A Characterization of \((X, kX)\)-Intersection Graphs

Leizhen Cai
Department of Computer Science and Engineering
The Chinese University of Hong Kong
Shatin, Hong Kong, China
lcai@cs.cuhk.edu.hk

Xuding Zhu
Department of Applied Mathematics
National Sun Yat-sen University
Kaohsing, Taiwan
zhu@math.nsysu.edu.tw

September 7, 2000

Abstract

For two fixed graphs \(X\) and \(Y\), the \((X, Y)\)-intersection graph of a graph \(G\) is a graph whose vertices are induced subgraphs of \(G\) isomorphic to \(Y\) and where two vertices are adjacent if their intersection in \(G\) contains an induced subgraph isomorphic to \(X\). A conformal \(k\)-graph is a simple hypergraph whose hyperedges are exactly \(k\)-cliques in the 2-section of the hypergraph.

Let \(kX\) denote the disjoint union of \(k\) copies of \(X\). We show that for any integer \(k \geq 2\) and connected graph \(X\) with no bipartite blocks, the family of \((X, kX)\)-intersection graphs coincides with the family of line graphs of conformal \(k\)-graphs. On the other hand, we obtain a Ramsey type result on vertex splitting and use it to prove that for any connected bipartite graph \(X\) with at least two vertices, the family of \((X, kX)\)-intersection graphs is strictly contained in the family of line graphs of conformal \(k\)-graphs.

*This work was partially supported by an Earmarked Research Grant from the Research Grants Council of Hong Kong.
1 Introduction

To represent the intersection of various induced subgraphs in a graph, Cai, Cornil and Proskurowski [2] introduced the following notion of \((X,Y)\)-intersection graphs. Let \((X,Y)\) be a pair of fixed graphs. For a graph \(G\), any induced subgraph in \(G\) isomorphic to a given graph \(G'\) is an induced \(G'\)-subgraph. The \((X,Y)\)-intersection graph of a graph \(G\), denoted \(I_{X,Y}(G)\), is a graph where

1. each vertex corresponds to a distinct induced \(Y\)-subgraph in \(G\), and
2. two vertices are adjacent iff the intersection of their corresponding induced \(Y\)-subgraphs contains an induced \(X\)-subgraph.

Various classes of graphs in the literature are special \((X,Y)\)-intersection graphs. The line graph \(L(G)\) of \(G\) is precisely the \((K_1,K_2)\)-intersection graph of \(G\), the edge intersection graph of triangles \(H_1(G)\) [11] of \(G\) is exactly the \((K_2,K_3)\)-intersection graph of \(G\), and the \(k\)-line graph [8] (also known as \(K_k\)-intersection graph [5] and \(k\)th interchange graph [6]) of \(G\) is the the same as the \((K_k-1,K_k)\)-intersection graph of \(G\). Furthermore, a strict \(G'\)-factor [7] of \(G\) corresponds to an independent set in the \((K_1,G')\)-intersection graph of \(G\).

As with line graphs, a natural question concerning \((X,Y)\)-intersection graphs is to characterize all graphs that are \((X,Y)\)-intersection graphs. It was shown in [2] that \((X,Y)\)-intersection graphs are line graphs of \(k\)-uniform hypergraphs if \(Y\) contains exactly \(k\) induced \(X\)-subgraphs, and line graphs of simple \(k\)-uniform hypergraphs if, in addition, these \(k\) induced \(X\)-subgraphs contain all vertices of \(Y\). This raises the issue of determining \((X,Y)\) for which the family of \((X,Y)\)-intersection graphs coincides with the family of line graphs of \((simple) k\)-uniform hypergraphs.

The case \(k = 2\) was studied in [1, 2]. The issue there was to characterize \((X,Y)\) for which the family of \((X,Y)\)-intersection graphs equals the family of line graphs or the family of line graphs of multigraphs. It was shown in [2] that for such \((X,Y)\) the two induced \(X\)-subgraphs inside \(Y\) must be connected in a highly symmetric manner. Furthermore, if these two induced \(X\)-subgraphs contain all vertices of \(Y\), the family of \((X,Y)\)-intersection graphs admits a forbidden induced subgraph characterization only when it coincides with the family of line graphs. For \(k \geq 3\), there are \((X,Y)\) pairs for which the family of \((X,Y)\)-intersection graphs coincides with the family of line graphs of \(k\)-uniform hypergraphs. However, it has been shown recently in [3] that no \((X,Y)\) pair makes the family of \((X,Y)\)-
intersection graphs equal to the family of line graphs of simple $k$-uniform hypergraphs, which is quite different from the case $k = 2$.

