_n$ , it must be a one-to-one mapping. Hence if  $f(a) = i$  and  $j \neq i$ , then there is a neighbor  $b$  of  $a$  such that  $f(b) = j$ . Therefore  $g(a) \neq j$  for any  $j \neq i$ . Thus  $g(a) = i = f(a)$  and  $f = g$ .

**LEMMA 5** *If both  $G$  and  $H$  have property (\*), then the Hajos sum  $G \oplus H$  also has property (\*).*

**Proof.** We need to show that every vertex  $a$  of  $G \oplus H$  has property (\*). Suppose  $G \oplus H$  is the graph as shown in Fig. 1.

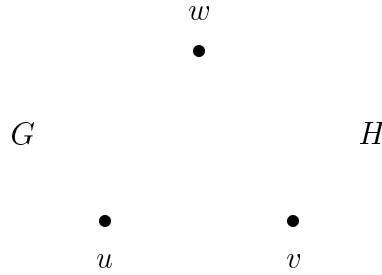


Fig. 1.  $G \oplus H$

Case 1.  $a \neq w$ . Without loss of generality, we assume that  $a \in V(H)$ . Now  $\mathcal{L}(K_n^{G \oplus H - a})$  is the union of the following subsets:

- $S_1 = \{f \in \mathcal{L}(K_n^{G \oplus H - a}) : f|_{G - w} \in \mathcal{L}(K_n^{G - w})\};$
- $S_2 = \{f \in \mathcal{L}(K_n^{G \oplus H - a}) : f \notin S_1 \text{ and } f|_{G - u} \in \mathcal{L}(K_n^{G - u})\};$
- $S_3 = \{f \in \mathcal{L}(K_n^{G \oplus H - a}) : f \notin S_1 \cup S_2 \text{ and } f|_{H - a} \in \mathcal{L}(K_n^{H - a})\};$
- $S_4 = \mathcal{L}(K_n^{G \oplus H - a}) \setminus S_1 \cup S_2 \cup S_3.$

**Claim 1:** Let  $S_1, S_2, S_3$  and  $S_4$  be the subsets of  $\mathcal{L}(K_n^{G \oplus H - a})$  defined as above. Then we have

- (1): There is no edge between  $S_i$  and  $S_j$  if  $i \neq j$ .
- (2): The subgraph  $S_2$  of  $\mathcal{L}(K_n^{G \oplus H - a})$  contains no edge.
- (3): There is a homomorphism of  $S_3$  to  $\mathcal{L}(K_n^{H - a})$ .
- (4): The subgraph  $S_4$  of  $\mathcal{L}(K_n^{G \oplus H - a})$  contains no edge.

**Proof of Claim 1:** (1). Suppose that  $f \in S_1$ ,  $g \notin S_1$  and  $f \sim g$ . Then  $g|G - w \sim g|G - w$  and  $g|G - w \sim f|G - w$  in  $K_n^{G - w}$ . Since  $G - w$  satisfies (2)\*, we have  $f|G - w = g|G - w$ . This contradicts the assumption that  $f \in S_1$  and  $g \notin S_1$ .

Similarly we can prove that there is no edge between  $S_2$  and  $S_3 \cup S_4$  and that there is no edge between  $S_3$  and  $S_4$ .

(2). Suppose to the contrary that there are two elements  $f, g$  of  $S_2$  such that  $f \sim g$ . Then  $f|G - w \sim f|G - w$  and  $f|G - w \sim g|G - w$ . Therefore  $g|G - w = f|G - w$  because  $G - w$  satisfies (2)\*.

Since  $f|G - w$  is a homomorphism and  $f|G - u$  is not, there is a neighbor  $b$  of  $w$  in  $G - u$  such that  $f(w) = f(b)$ . Thus  $g(b) = f(b) = f(w)$ , in contradiction to  $f \sim g$ . Therefore there is no edge between elements of  $S_2$ .

(3). For each  $f \in S_3$ , we must have  $f(u) = f(w)$ . Otherwise  $f$  would be a proper  $n$ -coloring of  $G$  which has chromatic  $\chi(G)$  greater than  $n$ . Therefore if  $f, g \in S_3$  and  $f \sim g$  then  $f(v) \neq g(u) = g(w)$  and  $f(w) = f(u) \neq g(v)$ . This implies that  $f|H - a \sim g|H - a$ . Thus the mapping which sends  $f$  to  $f|H - a$  is a homomorphism of the subgraph  $S_3$  of  $\mathcal{L}(K_n^{G \oplus H - a})$  to  $\mathcal{L}(K_n^{H - a})$ .

