Circular Distance Two Labeling and the λ-Number for Outerplanar Graphs

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Abstract

For a graph G, a *circular distance two labeling* with span k is a function $f : V(G) \to \{0, 1, 2, \cdots, k-1\}$ such that: 1) $2 \leq |f(u) - f(v)| \leq k-2$ if u and v are adjacent; and 2) $f(u) \neq f(v)$ if u and v are of distance two apart. For any graph G, let $\lambda_c(G)$ denote the smallest span of a circular distance two labeling for G, and let $\Delta(G)$ denote the maximum degree of G. We prove that, for any outerplanar graph G, $\lambda_c(G) = \Delta(G) + 3$ if $\Delta(G) \geq 15$; and $\lambda_c(G) \leq \Delta(G) + 4$ if $\Delta(G) \geq 11$. It is also shown that there are outerplanar graphs G with

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$\Delta(G) = 2, 3, 4, 5$ for which $\lambda_c(G) \geq \Delta(G) + 4$. Moreover, we prove that $\lambda_c(G) \leq \Delta(G) + 5$ for any triangulated outerplanar graph, and $\lambda_c(G) \leq \Delta(G)+7$ for any outerplanar graph. Immediate consequences of the results include that $\lambda(G) \leq \Delta(G)+2$ for any outerplanar graphs with $\Delta(G) \geq 15$, where $\lambda(G)$ is the minimum k of a k-$L(2,1)$-labeling (or distance two labeling) for G.

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1 Introduction

Introduced by Griggs and Yeh [6], the distance two labeling (or $L(2,1)$-labeling) for graphs is motivated from the channel assignment problem (cf. [7]). The task is to assign channels to some given stations or transmitters so that interference is avoided, and the span of all the channels used is minimized. To model this problem, we construct a graph G by representing each station by a vertex, and connecting two vertices by an edge if the corresponding stations are *close*.

Suppose that we are dealing with two levels of interference – strong and minor. Strong interference occurs between two close stations; to avoid it, the channels assigned to those stations must be separated by at least 2. Minor interference occurs between two stations that share a common close neighbor; to avoid it, channels assigned to such a pair of stations must be different. This motivates the concept of an $L(2,1)$-labeling of a graph G, which is defined to be a function $f : V(G) \to \{0, 1, 2, \cdots\}$ such that the following are satisfied:

- $|f(x) - f(y)| \geq 2$, if $x \sim y$; and

- $f(x) \neq f(y)$, if $d_G(x, y) = 2$.

Here, $d_G(u, v)$ denotes the distance between u and v in G.

An optimal assignment of the channels is to keep the frequency bandwidth as narrow as possible, i.e., to minimize the span – the difference between the maximum channel and the minimum channel assigned. We call an $L(2,1)$-labeling f of G a k-$L(2,1)$-labeling if $\max\{f(x) : x \in V(G)\} = k$. The numbers 0, 1, 2, $\cdots, k$ are called *colors*. We let the *λ-number* of G, $\lambda(G)$, be the minimum integer k such that G admits a k-$L(2,1)$-labeling. A $\lambda(G)$-$L(2,1)$-labeling is called *optimal*.

The $L(2,1)$-labeling has been studied extensively in the past decade; and the research has been focused on deriving upper bounds for $\lambda(G)$ for special classes of graphs G or designing algorithm for finding an optimal $L(2,1)$-labeling for a given graph G. (cf. [2, 3, 6, 4, 5, 9, 14, 15])

The colors $0, 1, \cdots, k$ used in a k-$L(2,1)$-labeling are not symmetric in the following sense: Suppose G admits a k-coloring. Let i, j be distinct colors, then, possibly, there exists some $x \in V(G)$, such that G admits a k-$L(2,1)$-coloring f with $f(x) = i$, but not a k-$L(2,1)$-coloring g with $g(x) = j$. For example, a star $K_{1,n}$ with center x admits an optimal $(n+1)$-$L(2,1)$-labeling, and any $(n+1)$-$L(2,1)$-labeling f that must have $f(x) = 0$ or $f(x) = n+1$. This kind of asymmetry in colors sometimes causes difficulties in discussion.

To avoid such asymmetry of colors in $L(2,1)$-labelings, in this article, we focus on the circular distance two labeling, in which the colors are symmetric. This concept was introduced and used in [10, 11, 12]. A *circular distance two labeling* with span k for a graph G is a function, $f : V(G) \to \{0, 1, 2, \cdots, k-1\}$, such that the following are satisfied:

- $2 \leq |f(x) - f(y)| \leq k - 2$, if $x \sim y$; and
- $f(x) \neq f(y)$, if $d_G(x, y) = 2$.

Let the *circular λ-number* of G, denoted by $\lambda_c(G)$, be the smallest span of a circular distance two labeling of G. It follows from the definition that for any graph G,

$$\lambda(G) + 1 \leq \lambda_c(G) \leq \lambda(G) + 2.$$

Denote by $\Delta(G)$ the maximum degree of G, or Δ when G is clear in the context. It is easy to see that for any graph G, $\lambda(G) \geq \Delta + 1$ and $\lambda_c(G) \geq \Delta + 3$.

It was shown in [6] that for any G, $\lambda(G) \leq \Delta^2 + 2\Delta$. This bound was improved to $\lambda(G) \leq \Delta^2 + \Delta$ by Chang and Kuo [2], where the proof actually shows that $\lambda_c(G) \leq \Delta^2 + \Delta + 1$. A still open conjecture [6] states that $\lambda(G) \leq \Delta^2$ for any graph G.

Determining $\lambda(G)$ is an NP-complete problem, even restricted to special classes of graphs, such as graphs with diameter 2 [6], planar graphs, bipartite graphs, chordal graph, or split graphs (cf. [1]). Research on the parameter $\lambda(G)$ has been concentrated on finding good upper bounds for $\lambda(G)$ for special classes of graphs. Below we list some of the known results on $\lambda(G)$ for some families of graphs (cf. [1]).