In this paper, we study $(X, kX)$-intersection graphs, where $kX$ denotes the disjoint union of $k$ copies of $X$. As observed in [2], for any $(X, Y)$, the family of $(X, Y)$-intersection graphs is exactly the same as the family of $(\overline{X}, \overline{Y})$-intersection graphs, where $\overline{X}$ and $\overline{Y}$, respectively, are the complement graphs of $X$ and $Y$. Therefore the family of line graphs is also the same as the family of $(K_1, 2K_1)$-intersection graphs, and the family of intersection graphs of $k$-cliques equals the family of $(K_1, kK_1)$-intersection graphs. Note that $kX$ contains $k$ $X$-subgraphs iff $X$ is connected, and the family of $(X, kX)$-intersection graphs is contained in the family of line graphs of simple $k$-uniform hypergraphs when $X$ is connected.

The case $k = 2$ for $(X, kX)$-intersection graphs has been studied by Cai [1]. He showed that the family of $(X, 2X)$-intersection graphs equals the family of line graphs whenever $X$ is a connected graph with no bipartite blocks, and that for any complete bipartite graph $X$ with at least two vertices, the family of $(X, 2X)$-intersection graphs is a strict subfamily of the family of line graphs. In this paper, we generalize his results to $(X, kX)$-intersection graphs for any $k \geq 2$. We show that for any integer $k \geq 2$ and connected graph $X$ with no bipartite blocks, the family of $(X, kX)$-intersection graphs coincides with the family of line graphs of conformal $k$-graphs. On the other hand, we obtain a Ramsey type result on vertex splitting and use it to prove that for any connected bipartite graph $X$ with at least two vertices, the family of $(X, kX)$-intersection graphs is strictly contained in the family of line graphs of conformal $k$-graphs. This settles two conjectures of Cai [1] in affirmative.

We define terms and fix notation in Section 2. In Section 3, we obtain a Ramsey type result on vertex splitting and use it to derive a necessary condition for $(X, 2X)$ to make the family of $(X, 2X)$-intersection graphs coincide with the family of line graphs. In Section 4, we prove a result on $X$-disjoint root graphs, which will be used in Section 5 to establish a close relation between $(X, 2X)$-intersection graphs and $(X, kX)$-intersection graphs, and to prove the main results of the paper.

2 Definitions and notation

In this paper, all graphs are simple undirected graphs. A hypergraph $H = (V, \mathcal{E})$ consists of a finite set $V$ of vertices and a family $\mathcal{E}$ of hyperedges, where each hyperedge is a nonempty subset of $V$ and the union of all hyperedges
(X, kX)-intersection graphs

equals V. A hypergraph H is simple if no hyperedge is contained in another hyperedge, and k-uniform if every hyperedge has k vertices. The 2-section \([H]_2\) of a hypergraph \(H = (V, E)\) is a graph on V such that two vertices are adjacent iff they are both contained in a hyperedge of H. A conformal k-graph H is a simple hypergraph whose hyperedges are exactly k-cliques of the 2-section \([H]_2\) of H. Note that conformal 2-graphs are precisely graphs with no isolated vertices. The line graph \(L(H)\) of a hypergraph H is a graph whose vertices are hyperedges of H and where two vertices are adjacent iff they have a nonempty intersection.

The following notion was introduced in [2] to facilitate the study of (X, Y)-intersection graphs. For a pair \((X, Y)\) of fixed graphs, the \((X, Y)\)-containment hypergraph of a graph G, denoted \(C_{X,Y}(G)\), is a hypergraph in which

1. each vertex corresponds to a distinct induced X-subgraph in G that is contained in some induced Y-subgraph of G,
2. each hyperedge corresponds to a distinct induced Y-subgraph in G, and
3. a vertex is contained in a hyperedge iff the induced X-subgraph corresponding to the vertex is contained in the induced Y-subgraph corresponding to the hyperedge.