(4). If  $f, g \in S_4$  and  $f \sim g$ , then it is easy to see that we should have  $f(u) = f(v) = g(u) = g(v)$ . This is a contradiction because  $uv \in E(G \oplus H)$ .

Therefore there is no edge between elements of  $S_4$ . This completes the proof of claim 1.

Summing up the discussion above, we have  $S_1 \rightarrow \mathcal{L}(K_n^{G-w}) \rightarrow K_n$ ,  $S_2 \rightarrow K_n$ ,  $S_3 \rightarrow \mathcal{L}(K_n^{H-a}) \rightarrow K_n$  and  $S_4 \rightarrow K_n$ . Furthermore  $S_1, S_2, S_3$  and  $S_4$  are isolated from each other. Therefore  $\mathcal{L}(K_n^{G \oplus H - a}) \rightarrow K_n$  and  $G \oplus H - a$  satisfies (1)\*.

Now we prove that  $G \oplus H - a$  also satisfies (2)\*. Suppose that  $f \sim g$  and  $f \sim f$  in  $\mathcal{L}(K_n^{G \oplus H - a})$ . Then  $f|G - w \sim g|G - w$  and  $f|G - w \sim f|G - w$  imply that  $f|G - w = g|G - w$ . Also  $f|G - u \sim g|G - u$  and  $f|G - u \sim f|G - u$  imply that  $f|G - u = g|G - u$ , and  $f(w) = g(w) = f(u) = g(u)$  implies that  $f(w) \neq g(v), f(w) \neq f(v)$  and that  $f(v) \neq g(w)$ . Thus  $f|H - a \sim f|H - a$  and  $g|H - a \sim f|H - a$ , which imply that  $f|H - a = g|H - a$ . Therefore  $f = g$  and hence  $G \oplus H - a$  satisfies (2)\* and  $a$  has property (\*).

Case 2.  $a = w$ . Then  $\mathcal{L}(K_n^{G \oplus H - w})$  is the union of the following subsets:

- $S_1 = \{f \in \mathcal{L}(K_n^{G \oplus H - w}) : f|G - w \in \mathcal{L}(K_n^{G-w})\}$ ;
- $S_2 = \{f \in \mathcal{L}(K_n^{G \oplus H - w}) : f \notin S_1 \text{ and } f|H - w \in \mathcal{L}(K_n^{H-w})\}$ ;
- $S_3 = \mathcal{L}(K_n^{G \oplus H - w}) \setminus S_1 \cup S_2$ .

As in the proof of Claim 1, we can show that there is no edge between  $S_i$  and  $S_j$  if  $i \neq j$ , and  $S_i \rightarrow K_n$  for  $i = 1, 2, 3$ . Therefore  $\mathcal{L}(K_n^{G \oplus H - a}) \rightarrow K_n$  and  $G \oplus H - a$  satisfies (1)\*.

To prove that  $G \oplus H - a$  satisfies (2)\*, let  $f \sim g$  and  $f \sim f$  in  $\mathcal{L}(K_n^{G \oplus H - a})$ . Then  $f|G - w \sim g|G - w$  and  $f|G - w \sim f|G - w$  imply that  $f|G - w = g|G - w$ , and  $f|H - w \sim f|H - w$  and  $g|H - w \sim f|H - w$  imply that  $f|H - w = g|H - w$ . Therefore  $f = g$  and hence  $G \oplus H - a$  satisfies (2)\* and  $a$  has property (\*).

**COROLLARY 6** *If  $G$  is constructed from copies of  $K_{n+1}$  by Hajos sums then  $G$  has property (\*).*

**LEMMA 7** *If  $G$  has property (\*) and  $G'$  is obtained from  $G$  by contracting a set  $T$  of independent vertices of  $G$  into a single vertex  $v_T$ , then  $G'$  has property (\*\*).*

**Proof.** First we show that if  $a \in V(G')$  and  $a \neq v_T$ , then  $a$  has property (\*). Define a mapping  $\phi : \mathcal{L}(K_n^{G'-a}) \mapsto \mathcal{L}(K_n^{G-a})$  as follows:

For each  $f \in \mathcal{L}(K_n^{G'-a})$ ,  $\phi(f) \in \mathcal{L}(K_n^{G-a})$  is the mapping defined by  $\phi(f)(y) = f(y)$  if  $y \notin T$  and  $\phi(f)(y) = f(v_T)$  if  $y \in T$ .

It is easy to see that  $\phi$  is a homomorphism. Therefore  $\mathcal{L}(K_n^{G'-a}) \rightarrow \mathcal{L}(K_n^{G-a}) \rightarrow K_n$ . So  $G' - a$  satisfies (1)\*.