Graphs	$\lambda(G)$	**Reference**
Trees	$\Delta + 1$ or $\Delta + 2$	Chang and Kuo[2]
Permutation	$\leq 4\Delta - 2$	Bodlaender et al. [1]
Chrodal	$\leq \frac{1}{4}(\Delta + 3)^2$	Sakai [14]
Diameter two	$\leq \Delta^2$	Griggs and Yeh [6]
Planar	$\leq 2.5\Delta + 90$	Molloy and Salavatipour [13]
Outerplanar (OP)	$\leq \Delta + 8$	[1]
Triangulated OP	$\leq \Delta + 6$	[1]

A graph is an outerplanar graph if it can be embedded in the plane in such a way that all the vertices lie on the infinite face. An outerplanar graph is *triangulated* if each finite face is a triangle.

In [1], the authors proposed the following conjecture:

Conjecture 1 *For any outerplanar graph G, $\lambda(G) \leq \Delta + 2$.*

The main result of this article is:

Theorem 1 *For any outerplanar graph G with $\Delta \geq 15$, $\lambda_c(G) = \Delta + 3$.*

An immediate consequence of Theorem 1 is the confirmation of Conjecture 1 for outerplanar graphs with large maximum degree.

Corollary 2 *For any outerplanar graph G with $\Delta \geq 15$, $\lambda(G) \leq \Delta + 2$.*

We note that the upper bound for $\lambda(G)$ in Corollary 2 is sharp. family of outerplanar graphs.

The condition that $\lambda \geq 15$ in these results seems annoying, however, the presence of the condition is not merely due to the proof techniques. We shall prove that there are outerplanar graphs G with $\Delta(G) = 2, 3, 4, 5$ such that $\lambda_c(G) \geq \Delta(G) + 4$. This shows that in Theorem 1, a condition like $\Delta \geq t$ for some $6 \leq t \leq 15$ is necessary.

In Section 4, we prove that if G is an outerplanar graph with $\Delta \geq 11$, then $\lambda_c(G) \leq \Delta + 4$ (implying $\lambda(G) \leq \Delta + 3$).

2 Structure of Outerplanar Graphs

We introduce some structure properties of outerplanar graphs, which play an important role in the proof of Theorem 1. Some of them are folklore and difficult to find specific reference.

Each outerplanar graph G can be transformed into a triangulated outerplanar graph G_T by adding some edges; G_T is called a *triangulation* of G. For an outerplanar graph G, there may exist many triangulations of G, however, we denote by G_T as an arbitrary but fixed triangulation.

Suppose G is a triangulated outerplanar graph. We define a level function l on $V(G)$ as follows. Initially, choose an edge $e = xy$ on the infinite face, call e the *root edge* and x, y the *root vertices*; and let $l(x) = 1$, $l(y) = 2$. Suppose the levels of some vertices of G have been defined. Choose a vertex u for which $l(u)$ is not defined yet, but u lies on a triangle (u, v, w) such that $l(v), l(w)$ are defined. Let $l(u) = \max\{l(v), l(w)\} + 1$.

As G is triangulated, once the root edge $e = xy$ is chosen, and $l(x)$ and $l(y)$ are fixed, this recursive process induces a unique level function l on $V(G)$.

Note that each non-root vertex u is adjacent to exactly two vertices, say w, v, whose levels are smaller than the level of u. The vertices v, w are called the *parents* of u. Furthermore, if $l(w) > l(v)$, then v is the *major parent* and w is the *minor parent* of u. For a non-root vertex u, we denote $f(u)$, $m(u)$ the major parent and the minor parent of u, respectively. It is clear from the definition that $f(u)$ and $m(u)$ are well-defined, and $l(u) = l(m(u)) + 1$.

By definition, $m(u)$ is adjacent to $f(u)$, for any u. In addition, as $l(f(u)) < l(m(u))$, we conclude that $f(u)$ is a parent of $m(u)$. The following observation can be verified easily:

$$l(m(u)) \geq l(f(u)) + 2 \text{ if and only if } f(u) = f(m(u)).$$

Or equivalently,

$$l(u) - l(f(u)) \geq 3 \text{ if and only if } f(u) = f(m(u)).$$

If v is a parent of u, then u is called a *son* of v. If v is the major (or minor, respectively) parent of u, then u is called a major (or minor, respectively) son of v. If $m(u) = m(u')$, then u and u' are *brothers*. Note that a vertex may have many sons. However, the following lemma shows that each vertex has at most one brother and at most two sons of the same level.

Lemma 3 *Suppose G is a triangulated outerplanar graph and l is the level function defined in the above. Let $v \in V(G)$ and let i be a positive integer. Then v has at most two sons u with $l(u) - l(v) = i$. In particular, v has at most 2 minor sons and at most one brother.*

Proof. We prove by induction on i. First, let $i = 1$. If u is a son of v with $l(u) - l(v) = 1$, then $m(u) = v$. Since v has only two parents, $f(v)$ and $m(v)$, the parents of u have only two possibilities: $\{v, f(v)\}$ or $\{v, m(v)\}$. As G is triangulated outerplanar, it is easy to see that for each pair of non-root adjacent vertices x, y, there is at most one vertex whose parents are x and y. Therefore, there are only two possibilities that u can be generated, completing the case $i = 1$.

Next, assume that v has at most two sons of level $k \geq l(v) + 1$; we need to show that v has at most two sons of level $k + 1$. Let u be a son of v with level $k + 1 \geq l(v) + 2$. Then $v = f(u)$, and $m(u)$ is a son of v of level k. By inductive hypothesis, v has at most two sons of level k. For each level k son v' of v, there is at most one son u of v with $m(u) = v'$. Therefore v has at most two sons of level $k + 1$.

In particular, if u' and u are brothers, then u and u' are both sons of $m(u)$, of level $l(m(u)) + 1$. Thus, by the special case as $i = 1$, each vertex has at most one brother. ∎

If G is a non-triangulated outerplanar graph, then we define the level function on a triangulation G_T of G, and view it as a level function on G. For each vertex u of G, the major (or minor, respectively) parent of u in G_T is called the major (or minor, respectively) parent of u in G. As G is non-triangulated, a vertex u may not be adjacent to its parents.