Clearly, \(C_{X,Y}(G)\) is a k-uniform hypergraph if Y contains exactly k induced X-subgraphs. Furthermore, \(I_{X,Y}(G) = L(C_{X,Y}(G))\).

Remark. The above definition of \((X, Y)\)-containment hypergraphs is slightly different from the one in [2]. In the definition of [2], each vertex of the hypergraph corresponds to an X-subgraph of G (not necessarily contained in a Y-subgraph). Hence, an \((X, Y)\)-containment hypergraph in [2] may contain isolated vertices, whereas such a hypergraph in this paper contains no isolated vertices. Also note that the \((K_1, P_4)\)-containment hypergraph of G is the same as the \(P_4\)-structure [4, 10] of G, which is an important concept in the study of perfect graphs.

For a k-uniform hypergraph H, any graph G satisfying \(C_{X,Y}(G) \cong H\) is a root graph of H. If, in addition, the induced X-subgraphs in G corresponding to vertices of H are mutually disjoint, then G is an X-disjoint root graph of H. An induced X-subgraph in a root graph is isolated if it is not contained in any induced Y-subgraph.

For any family \(\mathcal{H}\) of hypergraphs (graphs), \(L(\mathcal{H})\) denotes the family \(\{L(H) : H \in \mathcal{H}\}\). Let \(C_k\) denote the family of conformal k-graphs and \(G\).
the family of graphs. Note \( C_2 \neq \mathcal{G} \) but \( L(C_2) = L(\mathcal{G}) \). For a pair \((X, Y)\), \( \mathcal{I}_{X,Y} \) denotes the family of \((X, Y)\)-intersection graphs, and \( \mathcal{C}_{X,Y} \) the family of \((X, Y)\)-containment hypergraphs. For a family \( \mathcal{F} \) of graphs, \((X, Y)\) is an \( \mathcal{F} \)-generator (or a generator for \( \mathcal{F} \)) if \( \mathcal{I}_{X,Y} = \mathcal{F} \). We use \( K_n \) and \( K^n \), respectively, to denote the complete graph and complete \( k \)-uniform hypergraph on \( n \) vertices. For two disjoint graphs \( H \) and \( G \), \( H + G \) denotes the join of \( H \) and \( G \), i.e., the graph obtained from \( H \) and \( G \) by adding edges \( \{uv : u \in V(H) \text{ and } v \in V(G)\} \).

### 3 Vertex splitting

Let \( t \geq 2 \) be an integer. A vertex \( t \)-split of a graph \( G \) is any graph obtained from \( G \) by splitting each vertex of \( G \) into at most \( t \) vertices, i.e., replacing each vertex \( v \) by a set \( s(v) \), \( 1 \leq |s(v)| \leq t \), of new vertices, called the split-image of \( v \), and each edge \( uv \) by an edge joining a vertex in \( s(u) \) with a vertex in \( s(v) \). For a vertex \( t \)-split \( G' \) of \( G \), an induced subgraph \( H' \) of \( G' \) is inherited if the subgraph of \( G \) induced by the corresponding vertices of \( H' \) in \( G \) is isomorphic to \( H' \). In other words, an inherited induced \( H' \)-subgraph in \( G' \) is an induced \( H' \)-subgraph in \( G \) that was not destroyed in the splitting process. See Figure 1 for an example.

![Figure 1: A vertex 2-split with many induced \( P_4 \)-subgraphs, but only the top and bottom ones are inherited.](image)

Vertex splitting has a close connection with \((X, 2X)\)-type generators for line graphs. By exploring the relation between vertex splitting and \( X \)-disjoint root graphs, Cai [1] obtained the following necessary condition for \((X, 2X)\) to be a generator for line graphs. Note \( 2X = \overline{X} + X \) and recall \( \mathcal{I}_{X,Y} = \mathcal{I}_{\overline{X},Y} \).

**Theorem 3.1** [Cai [1]] Let \( X \) be a connected bipartite graph with \( t \geq 2 \) vertices. If there is a bipartite graph of which any vertex \( t \)-split contains...
an inherited induced $X$-subgraph, then neither $(X, 2X)$ nor $(\overline{X}, \overline{X} + X)$ is a generator for line graphs.