If  $f \sim f$  and  $f \sim g$  in  $\mathcal{L}(K_n^{G'-a})$ , then  $\phi(f) \sim \phi(f)$  and  $\phi(f) \sim \phi(g)$  in  $\mathcal{L}(K_n^{G-a})$ . Therefore  $\phi(f) = \phi(g)$  and hence  $f = g$ . Thus  $G' - a$  satisfies (2)\* and  $a$  has property (\*).

Now every edge of  $G'$  has at least one vertex which is not  $v_T$ . By lemma 3, every edge of  $G'$  has property (\*\*). Therefore  $G'$  has property (\*\*).

**LEMMA 8** *If both  $G$  and  $H$  have property (\*\*), then the Hajos sum  $G \oplus H$  also has property (\*\*).*

**Proof.** Suppose  $G \oplus H$  is the graph as shown in Fig. 1. We need to show that every edge  $ab$  of  $G \oplus H$  has property (\*\*).

Case 1.  $ab \neq uv$ . Without loss of generality, we assume that  $ab \in E(H)$ . Then  $\mathcal{L}(K_n^{G \oplus H - ab})$  is the union of the following subsets:

- $S_1 = \{f \in \mathcal{L}(K_n^{G \oplus H - ab}) : f|G \in \mathcal{L}(K_n^{G - uv})\};$
- $S_2 = \{f \in \mathcal{L}(K_n^{G \oplus H - ab}) : f \notin S_1 \text{ and } f|H \in \mathcal{L}(K_n^{H - ab})\};$
- $S_3 = \mathcal{L}(K_n^{G \oplus H - ab}) \setminus S_1 \cup S_2.$

As in the proof of Claim 1, we can show that there is no edge between  $S_i$  and  $S_j$  if  $i \neq j$ , and  $S_i \rightarrow K_n$  for  $i = 1, 2, 3$ . Therefore  $\mathcal{L}(K_n^{G \oplus H - ab}) \rightarrow K_n$  and  $G \oplus H - ab$  satisfies (1)\*\*.

To prove that  $G \oplus H - ab$  also satisfies (2)\*, let  $f \sim g$  and  $f \sim f$  in  $\mathcal{L}(K_n^{G \oplus H - ab})$ . Then  $f|G \sim g|G$  and  $f|G \sim f|G$  in  $K_n^{G - wu}$  imply that  $f|G = g|G$ . Since  $f$  is a homomorphism of  $G - wu$  to  $K_n$  and  $\chi(G) > n$ , we have  $f(w) = f(u) = g(w) = g(u)$ . This implies that  $f|H \sim f|H$  and  $f|H \sim g|H$  in  $K_n^{H - ab}$ . Therefore  $f|H = g|H$  and hence  $f = g$ .

Case 2.  $ab = uv$ . Then  $\mathcal{L}(K_n^{G \oplus H - uv})$  is the union of the following subsets:

- $S_1 = \{f \in \mathcal{L}(K_n^{G \oplus H - uv}) : f|G \in \mathcal{L}(K_n^{G - wu})\};$
- $S_2 = \{f \in \mathcal{L}(K_n^{G \oplus H - uv}) : f \notin S_1 \text{ and } f|H \in \mathcal{L}(K_n^{H - uv})\};$
- $S_3 = \mathcal{L}(K_n^{G \oplus H - ab}) \setminus S_1 \cup S_2.$

As in the proof of Claim 1, we can show that there is no edge between  $S_i$  and  $S_j$  if  $i \neq j$  and that  $S_i \rightarrow K_n$  for  $i = 1, 2, 3$ . Therefore  $\mathcal{L}(K_n^{G \oplus H - uv}) \rightarrow K_n$  and  $G \oplus H - uv$  satisfies (1)\*\*.

If  $f \sim g$  and  $f \sim f$  in  $\mathcal{L}(K_n^{G \oplus H - uv})$ , then  $f|G \sim g|G$  and  $f|G \sim f|G$  in  $K_n^{G - wu}$  imply that  $f|G = g|G$ . Also  $f|H \sim f|H$  and  $f|H \sim g|H$  in  $K_n^{H - uv}$  imply that  $f|H = g|H$ . Therefore  $f = g$  and  $G \oplus H - uv$  satisfies (2)\*\*.