Suppose $G = (V, E)$ is an outerplanar graph, for which a level function l has been defined. We define an ordering of $V(G)$ as follows: If $l(u) < l(u')$, then u precedes u'. If $l(u) = l(u)$ and $f(u)$ precedes $f(u')$, then u precedes u'. If $l(u) = l(u')$ and $f(u) = f(u')$, then u precedes u' if and only if $m(u)$ precedes $m(u')$. Obviously, such an ordering is uniquely determined by the level function l. For any $v \in V(G)$, the ordering also induces an ordering of the neighbors of v. In the remainder of the paper, the *j-th neighbor* of v is referred to such an ordering.

The following lemma lists some properties of an outerplanar graph with respect to the ordering and the level function l. These properties follow from Lemma 3; we omit the proof, which is straightforward.

Lemma 4 *Suppose v is a vertex of an outerplanar graph, and u is the sth neighbor of v.*

1. *If $s \geq 3$, then u is a son of v.*
2. *If $s \geq 5$, then $f(u) = v$.*
3. *If $s \geq 7$, then $f(m(u)) = f(u) = v$.*
4. *If $s \geq 9$, then $f(m(m(u))) = f(m(u)) = f(u)$.*
5. *If u' is the jth neighbor of v and $u \sim u'$, then $|j - s| \leq 2$.*

3 The Coloring Scheme: Proof of Theorem 1

The star $K_{1,\Delta}$ has $\lambda_c(K_{1,\Delta}) = \Delta+3$. This implies that $\lambda_c(G) \geq \Delta+3$ for any graph G of maximum degree Δ. Hence, to prove Theorem 1, it suffices to find a circular distance two labeling for G with colors from $\{0, 1, \cdots, \Delta+2\}$.

Let $V(G) = \{v_1, v_2, \cdots, v_n\}$, where the ordering is the one described in the previous section (induced from the level function l). We define a coloring scheme, $\phi : V \to \{0, 1, \cdots, \Delta + 2\}$, sequentially by this ordering of V.

The *colors* $\{0, 1, 2, \cdots, \Delta + 2\}$ are regarded *cyclically* – like colors on a circular palette – and all calculations are taken under modular $(\Delta+3)$. Let C be a subset of colors on the palette. A *segment* of C is a maximal interval of consecutive colors of C, i.e., a set I of colors of the form $I = \{j, j+1, \cdots, l\}$ such that $I \subset C$ and $j-1, l+1 \notin C$. The colors between two consecutive segments is called a *gap* of C. As we are working on a circular color palette (i.e. modular $(\Delta + 3)$), the number of gaps is the same as the number of segments.

An *attaching color* of C is a color j such that $j+1 \in C$ or $j-1 \in C$. A *filling* color of C is a color j such that $j \pm 1 \in C$ (i.e., $j+1 \in C$ and $j-1 \in C$). Denote by $A(C)$ and $F(C)$, respectively, the set of attaching colors and the set of filling colors of C.

Below, we introduce some notations used in the coloring scheme. Suppose we have colored all the vertices v_j for $j < t$, and we are about to color v_t. For $i \leq t$, let $\phi(v_i)$ be the color of v_i. Let w be a colored vertex. Set

$$C(w,t) = \{\phi(w), \phi(w)+1, \phi(w)-1\} \cup \{\phi(v_j) : j < t, v_j \sim w\};$$

$$C[w,t] = \{\phi(w), \phi(w)+1, \phi(w)-1\} \cup \{\phi(v_j) : j \leq t, v_j \sim w\};$$

$$C^*(v_t) = C(f(v_t), t);\ C^*[v_t] = C[f(v_t), t] \text{ and}$$

$$s(v_t) = |C^*[f(v_t), t]|.$$

According to the ordering of V, v_t has at most two colored neighbors, $m(v_t)$ and $f(v_t)$. If v_t is adjacent to both $m(v_t)$ and $f(v_t)$, then the colors in the set $C(f(v_t), t) \cup C(m(v_t), t)$ are forbidden for v_t; while other colors are legal. Because the total number of colors is $\Delta + 3$, so if v_t is adjacent to $f(v_t)$ and $m(v_t)$, then there is a legal color for v_t if and only if $|C^*(v_t) \cup C(m(v_t)), t)| \leq \Delta + 2$. Hence, the objective of the coloring scheme is to keep the number of forbidden colors for v_t to be at most $\Delta + 2$, for each t.

To this end, we define a sequential coloring scheme on $\{v_1, v_2, \cdots, v_n\}$ using colors $\{0, 1, 2, \cdots, \Delta+2\}$ such that, at each step t, the following four properties are maintained. In what follows, we let $w = f(v_t)$, and let β be the number of segments in $C^*[v_t]$. If w has degree Δ, then let u be its Δth neighbor; otherwise u does not exist and we simply ignore the parts involving u.

R1. If $w \not\sim m(u)$ or $s(v_t) \leq 12$, then $\beta \leq 5$; otherwise $\beta \leq 6$.

R2. If $s(v_t) \geq 8$, then $\phi(v_t) \in C^*[v_t]$.

R3. If $s(v_t) \geq 14$, then $\phi(v_t) \in C^*[v_t] \cap A(C^*[v_t])$.

R4. Assume w has degree Δ (i.e. u exists).
If $s(v_t) \geq 13$ and v_t precedes $m(u)$ in the ordering of V,
then there exists some $j^* \in F(C^*[v_t])$, which is legal for $m(u)$.
Moreover, if $w \not\sim m(u)$, then

$$j^* \in C^*[v_t] \cap F(C^*[v_t]) \quad \text{and} \quad j^* \neq \phi(w).$$

We proceed the proof by induction. Initially, let $\phi(v_1) = 0$, $\phi(v_2) = 2$ (note, v_1, v_2 are root vertices); so R1 is true while R2 - R4 are vacuous. Assume that R1 - R4 are satisfied after each $\phi(v_j)$ is defined, for any $j < t$; we shall prove that there is a legal color for v_t so that R1 - R4 are maintained. The following fact will be useful in the proof.

Claim 1 *If $f(m(v_t)) = f(v_t)$, then*

$$C(m(v_t), t) \subset \{\phi(m(v_t)), \phi(m(v_t)) \pm 1, \phi(f(m(v_t)), \phi(m(m(v_t))\}.$$

Proof. It suffices to show that $f(m(v_t)), m(m(v_t))$ are the only possible neighbors of $m(v_t)$ proceeding v_t. Assume to the contrary, there exists some $v_j \sim m(v_t)$, $j < t$, and $v_j \notin \{f(m(v_t)), m(m(v_t))\}$. Then v_j is a son of $m(v_t)$.