Cai [1] also conjectured that if $X$ is a connected bipartite graph with at least two vertices, then $(X, 2X)$ is not a generator for line graphs; and that for any bipartite graph $X$ and any integer $t \geq 2$, there exists a bipartite graph of which any vertex $t$-split contains an inherited induced $X$-subgraph.

Here we settle the above two conjectures in affirmative by relating vertex splitting to edge colouring and using the following theorem in Ramsey theory.

**Theorem 3.2** [Nešetřil and Rödl [9]] For every bipartite graph $B$ and every positive integer $n$, there is a bipartite graph $B'$ such that for any $n$-colouring of the edges of $B'$, $B'$ contains a monochromatic induced $B$-subgraph.

**Theorem 3.3** For every bipartite graph $B$ and every integer $t \geq 2$, there exists a bipartite graph $B'$ of which any vertex $t$-split contains an inherited induced $B$-subgraph.

**Proof.** Let $B'$ be the bipartite graph in Theorem 3.2 with $n = t^2$, and $G'$ be an arbitrary vertex $t$-split of $B'$. Let $(U, W)$ be a bipartition of $B'$. For each vertex $v$ of $B'$, let $v_1, v_2, \ldots, v_{t'}$, where $t' \leq t$, be the split-image of $v$ in $G'$. For each edge $uw$ of $B'$, where $u \in U$ and $w \in W$, colour it with colour $(i, j)$ if the corresponding edge of $uv$ in $G'$ is $u_iw_j$. This gives us an $n$-colouring of the edges of $B'$. By Theorem 3.2, $B'$ contains a monochromatic induced $B$-subgraph, which corresponds to an inherited induced $B$-subgraph in $G'$.

It follows from Theorem 3.1 and Theorem 3.3 that $(X, 2X)$ is not a generator for line graphs whenever $X$ is a connected bipartite graph with at least two vertices. We will generalize this to $(X, kX)$-intersection graphs in Section 5.

### 4 Disjoint root graphs

The concept of $X$-disjoint root graphs plays an important role in the study of $(X, 2X)$-type generators for line graphs. It was proved in [1] that in order for $(X, 2X)$ to be a generator for line graphs, certain graphs must have $X$-disjoint root graphs. In this section, we generalize this result to $(X, kX)$-type generators for $L(C_k)$, which will be used in the next section to establish a
relation between \((X, 2X)\)-type generators for line graphs and \((X, kX)\)-type generators for \(L(\mathcal{C}_k)\).

To begin with, we present a result on line graphs of conformal \(k\)-graphs. In general, two nonisomorphic conformal \(k\)-graphs may have isomorphic line graphs. However, some conformal \(k\)-graph \(H\) can be uniquely determined by its line graph \(L(H)\), i.e., if \(H'\) is a \(k\)-uniform hypergraph with \(L(H') \cong L(H)\), then \(H' \cong H\). A simple example is the complete \(k\)-uniform hypergraph \(K^k_n\) with \(n \geq 2k^2\), which follows from a general result in [3]. Here we construct a family of conformal \(k\)-graphs in which each hypergraph is uniquely determined by its line graph. This family of conformal \(k\)-graphs will be used later in proving a useful theorem on \(X\)-disjoint root graphs.

Let \(G\) be a graph on \(n\) vertices \(v_1, v_2, \ldots, v_n\), and \(G_1, G_2, \ldots, G_n\) be \(n\) pairwise disjoint graphs. The composition \(G[G_1, G_2, \ldots, G_n]\) is the graph obtained from \(G\) by replacing vertex \(v_i\) with graph \(G_i\) and adding all edges between \(G_i\) and \(G_j\) whenever \(v_i\) is adjacent to \(v_j\) in \(G\). To be precise, \(G[G_1, G_2, \ldots, G_n]\) has vertex set \(\bigcup_{i=1}^n \{(v_i, u) : u \in V(G_i)\}\) where two vertices \((v_i, u), (v_j, u')\) are adjacent iff either \(v_iv_j \in E(G)\) or \(i = j\) and \(uu' \in E(G_i)\).