**COROLLARY 9** *If  $G$  is constructed from copies of  $K_{n+1}$  by Hajos sums and at most one contraction of an independent set, then  $G$  has property (\*\*).*

**LEMMA 10** *If  $G$  has property (\*\*), then  $G$  is persistent and for any persistent graph  $H$ , the Hajos sum  $G \oplus H$  is also persistent. Or equivalently, if  $G$  has property (\*\*), then  $G$  is strongly persistent.*

**Proof.** Since  $G$  has property (\*\*),  $\mathcal{L}(K_n^{G-ab}) \rightarrow K_n$  for any given edge  $ab$  of  $G$ . Furthermore any element  $f \in V(K_n^{G-ab}) \setminus V(\mathcal{L}(K_n^{G-ab}))$  is an isolated loop. The graph  $K_n^G$  has the same vertex set as  $K_n^{G-ab}$  and less edges. So  $K_n^G$  can be viewed as a subgraph of  $K_n^{G-ab}$ . When we consider  $f \in V(K_n^{G-ab}) \setminus V(\mathcal{L}(K_n^{G-ab}))$  as a vertex of  $K_n^G$ , the loop disappears because  $\chi(G) > n$ . Therefore  $K_n^G \rightarrow K_n$  and hence  $G$  is persistent by Theorem 2.

To prove that  $G \oplus H$  is persistent, let  $G \oplus H$  be the graph as shown in Fig. 1. Again we only need to prove that  $K_n^{G \oplus H} \rightarrow K_n$  by Theorem 2.  $K_n^{G \oplus H}$  is the union of  $S_1 = \{f \in K_n^{G \oplus H} : f|G \in \mathcal{L}(K_n^{G-wu})\}$  and  $S_2 = K_n^{G \oplus H} \setminus S_1$ .

Obviously  $S_1 \rightarrow \mathcal{L}(K_n^{G-wu}) \rightarrow K_n$  because  $G-wu$  satisfies (1)\*\*. Suppose that  $f \in S_2$ . Then  $f|G$  is a homomorphism of  $G-wu$  to  $K_n$ , i.e.,  $f|G \sim f|G$  in  $K_n^{G-wu}$ . Therefore  $f$  is not adjacent to any elements of  $S_1$  because  $G-wu$  satisfies (2)\*\*. Also we have  $f(w) = f(u)$  because otherwise  $f$  would be a proper  $n$ -coloring of  $G$ . Therefore  $f \sim g$  and  $f, g \in S_2$  imply that  $f|H \sim g|H$  in  $K_n^H$ . Hence  $S_2 \rightarrow K_n^H \rightarrow K_n$  because  $H$  is persistent. So  $K_n^{G \oplus H} \rightarrow K_n$  and  $G \oplus H$  is persistent.

**LEMMA 11** *If a graph  $G$  is strongly persistent and  $G'$  is a graph obtained from  $G$  by adding vertices and edges, then  $G'$  is also strongly persistent.*

The proof of this lemma is straight forward and omitted.

Suppose that a graph  $G'$  is obtained from  $G$  by first adding vertices and edges and then taking a Hajos sum or contraction. It is easy to see that either  $G'$  can be obtained from  $G$  by first taking a Hajos sum or contraction and then adding vertices and edges, or  $G'$  can be obtained from  $G$  by simply adding vertices and edges.

Now we can finish the proof of Theorem 1 as follows:

Suppose  $G$  is obtained from copies of  $K_{n+1}$  by Hajos sums, adding vertices and edges and at most one contraction. Then by the above argument,  $G$  can be obtained from copies of  $K_{n+1}$  by first taking Hajos sums and at most one contraction, then adding vertices and edges. By corollary 9 and lemmas 10 and 11, such a graph is strongly persistent.

Suppose both  $G$  and  $H$  are strongly persistent. We shall prove that  $G \oplus H$  is also strongly persistent. Let  $Q$  be a persistent graph, we need to show that  $(G \oplus H) \oplus Q$  is persistent. This is true because  $(G \oplus H) \oplus Q$  either equals  $G \oplus (H \oplus Q)$  or equals  $H \oplus (G \oplus Q)$  by choosing the corresponding edges appropriately. To be precise, let  $(G \oplus H) \oplus Q = (G \oplus_{ab-cd} H) \oplus_{xy-uv} Q$ . If  $\{x, y\} \neq \{b, d\}$ , then  $(G \oplus_{ab-cd} H) \oplus_{xy-uv} Q = G \oplus_{ab-cd} (H \oplus_{xy-uv} Q)$ . If  $\{x, y\} = \{b, d\}$ , we assume that  $x = b$  and  $y = d$ . Then  $(G \oplus_{ab-cd} H) \oplus_{xy-uv} Q = (G \oplus_{ba-uv} Q) \oplus_{av-cd} H$ .