Since $l(v_t) - l(m(v_t)) = 1$, and $j < t$, it must be that $l(v_j) - l(m(v_t)) = 1$. Thus, $m(v_j) = m(v_t)$ and $l(v_j) = l(v_t)$, so v_j is a brother of v_t. Because $f(v_t) = f(m(v_t))$, one has $f(v_j) = m(m(v_t))$. This implies that $l(f(v_t)) < l(f(v_j))$, so v_t precedes v_j in the ordering, contrary to our assumption. ∎

Note that if $s(v_t) \geq 10$, then it follows from Lemma 4 that $f(m(v_t)) = f(v_t)$, and hence the conclusion of Claim 1 holds.

If $s(v_t) \leq 7$, then R2 - R4 are vacuous. Moreover, since $C^*[v_t]$ contains three consecutive colors, $\phi(w)$, $\phi(w) \pm 1$, it follows that $\beta \leq 5$, so R1 holds.

Assume $s(v_t) \geq 8$. We consider two cases.

Case 1. $v_t \not\sim w$

Let t' be the largest index such that $t' < t$ and $v_{t'} \sim w$. Then $v_{t'}$ is the jth neighbor of w, where $j = s(v_t) - 3 \geq 5$. By (2) of Lemma 4, $w = f(v_{t'})$ and hence $C^*(v_t) = C^*[v_t] = C^*[v_{t'}]$. By inductive hypothesis, R1 holds.

The forbidden colors for v_t are included in the set

$$C' = \{\phi(m(v_t)), \phi(m(v_t)) \pm 1, \phi(m(m(v_t))), \phi(f(m(v_t))), \phi(x)\},$$

where x is a possible already colored brother of v_t. So, there are at most 6 forbidden colors for v_t. Note that $\phi(f(v_t))$ might be a forbidden color for v_t, however, as $f(v_t)$ is a parent of $m(v_t)$, i.e., $f(v_t)$ is either $m(m(v_t))$ or $f(m(v_t))$, one concludes $\phi(f(v_t)) \in C'$.

Assume $8 \leq s(v_t) \leq 12$. Then R3, R4 are vacuous, and $|C^*(v_t)| - |C(m(v_t), t)| \geq 8 - 6 = 2$. Color v_t with a color $j \in C^*(v_t) - C(m(v_t), t)$, then R2 holds.

Assume $s(v_t) = 13$. By inductive hypothesis for t', there exists a filling color j^* of $C^*[v_{t'}] = C^*[v_t]$ which is legal for $m(u)$. Because $s(v_t) = 13$, by Claim 1, v_t has at most 5 forbidden colors. Hence, there exists some $j \in C^*(v_t) - \{j^*, j^* \pm 1\}$ and j is legal for v_t. Let $\phi(v_t) = j$. Then j^* is still legal for $m(u)$. So R2 - R4 hold.

Assume $s(v_t) \geq 14$. By Claim 1, v_t has at most 5 forbidden colors. By inductive hypothesis, $C^*[v_t]$ has at most 6 segments and at most 5 of them are singletons. It follows from definition that $C^*[v_t] - A(C^*[v_t])$ are the singleton segments of $C^*[v_t]$. Therefore $|C^*[v_t] \cap A(C^*[v_t])| \geq |C^*[v_t]| - 5 \geq 9$, implying

$$|C^*[v_t] \cap A(C^*[v_t]) - \{j^*, j^* \pm 1\}| \geq 6.$$

Hence, there exists some $j \in C^*[v_t] \cap A(C^*[v_t]) - \{j^*, j^* \pm 1\}$ and j is legal for v_t. Let $\phi(v_t) = j$. Then j^* is kept legal for $m(u)$, and R2 - R4 hold. This completes the proof of Case 1.

Case 2. $v_t \sim w$

Then $s(v_t) = |C^*[v_t]| = |C^*(v_t)| + 1$. By definition, R2 holds.

Denote by Q the set of forbidden colors of v_t, excluding the ones in $C^*(v_t)$. Thus $Q \subseteq C(m(v_t), t) - \{\phi(f(v_t))\}$, and $|Q| \leq 5$.

Assume $8 \leq s(v_t) \leq 12$. We only need to verify R1. It suffices to show that if $C^*(v_t)$ has 5 segments, then $C^*[v_t]$ has at most 5 segments. Assume $C^*(v_t)$ has 5 segments, and hence has 5 gaps. Each gap contains at least 1 element from the set $A(C^*(v_t)) - C^*(v_t)$. Because $|C^*(v_t)| \leq 11$, and the total number of colors is $\Delta + 3 \geq 18$, there is at least one gap which contains two elements from the set $A(C^*(v_t)) - C^*(v_t)$. Hence, we have $|A(C^*(v_t)) - C^*(v_t)| \geq 6$, and $A(C^*(v_t)) - C^*(v_t) - Q \neq \emptyset$ (as $|Q| \leq 5$). Let $j \in A(C^*(v_t)) - C^*(v_t) - Q$. Then j is a legal color for v_t. Let $\phi(v_t) = j$. As $j \in A(C^*(v_t))$, the number of segments is not increased, so R1 holds.

Assume $s(v_t) \geq 13$. Because $s(m(v_t)) \geq 11$ and $s(m(m(v_t))) \geq 9$, by inductive hypothesis, we have $\phi(m(v_t))$, $\phi(m(m(v_t))) \in C^*(v_t)$. By Lemma 4, $f(m(v_t)) = w$. Thus by Claim 1,

$$Q \subseteq \{\phi(m(v_t)) \pm 1\}. \qquad (*)$$

So $|Q| \leq 2$. Moreover, if $|Q| = 2$, then the two colors of Q are in special positions, namely, they are at the two sides of a color, $\phi(m(v_t))$, in $C^*(v_t)$.