**Lemma 4.1** Let \(G\) be an arbitrary graph on \(n\) vertices \(v_1, v_2, \ldots, v_n\), and let \(H\) be the conformal \(k\)-graph whose 2-section is isomorphic to the composition \(G[K_{t_1}, K_{t_2}, \ldots, K_{t_n}]\), where each \(t_i \geq 2k^2\). For any \(k\)-uniform hypergraph \(H'\), if \(L(H') \cong L(H)\) then \(H' \cong H\).

**Proof.** Let \(G'\) denote \(G[K_{t_1}, K_{t_2}, \ldots, K_{t_n}]\). First, since each edge of \(G'\) is contained in a \(k\)-clique, these is indeed a unique conformal \(k\)-graph \(H\) with \([H]_2 \cong G'\).

By the definition of conformal \(k\)-graphs, the complete \(k\)-uniform hypergraph \(K^k_{t_i}\) is an induced subhypergraph of \(H\). Therefore, each \(L(K^k_{t_i})\) is an induced subgraph of \(L(H)\). Since \(L(H') \cong L(H)\), \(H'\) contains a subhypergraph \(A_i\) such that \(L(A_i) \cong L(K^k_{t_i})\). Therefore \(A_i \cong K^k_{t_i}\) for each \(i\) as \(A_i\) is a \(k\)-uniform hypergraph and \(K^k_{t_i}\) has at least \(2k^2\) vertices [3]. This implies that each \(A_i\) is an induced subhypergraph of \(H'\). Since \(K^k_{t_1}, K^k_{t_2}, \ldots, K^k_{t_n}\) are pairwise disjoint, \(A_1, A_2, \ldots, A_n\) are also pairwise disjoint. It follows that \(A_1 \cup \cdots \cup A_n\) is isomorphic to \(K^k_{t_1} \cup \cdots \cup K^k_{t_n}\).

Since the 2-section of \(H\) is isomorphic to \(G'\), each induced subhypergraph \(H'[V(A_i) \cup V(A_j)]\), where \(i \neq j\), equals either \(K^k_{t_i+t_j}\) or \(K^k_{t_i} \cup K^k_{t_j}\). Therefore for any \(i \neq j\), \(H'[V(A_i) \cup V(A_j)] \cong K^k_{t_i+t_j}\) iff \(H'[V(K_{t_i}) \cup V(K_{t_j})] \cong K^k_{t_i+t_j}\) and \(H'[V(A_i) \cup V(A_j)] \cong K^k_{t_i} \cup K^k_{t_j}\) iff \(H'[V(K_i) \cup V(K_j)] \cong K^k_{t_i} \cup K^k_{t_j}\).
Therefore $H' \cong H$ since $H'$ contains the same number of hyperedges as $H$.

We now consider $(X, kX)$-intersection graphs. We show that for any connected $X$, an $(X, kX)$-containment hypergraph is always a conformal $k$-graph. Therefore, an $(X, kX)$-intersection graph is isomorphic to the line graph of some conformal $k$-graph. This motivates us to determine $(X, kX)$ for which the family of $(X, kX)$-intersection graphs exactly equals the family of line graphs of conformal $k$-graphs.

**Lemma 4.2** For any integer $k \geq 2$ and connected graph $X$, $C_{X, kX} \subseteq C_k$.

**Proof.** Let $G$ be an arbitrary graph. Then the $(X, kX)$-containment hypergraph $H$ of $G$ is a simple $k$-uniform hypergraph since $kX$ contains exactly $k$ induced $X$-subgraphs when $X$ is connected. Observe that each edge in the 2-section $[H]_2$ of $H$ corresponds to two disjoint induced $X$-subgraphs in $G$ and there is no edge between these two $X$-subgraphs. Let $K$ be an arbitrary $k$-clique in $[H]_2$, and $S$ be the set of induced $X$-subgraphs in $G$ corresponding to vertices in $K$. Then any two induced $X$-subgraphs in $S$ are disjoint and not connected by edges. This implies that vertices in $S$ induce a $kX$-subgraph in $G$, which corresponds to a hyperedge in $H$. Therefore $H$ is a conformal $k$-graph.