It was observed in [6] that contractions preserve persistence. By Theorem 1 and this observation, we have

**THEOREM 12** *If  $G$  is a graph constructed from copies of  $K_{n+1}$  by the three types of operations in such a way that whenever a Hajos sum is performed, one of the factor graphs can be constructed from copies of  $K_{n+1}$  by Hajos sums, adding vertices and edges and at most one contraction, then  $G$  is persistent.*

### 3 Unique colorability

A graph  $G$  is uniquely  $n$ -colorable if there is a unique partition of  $V(G)$  into  $n$  independent sets.

**THEOREM 13** [8] *If  $G$  is a connected graph with  $\chi(G) > n$ , then  $K_n \times G$  is uniquely  $n$ -colorable.*

D. Duffus, B. Sands and R. Woodrow [6] noticed a relation between uniquely colorable graphs and Hedetniemi's conjecture. As generalizations of Theorem 13, they proposed the following two conjectures concerning unique colorability:

- $(A_n)$  : For all uniquely colorable graphs  $G$  and  $H$ , each proper  $n$ -coloring of  $G \times H$  is induced by  $G$  or by  $H$ , i.e., if  $\phi : V(G \times H) \mapsto$

$\{1, 2, \dots, n\}$  is a proper  $n$ -coloring of  $G \times H$ , then either  $\phi((g, h)) = \phi((g, h'))$  for all  $g \in V(G)$  or  $\phi((g, h)) = \phi((g', h))$  for all  $h \in V(H)$ .

- $(B_n)$  : For all uniquely  $n$ -colorable graphs  $G$  and all connected graphs  $H$  with  $\chi(H) > n$ ,  $G \times H$  is uniquely  $n$ -colorable.

Let  $(C_{n+1})$  be the statement that  $\chi(G) = \chi(H) = n + 1$  implies that  $\chi(G \times H) = n + 1$ . It was proved in [6] that  $(A_n) \Rightarrow (B_n) \Rightarrow (C_{n+1})$ . By proving a special case of  $(A_n)$ , they established the following special case of Hedetniemi's conjecture:

**THEOREM 14** [6] *If  $G$  and  $H$  are connected  $n + 1$ -chromatic graphs such that  $K_n \rightarrow G$  and  $K_n \rightarrow H$ , then  $\chi(G \times H) = n + 1$ .*

We propose a conjecture concerning the coloring graph of uniquely colorable graphs which is between  $(B_n)$  and  $(C_{n+1})$ .

- $(D_n)$  : For all uniquely  $n$ -colorable graphs  $G$ ,  $\mathcal{L}(K_n^G)$  is  $n$ -colorable.

**THEOREM 15** *The conjecture  $(D_n)$  is between conjecture  $(B_n)$  and conjecture  $(C_{n+1})$ , i.e.,  $(B_n) \Rightarrow (D_n) \Rightarrow (C_{n+1})$ .*

**Proof.** Suppose  $(B_n)$  is true and  $(D_n)$  is false. Let  $G$  be a uniquely  $n$ -colorable graph with  $\mathcal{L}(K_n^G)$  not  $n$ -colorable and let  $H$  be a connected component of  $\mathcal{L}(K_n^G)$  such that  $\chi(H) > n$ . By  $(B_n)$ ,  $G \times H$  is uniquely  $n$ -colorable. However the  $n$ -coloring  $\phi : V(G \times H) \mapsto V(K_n)$  defined by

$$\phi((x, f)) = f(x)$$

is a proper  $n$ -coloring of  $G \times H$  which is different from the proper  $n$ -coloring induced by the proper  $n$ -coloring of  $G$ . For otherwise  $f(x) = g(x)$  for all  $f, g \in V(H)$  and all  $x \in V(G)$ , i.e.,  $H$  contains a single vertex. This contradicts the assumption that  $\chi(H) > n$ . Thus  $(B_n) \Rightarrow (C_{n+1})$ .

Suppose that  $(D_n)$  is true. For any connected graph  $H$  with  $\chi(H) > n$ , the graph  $K_n \times H$  is uniquely  $n$ -colorable by Theorem 13. Therefore  $\mathcal{L}(K_n^{K_n \times H}) \rightarrow K_n$ .

Since  $\mathcal{L}(K_n^{K_n}) \rightarrow K_n$  and  $\chi(H) > n$ , we know that  $(\mathcal{L}(K_n^{K_n})^H)$  has no loops, for otherwise we would have  $H \rightarrow \mathcal{L}(K_n^{K_n}) \rightarrow K_n$ . Therefore we have  $(\mathcal{L}(K_n^{K_n}))^H \rightarrow \mathcal{L}(K_n^{K_n \times H})$ . (The mapping  $\psi : V((\mathcal{L}(K_n^{K_n}))^H) \mapsto V(\mathcal{L}(K_n^{K_n \times H}))$  defined by  $\psi(f)(i, h) = f(h)(i)$  is the required homomorphism).