Assume $s(v_t) = 13$. Note that if u exists, then $v_t \not\sim m(u)$ (for if u exists, then $s(m(u)) \geq 16$ as $s(u) = \Delta + 3 \geq 18$). We discuss two sub-cases separately:

Case 2.A. $s(v_t) = 13$ and, $m(u) \not\sim w$ or u does not exist

First, we define $\phi(v_t)$. As $|C^*(v_t)| = 12$, by inductive hypothesis, $C^*(v_t)$ contains at most 5 segments. Since the total number of colors is at least 18, and because of the special positions of the two colors in Q (if $|Q| = 2$), one can easily verify that there exists $j \in A(C^*(v_t)) - C^*(v_t) - Q$. Let $\phi(v_t) = j$. Then $\beta \leq 5$, and R1 is true.

Secondly, assume u exists. We need to find some $j^* \in F(C^*[v_t]) \cap C^*[v_t]$, $j^* \neq \phi(w)$, and j^* is legal for $m(u)$. Since $C^*[v_t]$ has at most 5 segments, so $|C^*[v_t] \cap F(C^*[v_t])| \geq 3$ (as $|C^*[v_t]| = 13$ and only the two ends of each segment are not filling colors). Let

$$j^* \in C^*[v_t] \cap F(C^*[v_t]) - \{\phi(w), \phi(m(m(m(u)))\}.$$

Note that: 1) $m(m(m(u))$ might be distance two away from $m(u)$ and has been colored already; 2) $m(m(v_t))$ is not colored yet, at this step, as $s(m(m(v_t)) \geq 14$; 3) $m(u) \not\sim v_t$. Therefore, j^* is legal for $m(u)$, and R4 is satisfied.

Case 2.B. $s(v_t) = 13$ and $m(u) \sim w$

We need to find $\phi(v_t)$, and $j^* \in F(C^*[v_t])$ such that j^* is legal for $m(u)$.

Suppose there exists some $i \in F(C^*(v_t)) - C^*(v_t)$. Let $j^* = i$. We then need to assign a color to v_t so that j^* is kept legal for $m(u)$. By (*) and the facts that $|C^*(v_t)| = 12$ and the total number of colors is at least 18, there exists $j \in A(C^*(v_t)) - C^*(v_t) - \{j^*\} - Q$. Let $\phi(v_t) = j$. Then R1 and R4 hold, as $v_t \not\sim m(u)$.

Assume that $F(C^*(v_t)) \subseteq C^*(v_t)$. Then every gap in $C^*(v_t)$ has at least two elements. By (*) and the facts that $|C^*(v_t)| = 12$ and there are at least 18 colors, one can find a gap $\{k, k+1, \cdots, k+i\}$ of $C^*(v_t)$ such that $i \geq 1$ and $k+1$ is legal for v_t (recall that besides $C^*(v_t)$, v_t has at most two forbidden colors which are in special positions). Let $j^* = k$ and $\phi(v_t) = k+1$. Then j^* satisfies the requirements of R4. Moreover, $C^*[v_t]$ contains at most 6 segments. So R1 and R4 hold. This completes the proof for the case $s(v_t) = 13$.

Next, assume $s(v_t) = 14$ or 15. Then $|C^*(v_t)| = 13, 14$. We need to find a legal color j for v_t such that $j \in A(C^*(v_t)) - C^*(v_t)$ and j^* is kept legal for $m(u)$.

If $m(u) \sim w$, then $\{j^*\}$ is a gap in $C^*(v_t)$ (see Case 2.B above), and hence $C^*(v_t)$ has at least two gaps. If $C^*(v_t)$ has more than two gaps, then $|A(C^*(v_t)) - C^*(v_t)| \geq 4$ and hence $A(C^*(v_t)) - C^*(v_t) - Q - \{j^*\} \neq \emptyset$. If $C^*(v_t)$ has exactly two gaps, then the gap other than $\{j^*\}$ must contain at least 3 elements. It is impossible that $\phi(m(v_t)) \pm 1$ be the two ends of this gap, as $\phi(m(v_t)) \in C^*(v_t)$. Therefore, there exists some $j \in A(C^*(v_t)) - C^*(v_t) - Q - \{j^*\}$. Let $\phi(v_t) = j$. As $j^* \pm 1 \in C^*(v_t)$, j^* is still legal for $m(u)$. So R1, R3 and R4 hold.

If $m(u) \not\sim w$, then $j^*, j^* \pm 1 \in C^*(v_t)$ (see Case 2.A above). Again, since $|Q| \leq 2$ and due to the special positions of Q (if $|Q| = 2$), there exists $j \in A(C^*(v_t)) - C^*(v_t) - Q$. Let $\phi(v_t) = j$. Again, R1, R3 and R4 hold.

Note that if u exists, since $v_t \sim w$, it must be either $v_t = u$ or v_t proceeds u, hence $s(v_t) \leq \Delta + 2$ if $v_t \neq u$; and $s(v_t) = \Delta + 3$ only occurs when $v_t = u$.

Assume $16 \leq s(v_t) \leq \Delta+2$. Then $s(m(v_t)) \geq 14$. By inductive hypothesis and R3, $\phi(m(v_t)) \in A(C^*(v_t)) \cap C^*(v_t)$, so $|Q| \leq 1$.

Suppose $v_t \neq m(u)$ and v_t proceeds $m(u)$. If $s(v_t) \leq \Delta + 1$ (i.e. $15 \leq |C^*(v_t)| \leq \Delta$), then there exists $j \in A(C^*(v_t)) - C^*(v_t) - \{j^*\} - Q$, as the total number of colors is at least $\Delta + 3$, and $|Q| \leq 1$. Let $\phi(v_t) = j$. If $s(v_t) = \Delta + 2$, then $m(u) \not\sim w$ (as v_t proceeds $m(u)$), and hence $j^* \in C^*(v_t)$. So, there exists $j \in A(C^*(v_t)) - C^*(v_t) - Q$. Let $\phi(v_t) = j$. In addition, one can easily see that in any of these two situations ($s(v_t) = \Delta+1$ or $\Delta+2$), after $\phi(v_t) = j$ is defined, the color j^* is kept legal for $m(u)$, as $j^* \pm 1 \in C^*(v_t)$. So R1, R3, R4 hold.