**Corollary 4.3** For any integer $k \geq 2$ and connected graph $X$, $I_{X, kX} \subseteq L(C_k)$.

To prove that $(X, kX)$ is an $L(C_k)$-generator, we need only construct an $X$-disjoint root graph $G$ for an arbitrary conformal $k$-graph $H$. On the other hand, to prove that $(X, kX)$ is not an $L(C_k)$-generator, we need to show that the line graph of some conformal $k$-graph $H$ is not an $(X, kX)$-intersection graph. This appears to be very difficult because induced $X$-subgraphs in possible root graphs of $H$ can interwine in a complicated manner. Here we show that we need only consider $X$-disjoint root graphs.

Note $(X, kX)$ being an $L(C_k)$-generator only guarantees that for any conformal $k$-graph $H$, there is a graph $R$ satisfying $I_{X, kX}(R) \cong L(H)$. However, because of the lack of the one-to-one correspondence between conformal $k$-graphs and their line graphs, it is not even clear if $H$ has a root graph. To tackle this, we construct from $H$ a conformal $k$-graph $H'$ of Lemma 4.1, and use a root graph of $H'$ to produce an $X$-disjoint root graph of $H$. 
Theorem 4.4 For any integer \( k \geq 2 \) and connected graph \( X \), \((X, kX)\) is an \( L(C_k)\)-generator iff every conformal \( k \)-graph has an \( X \)-disjoint root graph that contains no isolated induced \( X \)-subgraphs.

Proof. The sufficiency of the theorem follows from Corollary 4.3 and the fact that \( I_{X,kX}(G) = L(C_{X,kX}(G)) \). We now prove the necessity. Let \( H \) be an arbitrary conformal \( k \)-graph on \( n \) vertices \( v_1, v_2, \ldots, v_n \) and \( G \) be the 2-section of \( H \). Let \( m \) be the number of vertices in \( X \). Set \( t_n = 2k^2 + (mn)^2 \) and let \( t_i, 1 \leq i < n \), be a finite integer satisfying \( t_i > (mn)^2 + m \sum_{j=i+1}^{n} t_j \). (The reason for setting \( t_i \) in the way will become clear later in the proof.) Denote by \( G' \) the composition \( G[G_1, G_2, \ldots, G_n] \) with \( G_i \) being the complete graph on \( t_i \) vertices, and let \( H' \) be the conformal \( k \)-graph corresponding to \( G' \).

Since \((X, kX)\) is an \( L(C_k)\)-generator, there is a graph \( R' \) whose \((X, kX)\)-intersection graph \( I_{X,kX}(R') \) is isomorphic to the line graph \( L(H') \) of \( H' \). It follows from Lemma 4.1 that the \((X, kX)\)-containment hypergraph \( C_{X,kX}(R') \) of \( R' \) is isomorphic to \( H' \). We now use \( R' \) to construct an \( X \)-disjoint root graph \( R \) of \( H \).

Each vertex of \( H' \) corresponds to an induced \( X \)-subgraph in \( R' \). Let \( S_i \) denote the set of induced \( X \)-subgraphs in \( R' \) that correspond to the vertices in the complete graph \( G_i \). Since vertices of any two induced \( X \)-subgraphs in \( S_i \) induce a \( 2X \)-subgraph in \( R' \), we deduce the following two facts:

1. Inside each \( S_i \), every pair of induced \( X \)-subgraphs are disjoint and not connected by edges.

2. For any \( i \neq j \), an induced \( X \)-subgraph in \( S_i \) can share vertices with at most \( m \) induced \( X \)-subgraphs in \( S_j \).

To construct an \( X \)-disjoint root graph \( R \) of \( H \), we use the following algorithm to choose one induced \( X \)-subgraph \( X_i \) from each \( S_i \) to form a set \( S^* = \{X_i : 1 \leq i \leq n\} \) of disjoint induced \( X \)-subgraphs such that no induced \( X \)-subgraph in \( \bigcup_{i=1}^{n} S_i - S^* \) intersects more than one induced \( X \)-subgraph in \( S^* \).

For each value of \( i \) from 1 to \( n \) in increasing order perform the following two steps. Initially, all induced \( X \)-subgraphs are unmarked.