The constant maps in  $\mathcal{L}(K_n^{K_n})$  induce a complete graph of order  $n$ . Therefore  $K_n \rightarrow \mathcal{L}(K_n^{K_n})$  and hence  $K_n^H \rightarrow (\mathcal{L}(K_n^{K_n}))^H \rightarrow \mathcal{L}(K_n^{K_n \times H}) \rightarrow K_n$ . By Theorem 2,  $H$  is persistent. So  $(D_n)$  implies  $(C_{n+1})$ .

**THEOREM 16** *Conjecture  $(A_n)$  holds if and only if for all uniquely  $n$ -colorable graphs  $G$ , we have  $\mathcal{L}(K_n^G) \rightarrow K_n$  and  $\mathcal{L}(K_n^G)$  contains no uniquely  $n$ -colorable subgraphs other than the complete graph of order  $n$  induced by the constant maps.*

**Proof.** “only if”: Since  $(A_n) \Rightarrow (B_n) \Rightarrow (D_n)$ , we only have to show that when  $(A_n)$  holds, the graph  $\mathcal{L}(K_n^G)$  contains no uniquely  $n$ -colorable subgraphs other than the complete subgraph of order  $n$  induced by the constant maps.

Suppose  $Y$  is a uniquely  $n$ -colorable subgraph of  $\mathcal{L}(K_n^G)$ . Then by  $(A_n)$ , the proper  $n$ -coloring  $\psi : V(G \times Y) \mapsto V(K_n)$  defined by  $\psi(g, f) = f(g)$  is either induced by the unique  $n$ -coloring of  $G$  or induced by the unique  $n$ -coloring of  $Y$ . If  $\psi$  is induced by the coloring of  $G$ , then for all  $f, f' \in V(Y)$ , we have  $f(g) = f'(g)$  for all  $g \in V(G)$ , i.e.,  $Y$  contains only one element, contradicting the assumption that  $Y$  is uniquely  $n$ -colorable if  $n \geq 2$ . (When  $n = 1$ ,  $K_n^G$  contains only a single constant mapping). If  $\psi$  is induced by the coloring of  $Y$ , then for all  $g, g' \in V(G)$ , we have  $f(g) = f(g')$  for all  $f \in V(Y)$ , i.e.,  $Y$  contains only constant maps.

“if”: Let  $G$  and  $H$  be uniquely  $n$ -colorable graphs and let  $\psi$  be a homomorphism of  $G \times H$  to  $K_n$ . Then the mapping  $\phi : V(H) \mapsto K_n^G$  defined

by  $\phi(h)(g) = \psi(g, h)$  is a homomorphism of  $H$  to  $K_n^G$ . Now either we have  $\phi(H) \subset \mathcal{L}(K_n^G)$  or  $\phi(y)$  is a homomorphism of  $G$  to  $K_n$  for some  $y \in V(H)$ .

If  $\phi(H) \subset \mathcal{L}(K_n^G)$ , then  $\phi(H)$  is a uniquely  $n$ -colorable subgraph of  $\mathcal{L}(K_n^G)$ . Hence  $\phi(H)$  is the complete graph of order  $n$  induced by the constant maps. So for all  $h \in V(H)$ , we have  $\phi(h)(g) = \phi(h)(g')$  for all  $g, g' \in V(G)$ . Therefore  $\psi(g, h) = \psi(g', h)$  for all  $g, g' \in V(G)$  and  $h \in V(H)$ , which implies that  $\psi$  is induced by a proper  $n$ -coloring of  $H$ .

If  $\phi(y)$  is a homomorphism of  $G$  to  $K_n$  for some  $y \in V(H)$ , then  $\phi(y)$  is an isolated loop in  $K_n^G$  because  $G$  is uniquely  $n$ -colorable. Therefore  $\phi(H) = \{\phi(y)\}$  and hence  $\psi$  is induced by a proper  $n$ -coloring of  $G$ .

## 4 Some comments

In order to prove Theorem 1, we first proved that  $K_{n+1}$  has property (\*), and then showed that this property is preserved by Hajos sums. However if we perform a contraction, this property may be lost. For example, the Hajos sum of two triangles is a pentagon, which has property (\*). If we contract two non-adjacent vertices of this pentagon, the resulting graph has no longer property (\*). However if we perform one contraction, although property (\*) may be lost, the resulting graphs still have property (\*\*), which is weaker than property (\*) but strong enough for our purpose. Property (\*\*) is also preserved by Hajos sums. Again it is not preserved by contractions.