If $v_t = m(u)$, let $\phi(v_t) = j^*$. If $m(u)$ proceeds v_t, then $s(v_t) = \Delta + 2$, $|C^*(v_t)| = \Delta + 1$, and $j^* \in C^*(v_t)$. As $|Q| \leq 1$, there exists $j \in A(C^*(v_t)) - C^*(v_t) - Q$. Let $\phi(v_t) = j$. Then R1, R3 and R4 are true.

Assume $s(v_t) = \Delta + 3$, then $v_t = u$. By inductive hypothesis, $\phi(m(v_t)) = j^* \in F(C^*(v_t)) \cap C^*(v_t)$, implying $Q = \emptyset$. Hence, all the forbidden colors for v_t are included in $C^*(v_t)$. As $|C^*(v_t)| = \Delta + 2$, we conclude that there is a legal color for v_t. This completes the proof of the correctness of the coloring scheme.

4 Outerplanar Graphs with Small Maximum Degrees

For outerplanar graphs with small maximum degrees, the equality of Theorem 1 does not always hold.

Theorem 5 *Let G_1, G_2, G_3, G_4 be the graphs as shown in Figure 1 above (ignore the labels of vertices of G_4 at the moment). Then $\Delta(G_i) = i+1$, and $\lambda_c(G_i) \geq \Delta(G_i) + 4$.*

Proof. We leave it to the readers to verify that for G_1, G_2 and G_3, $\lambda_c(G_i) \geq \Delta(G_i) + 4$.

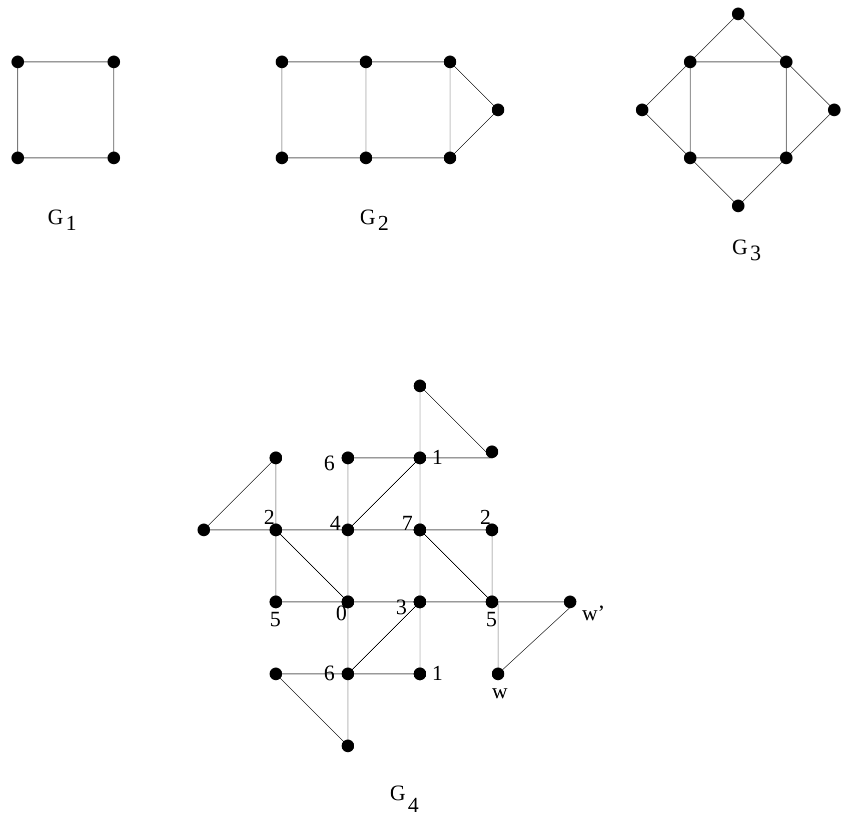

Figure 1: Graphs G_1, G_2, G_3, G_4

For graph G_4, we first consider the subgraph H depicted in Figure 2. We prove that $\lambda_c(H) = \Delta + 3$, and there is only one (up to symmetry) optimal circular distance two labeling for H.

Let f be an optimal labeling for H, $f : V(H) \to \{0, 1, 2, \cdots, 7\}$. For any edge $e = uv \in E(H)$, define the weight $w(uv)$ to be the circular difference between $f(u)$ and $f(v)$ modular 8, that is,

$$w(uv) = \min\{|f(u) - f(v)|,\ 8 - |f(u) - f(v)|\}.$$

Note that, for any $e \in E(H)$, we have $2 \leq w(e) \leq 4$. Let e_1, e_2, e_3, e_4 be the edges as shown in Figure 2(a).

Claim 2 *There exists an $i \in \{1,2,3,4\}$, such that $w(e_i) = 2$.*

Proof. Assume to the contrary that $w(e_i) = 3, 4$ for $i = 1, 2, 3, 4$. Due to the symmetry of the colors and the symmetric structure of H, we assume, without loss of generality, $f(v_1) = 0$, and $f(u_1) = 3$ or 4. If $f(u_1) = 3$, referring to Fig. 2(a), the colors assigned to x and v_2 must be distinct, and have circular difference at least 2 to both 0 and 3. The only possibility is that $\{f(x), f(v_2)\} = \{5, 6\}$. This forces that $\{f(v_4), f(u_4)\} = \{2, 4\}$, and hence $w(e_4) = 2$, contradicting the hypothesis. The argument for the case $f(u_1) = 4$ is similar. ∎

Thus, without loss of generality, assume $f(v_1) = 0$ and $f(u_1) = 2$. As the neighbors of v_1 in H are in $\{v_2, u_1, x, u_4, v_4\}$, one has $\{f(v_2), f(x), f(u_4), f(v_4)\}$ $= \{3, 4, 5, 6\}$. Note that the circular difference between $f(v_4)$ and $f(u_4)$ must be at least 2. Thus, there are the following eight possibilities:

	$f(x)$	$f(v_2)$	$f(v_4)$	$f(u_4)$
Case 1	4	6	3	5
Case 2	4	6	5	3
Case 3	6	4	3	5
Case 4	6	4	5	3
Case 5	4	5	3	6
Case 6	4	5	6	3
Case 7	5	4	6	3
Case 8	5	4	3	6

It is a routine to verify that the coloring in Cases 1 - 7 cannot be extended to the whole graph H. We work out Cases 1 and 5, and leave the others to the reader, as the arguments are similar.