Step 1. Choose from \( S_i \) an unmarked induced \( X \)-subgraph \( X_i \) that is disjoint from all induced \( X \)-subgraphs in \( \bigcup_{j=i+1}^{n} S_j \).
Step 2. Let $S'$ be the set of induced $X$-subgraphs (marked or unmarked) in $\bigcup_{j=1}^{n} S_j$ that share vertices with $X_i$. For each induced $X$-subgraph in $\bigcup_{j=i+1}^{n} S_j$, mark it if it shares vertices with an induced $X$-subgraph in $S'$. It is clear that if $X_1, \ldots, X_i$ are selected in the first $i$-iterations, these induced $X$-subgraphs are mutually disjoint and disjoint from all induced $X$-subgraphs in $\bigcup_{j=1}^{i-1} S_j$. Furthermore, no induced $X$-subgraphs inside $\bigcup_{j=1}^{i} S_j - \{X_1, \ldots, X_i\}$ intersect more than one induced $X$-subgraph in $X_1, \ldots, X_i$. Therefore after the $n$-th iteration, the set $S^*$ of selected induced $X$-subgraphs has the required property. It remains to be shown that indeed one induced $X$-subgraph is chosen at Step 1 of each iteration. Recall that $S_1$ contains $t_1 \geq (mn)^2 + m \sum_{j=2}^{n} t_j$ induced $X$-subgraphs and none of them are marked. Since each induced $X$-subgraph in $\bigcup_{j=2}^{n} S_j$ intersects at most $m$ induced $X$-subgraphs in $S_1$, there is an induced $X$-subgraph in $S_1$ that is disjoint from all induced $X$-subgraphs in $\bigcup_{j=2}^{n} S_j$, and hence all induced $X$-subgraphs in $\bigcup_{j=1}^{n} S_j - X_1$.

Assume that $i-1$ induced $X$-subgraphs $X_1, \ldots, X_{i-1}$ have been selected. Consider the situation right before the execution of Step 1 of the $i$-th iteration. Each selected induced $X$-subgraph $X_j$, $1 \leq j < i$, intersects at most $m$ induced $X$-subgraphs in each $S_j$, $j' < j$. Hence the total number of induced $X$-subgraphs in $S_1, \ldots, S_{j-1}$ that share vertices with $X_j$ is at most $m(j-1)$. Each of these $m(j-1)$ induced $X$-subgraphs share vertices with at most $m$ induced $X$-subgraphs in $S_i$. Therefore at most $m^2(j-1)$ unmarked induced $X$-subgraphs in $S_i$ became marked in Step 2 right after $X_j$ was selected. This implies that the total number of marked induced $X$-subgraphs in $S_i$ is at most $m^2(i-1)(i-2)/2$, which is less than $(mn)^2$. Therefore $S_i$ contains more than $t_i - (mn)^2 > m \sum_{j=i+1}^{n} t_j$ unmarked induced $X$-subgraphs. Since $\bigcup_{j=i+1}^{n} S_j$ contains $\sum_{j=i+1}^{n} t_j$ induced $X$-subgraphs, and each induced $X$-subgraph intersects at most $m$ induced $X$-subgraphs in $S_i$, $S_i$ has at least one unmarked induced $X$-subgraph that is disjoint from all induced $X$-subgraphs in $\bigcup_{j=i+1}^{n} S_j$. Therefore the algorithm indeed constructs a set $S^*$ of $n$ disjoint induced $X$-subgraphs with the required property.

Let $R$ be the subgraph of $R'$ induced by all vertices of induced $X$-subgraphs of $S^*$. By the construction of $G'$ and the choice of $S^*$, it is clear that, for any $i \neq j$, $V(X_i) \cup V(X_j)$ induces a $2X$-subgraph in $R$ iff $v_iv_j$ is an edge in $G$. Therefore the vertices of any $k$ $X$-subgraphs from $S^*$ induce a $kX$-subgraph in $R$ iff the corresponding vertices of these $k$ $X$-subgraphs in $G$ form a $k$-clique. Furthermore, by the construction of $S^*$, no induced $X$-subgraph in $\bigcup_{i=1}^{n} S_i - S^*$ can share vertices with more than one induced $X$-subgraph in $S^*$, implying that $R$ contains no induced $X$-
subgraph in $\bigcup_{i=1}^{n} S_i - S^*$. Therefore $C_{X,kX}(R) \cong H$ and we have obtained an $X$-disjoint root graph of $H$.