Since property (\*\*) is too strong, it is natural to look for some weaker ones. It is not difficult to prove (similar to the proof of lemma 8) that the following property is preserved by Hajos sums and it implies Hedetniemi's conjecture.

We say a graph  $G$  with  $\chi(G) > n$  has property (\*\*)' if for every edge  $ab$  of  $G$ , we have:

- (1)' : The graph  $\mathcal{L}(K_n^{G-ab})$  is  $n$ -colorable.

- (2)' : If  $f$  is a loop in  $K_n^{G-ab}$  and  $g$  is connected to  $f$ , then  $f(a) = g(a) = f(b) = g(b)$ .

Although this property is weaker than property (\*\*), it is still stronger than Hedetniemi's conjecture. The following is an argument to show that (1)' implies  $(D_n)$ , while  $(D_n)$  has already been proved to imply Hedetniemi's conjecture.

Let  $H$  be a uniquely  $n$ -colorable graph. If  $H$  is the complete graph  $K_n$ , then  $(D_n)$  holds trivially for  $H$ . If  $H$  is not complete, then add an edge  $ab$  to two non-adjacent vertices  $a, b$  of  $H$  which are in the same color class in the unique  $n$ -color partition. Then the resulting graph  $H'$  is of chromatic number  $n + 1$ . Hence every edge of  $H'$  has property (\*\*)' . In particular we have  $\mathcal{L}(K_n^{H'-ab}) = \mathcal{L}(K_n^H) \rightarrow K_n$ . So  $(D_n)$  holds.

Property (\*\*)' is also stronger than the following property  $(C_{n+1}^*)$  introduced by D. Duffus, B. Sands and R. Woodrow in [6]:

- $(C_{n+1}^*)$ : An  $n + 1$ -chromatic graph  $G$  is said to have property  $(C_{n+1}^*)$  if for all edges  $ab$  of  $G$ , for all connected  $n + 1$ -chromatic graphs  $H$ , for all proper  $n$ -colorings  $\phi$  of  $(G - ab) \times H$ , and for all  $h \in V(H)$ ,  $\phi(a, h) = \phi(b, h)$ .

**THEOREM 17** *If  $G$  is an  $n + 1$ -chromatic graph and has property (\*\*)', then  $G$  satisfies condition  $(C_{n+1}^*)$ .*

**Proof.** Suppose that  $G$  has property (\*\*)',  $ab$  is an edge of  $G$ ,  $H$  is a connected  $n + 1$ -chromatic graph and that  $\phi$  is a proper  $n$ -coloring of  $(G - ab) \times H$ . Define  $\psi : V(H) \mapsto V(K_n^{G-ab})$  as  $\psi(h) = \phi_h$ , where  $\phi_h(x) = \phi(x, h)$ . It is easy to verify that  $\psi$  is a homomorphism of  $H$  to  $K_n^{G-ab}$ . Since  $\mathcal{L}(K_n^{G-ab}) \rightarrow K_n$ , we know that  $\psi(H)$  is not contained in  $\mathcal{L}(K_n^{G-ab})$ , for otherwise we would have  $H \rightarrow K_n$ . Therefore  $\psi(h_0)$  is a loop in  $K_n^{G-ab}$  for some  $h_0 \in V(H)$ . Since for all  $h \in V(H)$ ,  $\psi(h)$  is connected to  $\psi(h_0)$ , we have  $\psi(h)(a) = \psi(h_0)(a) = \psi(h_0)(b) = \psi(h)(b)$ , i.e.,  $\phi(h, a) = \phi(h, b)$  for all  $h \in V(H)$ .

El-Zahar constructed a 6-chromatic graph which is persistent but fails to satisfy  $(C_6^*)$ , and one can add vertices and edges to this graph to obtain  $n$ -chromatic graphs violating  $(C_n^*)$  for all  $n \geq 6$ , [6]. In light of such an example, we propose another property which is still weaker:

- $(C_{n+1}^{**})$  : An  $n + 1$ -chromatic graph  $G$  is said to have property  $(C_{n+1}^{**})$  if for all edges  $ab$  of  $G$ , for all vertex critical  $n + 1$ -chromatic graphs  $H$ , for all proper  $n$ -colorings  $\phi$  of  $(G - ab) \times H$ , and for all  $h \in V(H)$ ,  $\phi(a, h) = \phi(b, h)$ .