For Case 1, the colors of the vertices are as shown in Figure 2(ii). By considering the neighbors of v_2 (which is colored by 6), we conclude that $\{a, b, c\} = \{1, 3, 4\}$. By considering the neighbors of v_4 (which is colored by

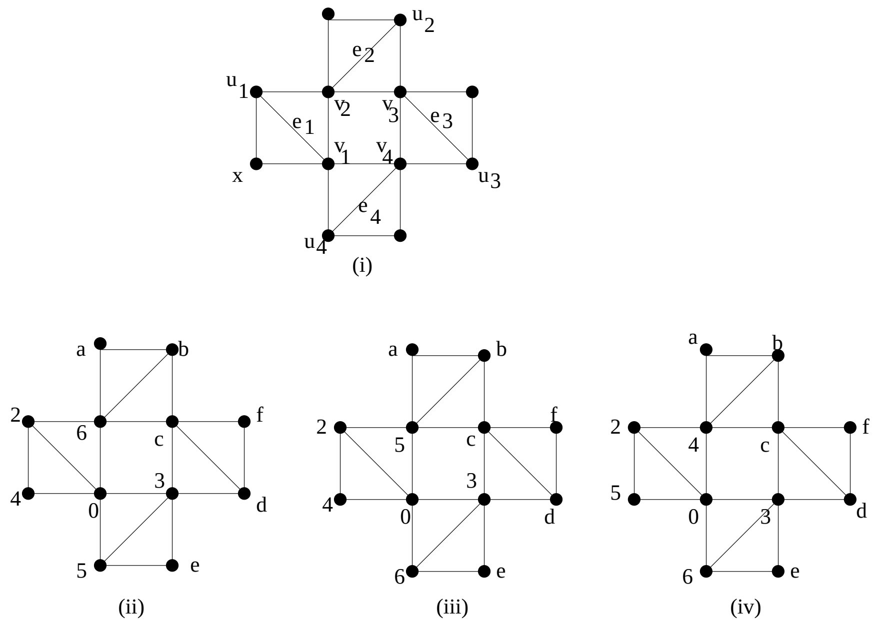

Figure 2: The subgraph H of G_4 and some possible labelings

3), we conclude that $\{c, d, e\} = \{1, 6, 7\}$. So $c = 1$ and $\{a, b\} = \{3, 4\}$. This is illegal, as $\{3, 4\}$ can not be assigned to adjacent vertices.

For Case 5, the colors of the vertices are as shown in Figure 2(iii). By considering the neighbors of v_4 (which is colored by 3), we conclude that $\{c, d, e\} = \{1, 5, 7\}$. This is impossible as none of c, d and e could be 5 (since e is adjacent to a vertex colored by 6, c is adjacent to a vertex colored by 5, and d has distance 2 to a vertex colored by 5).

Next we show that Case 8 extends to a unique coloring for H. The colors of the vertices are as shown in Figure 2(iv). By considering the neighbors of v_2 (which is colored by 4), we conclude that $\{a, b, c\} = \{1, 6, 7\}$. By considering the neighbors of v_4 (which is colored by 3), we conclude that $\{c, d, e\} = \{1, 5, 7\}$. But $c \neq 1$, for otherwise $\{a, b\} = \{6, 7\}$ which is impossible for adjacent vertices. Thus $c = 7$. This forces $a = 6, b = 1, e = 1, d = 5, f = 2$, which gives the coloring as shown in the center part of G_4.

If there is a labeling for G_4 with span 8, then this labeling must be an extension of the labels as depicted in Figure 1. However, it is easy to see that the coloring cannot be extended to vertices w, w'. Hence, $\lambda_c(G_4) > \Delta + 3$. ∎

With the modification of the coloring scheme used in the proof of Theorem 1, by deleting the filling colors, we obtain an upper bound of $\lambda_c(G)$ for outerplanar graphs with maximum degree at least 11.

Theorem 6 *For any outerplanar graph G with maximum degree $\Delta \geq 11$, $\lambda_c(G) \leq \Delta + 4$.*

Proof. Let G be an outerplanar graph with $\Delta \geq 11$. Similar to the proof of Theorem 1, we give a sequential coloring scheme, on the ordering of $V(G)$, using colors from $\{0, 1, 2, \cdots, \Delta + 3\}$. We prove that the following three properties are maintained in each step:

> **R1.** $\beta \leq 5$.
>
> **R2.** If $s(v_t) \geq 8$, then $\phi(v_t) \in C^*[v_t]$.
>
> **R3.** If $s(v_t) \geq 12$, then $\phi(v_t) \in A(C^*[v_t]) \cap C^*[v_t]$

Case 1. $v_t \not\sim w$

By inductive hypothesis, R1 holds.

As v_t has at most 6 forbidden colors (cf. Case 1 in the previous section), there exists $j \in C^*(v_t) = C^*[v_t]$ such that j is legal for v_t, if $s(v_t) \geq 8$. So R2 is true.

If $s(v_t) \geq 12$, then by inductive hypothesis and R1, $C^*(v_t)$ has at most 5 segments, so $|A(C^*(v_t)) \cap C^*(v_t)| \geq 8$. As v_t has at most 6 forbidden colors, there exists $j \in A(C^*(v_t)) \cap C^*(v_t)$ which is legal for v_t. So R3 holds.

Case 2. $v_t \sim w$

Then R2 holds automatically.

Assume $C^*[v_t] \geq 8$. Note that if $|C^*(v_t)| = 7$ then $C^*(v_t)$ has at most 5 segments, as $\{\phi(w), \phi(w) \pm 1\} \subseteq C^*(v_t)$.

Let Q be the set of forbidden color of v_t not contained in $C^*(v_t)$ (same as defined in Case 2 of the previous section).

If $s(v_t) = 8, 9$, then $|C^*(v_t)| = 7, 8$, and

$$Q \subseteq \{\phi(m(v_t)), \phi(m(v_t)) \pm 1, \phi(m(m(v_t))), \phi(x)\},$$

where x is a possible (already) colored brother of v_t. Because there are at least 14 colors, there exists some $j \notin C^*(v_t) \cup Q$. If $C^*(v_t)$ contains less than 5 segments, let $\phi(v_t) = j$. Suppose $C^*(v_t)$ contains 5 segments (so 5 gaps). Then $|A(C^*(v_t)) - C^*(v_t)| \geq 6$, as there are at least 14 colors. Since $|Q| \leq 5$, there exists $j \in A(C^*(v_t)) - C^*(v_t) - Q$. Let $\phi(v_t) = j$. Then R1 holds (R3 is vacuous).

If $s(v_t) = 10, 11$, then $s(m(v_t)) \geq 8$. By inductive hypothesis and Claim 1 in the previous section,

$$Q \subseteq \{\phi(m(v_t)) \pm 1, \phi(m(m(v_t)))\}.$$

So $|Q| \leq 3$. By a similar argument for $s(v_t) = 7, 8$ in the above, and the fact that $|A(C^*(v_t)) - C^*(v_t)| \geq 5$ (if $C^*(v_t)$ contains 5 segments), one can show that R1 hold (R3 is vacuous).

Assume $12 \leq s(v_t) \leq \Delta + 2$. Then $|C^*(v_t)| \leq \Delta + 1$, $s(m(v_t)) \geq 10$, and $s(m(m(v_t))) \geq 8$. By inductive hypothesis, $\phi(m(v_t))$, $\phi(m(m(v_t))) \in C^*(v_t)$. Hence by Claim 1,

$$Q \subseteq \{\phi(m(v_t)) \pm 1\}.$$

So $|Q| \leq 2$. Note that if $|Q| = 2$, then the two colors of Q are in special positions, i.e., they are at the two sides of a color, $\phi(m(v_t))$, in $C^*(v_t)$. As there are at least $\Delta+4$ colors, one can find some $j \in A(C^*(v_t)) - C^*(v_t) - Q$. (This can be easily seen if $C^*(v_t)$ contains only one segments; if $C^*(v_t)$ contains more than one segment, then $|A(C^*(v_t)) - C^*(v_t)| \geq 3$.) Let $\phi(v_t) = j$. Then R1 and R3 hold.

Assume $s(v_t) = \Delta + 3 \geq 14$. Then $s(m(v_t)) \geq 12$. By R3, $\phi(m(v_t))$ is an attaching color of $C^*(v_t)$, implying $|Q| = 1$. Thus, one can find a legal color for v_t, as $|C^*(v_t) \cup Q| \leq \Delta + 3$ and there are at least $\Delta+4$ colors. This completes the proof, as $s(v_t) \leq \Delta + 3$ for all t. ∎

For any outerplanar graph, we have the following general result:

Theorem 7 *For any triangulated outerplanar graph G, $\lambda_c(G) \leq \Delta(G) + 5$. For any outerplanar graph, $\lambda_c(G) \leq \Delta(G) + 7$.*

Proof. Suppose $G = (V, E)$ is a triangulated outerplanar graph. Let $v_1, v_2, \cdots, v_n$ be an ordering of V as defined in Section 2.

We color the vertices of G sequentially, by the ordering $v_1, v_2, \cdots, v_j$, except the ordering for a pair of brothers might be reversed.

We prove by induction that one can color the vertices of G with colors $\{0, 1, \cdots, \Delta + 4\}$ in such a way that, at each step, the following hold: (We adopt the same notations from Section 3.)

R1. $\phi(v_t)$ is an attaching color of either $C(f(v_t), t)$ or $C(m(v_t), t)$.

Initially, let $\phi(v_1) = 0$, $\phi(v_2) = 2$, $\phi(v_3) = 4$, so R1 is true. Suppose we want to color a vertex v_t. Let $F(v_t)$ be the set of possible forbidden colors for v_t. If v_t has no brother that is colored already, then

$$F(v_t) = C(f(v_t), t) \cup C(m(v_t), t) = C(f(v_t), t) \cup \{\phi(m(v_t)) \pm 1\},$$

since all the colored neighbors of $m(v_t)$, including $m(v_t)$ itself, are colored neighbors of $f(v_t)$. Thus $|F(v_t)| \leq \Delta + 4$. Hence, there exists a color j which is attaching to $F(v_t)$ and is legal for v_t. Color v_t by such a color, so R1 is true.

Now assume v_t has a brother x, and both v_t and x are not colored yet; while all the vertices with levels lower than v_t and x (note that $l(x) = l(v_t)$) are colored. The priority of coloring v_t and x is determined by the following: Let $m(x) = m(v_t) = v_j$. Without loss of generality, assume that $f(x) = f(m(v_t)) = f(v_j)$ and $f(v_t) = m(m(v_t)) = m(v_j)$. By inductive hypothesis, $\phi(v_j)$ is attaching to either $C(m(v_j), j)$ or $C(f(v_j), j)$. Assume $\phi(v_j)$ is attaching to $C(m(v_j), j)$ (the other case is similar, by reversing the coloring order of x and v_t). We color x before v_t, by a legal color attaching to $F(x)$ (by the previous paragraph).

Now we find a legal color for v_t. Because $\phi(v_j)$ is attaching to $C(m(v_j), j)$, and since $m(v_j) = f(v_t)$, so $F(v_t)$ contains $C(f(v_t), t) \cup \{\phi(x)\}$ and at most one of $\phi(v_j) \pm 1$. Hence, $|F(v_t)| \leq \Delta + 4$, and so there is a legal color for v_t that is attaching to $F(v_t)$. This completes the proof for the first part of the theorem.

The second part can be proved similarly, by using a triangulation on G; with the only exception that a vertex u may not be adjacent to $m(u)$ and/or $f(u)$, resulting in the extra 2 colors needed in the bound. ∎

Corollary 8

$$\lambda(G) \leq \begin{cases} \Delta + 3, & \textit{if } G \textit{ is outerplanar with } \Delta(G) \geq 10; \\ \Delta + 6, & \textit{if } G \textit{ is outerplanar;} \\ \Delta + 4, & \textit{if } G \textit{ is triangulated outerplanar.} \end{cases}$$

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