This property can be proved to be preserved by Hajos sums, and it implies Hedetniemi's conjecture. Also the example mentioned above has this property.

In the process of forming Hajos sums and contractions, the good properties of  $K_{n+1}$  are lost gradually. In order to establish Hedetniemi's conjecture, what we need to prove is that  $K_{n+1}$  is good enough so that no matter how many (finite) operations are performed, the resulting graph is still persistent.

Let  $\mathcal{N}_1 = \{G : \chi(G) > n \text{ and } G \text{ is persistent}\}$ , and for  $k \geq 1$ , let  $\mathcal{N}_{2k} = \{G \in \mathcal{N}_{2k-1} : \forall G' \in \mathcal{N}_{2k-1}, G \oplus G' \in \mathcal{N}_{2k-1}\}$ ,  $\mathcal{N}_{2k+1} = \{G \in \mathcal{N}_{2k} : \text{any contraction of } G \text{ is still in } \mathcal{N}_{2k}\}$ . Hedetniemi's conjecture is equivalent to asserting that  $K_{n+1} \in \mathcal{N}_i$  for all integer  $i$ . By noting that the Hajos sum is associative in a certain sense, the result of D. Duffus, B. Sands and R. Woodrow in [6] is that  $K_{n+1} \in \mathcal{N}_2$ . The main result of this paper is still weaker than the assertion that  $K_{n+1} \in \mathcal{N}_4$ .

## References

- [1] C. Berge, *Graphs and hypergraphs*, North-Holland publishing company, 1973.
- [2] B. Bollobas, *Uniquely colorable graphs*, J. Comb. Th.(B), 25(1978), 54-61.

- [3] B. Bollobas and N. Sauer, *Uniquely colorable graphs with large girth*, Canada J. Math., 28(1972), 1340-1344.
- [4] M. Borowiecki, *On chromatic number of products of two graphs*, Colloq. Math., 25(1972), 49-52.
- [5] S. Burr, P. Erdős and L. Lovász, *On graphs of Ramsey type*, Ars Combinatoria, 1(1976), 167-190.
- [6] D. Duffus, B. Sands and R. Woodrow, *On the chromatic number of the products of graphs*, J. Graph Theory, 9(1985), 487-495.
- [7] E. El-Zahar and N. Sauer, *The chromatic number of the product of two 4-chromatic graphs is 4*, Combinatorica, 5(1985), 121-126.
- [8] D. Greenwell and L. Lovász, *Applications of product colouring*, Acta Math. Acad. Sci. Hungar. 25(1974), 335-340.
- [9] A. Hajnal, *The chromatic number of the product of two alpha-one chromatic graphs can be countable*, Combinatorica, 5(1985), 137-139.
- [10] G. Hajós, *Über eine Konstruktion nicht  $n$ -färbbarer Graphen*, Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math. Natur. Reihe 10(1961), 116-117.
- [11] R. Häggkvist, P. Hell, D.J. Miller and V. Neumann Lara, *On multiplicative graphs and the product conjecture*, Combinatorica, 8(1988), 71-81.
- [12] S. T. Hedetniemi, *Homomorphisms and graph automata*, University of Michigan Technical Report 03105-44-T, 1966.
- [13] P. Hell and D. J. Miller, *Graphs with forbidden homomorphic images*, Ann. N. Y. Acad. Sci., 319(1979), 270-280.
- [14] P. Hell, H. Zhou and X. Zhu, *Multiplicativity of oriented cycles*, Submitted to J. Comb. Th.
- [15] D. J. Miller, *The categorical product of graphs*, Canada J. Math., 20(1968), 1511-1521.

- [16] S. Poljak and V. Rödl, *On the arc-chromatic number of a digraph*, J. Graph Theory, 3(1981), 339-350.
- [17] J. Nešetřil and A. Pultr, *On classes of relations and graphs determined by subobjects and factorobjects*, Discrete Math. 22 (1978), 287-300.
- [18] D. Thurzík, *A note on chromatic number of direct product of graphs*, Comment Math. Univ. Carolinae, 24(1983), 461-463.
- [19] E. Welzl, *Symmetric graphs and interpretations*, J. Comb. Th. (B), 37(1984), 235-244.
- [20] X. Zhu, *A simple proof of the multiplicativity of directed cycles of prime power length*, Discrete Applied Math., 37(1992).
- [21] X. Zhu, *Multiplicative structures*, Ph.D. thesis, The University of Calgary, 1990.
- [22] X. Zhu, *On the chromatic number of the product of hypergraphs*, Ars Combinatoria, to appear.