Now we show that actually $H$ has an $X$-disjoint root graph with no isolated induced $X$-subgraphs. Let $H^*$ be the conformal $k$-graph whose 2-section is isomorphic to $[H]_{2+K_{k-1}}$. Then, as we have just proved, $H^*$ has an $X$-disjoint root graph $R^*$. Let $V^*$ denote the vertices of $X$-subgraphs in $R^*$ that correspond to vertices of $H$, and let $R = R^*[V^*]$. Then $C_{X,kX}(R) \cong H$. Suppose that $R$ contains an isolated induced $X$-subgraph. Then it forms an induced $kX$-subgraph with the $k - 1$ disjoint induced $X$-subgraphs corresponding to $K_{k-1}$, contradicting to the fact that $C_{X,kX}(R^*) \cong H^*$. Therefore $H$ has an $X$-disjoint root graph with no isolated induced $X$-subgraphs.

5 Generators for line graphs of conformal $k$-graphs

Having the tools from the previous sections, we now characterize $(X,kX)$-type generators for line graphs of conformal $k$-graphs. First, we use Theorem 4.4 to establish the following relation between $L(G)$-generators and $L(C_k)$-generators, which enables us to concentrate on $(X,2X)$ when studying $(X,kX)$-type generators for $L(C_k)$.

**Theorem 5.1** For any integer $k \geq 3$, $(X,kX)$ is an $L(C_k)$-generator iff $(X,2X)$ is an $L(G)$-generator.

**Proof.** Suppose that $(X,2X)$ is an $L(G)$-generator and let $H$ be an arbitrary conformal $k$-graph. By Theorem 4.4, the 2-section $[H]_2$ of $H$ has an $X$-disjoint root graph $R$, i.e., $C_{X,2X}(R) \cong [H]_2$. For each $k$-clique of $[H]_2$, its corresponding induced $X$-subgraphs in $R$ are mutually disjoint and not connected by edges. Since each $k$-clique of $[H]_2$ corresponds to a hyper-edge of $H$, $C_{X,kX}(R) \cong H$ and hence $I_{X,kX}(R) \cong L(H)$. It follows from Corollary 4.3 that $(X,kX)$ is an $L(C_k)$-generator.

Conversely, suppose that $(X,kX)$ is an $L(C_k)$-generator and let $G$ be an arbitrary graph. Let $G' = G + K_{k-1}$. Since every edge of $G'$ is contained in some $k$-clique, there is a conformal $k$-graph $H'$ corresponding to $G'$, i.e., $[H']_2 = G'$. By Theorem 4.4, $H'$ has an $X$-disjoint root graph $R'$ with no isolated induced $X$-subgraphs. For each vertex $v$ of $G$, let $X_v$ be its corresponding induced $X$-subgraph in $R'$. Let $R$ denote the subgraph of $R'$ induced by $\bigcup_{v \in V(G)} V(X_v)$. Then the set of all induced $X$-subgraphs in $R$ is precisely $\{X_v : v \in V(G)\}$ since $R'$ contains no isolated induced $X$-subgraphs and induced $X$-subgraphs in $R'$ are mutually disjoint.
Let $u$ and $v$ be two arbitrary vertices of $G$. We claim that $uv$ is an edge of $G$ if $R[V(X_u) \cup V(X_v)] \cong 2X$. If $uv$ is an edge of $G$, then $uv$ is contained in a $k$-clique of $G'$ and thus $u$ and $v$ are contained in a hyperedge of $H'$. Therefore no edges connecting $X_u$ with $X_v$ in $R'$ and hence in $R$, implying $R[V(X_u) \cup V(X_v)] \cong 2X$.

If $u$ is not adjacent to $v$, then no hyperedge in $H'$ contains both $u$ and $v$ since $[H']_2 = G'$ and $G$ is an induced subgraph of $G'$. Suppose that $X_u$ and $X_v$ are not connected by edges in $R'$. Let $S$ denote the set of induced $X$-subgraphs in $R'$ that correspond to the vertices of the complete graph $K_{k-1}$ of $G'$. Since $V(K_{k-1}) \cup \{u\}$ and $V(K_{k-1}) \cup \{v\}$ are hyperedges in $H'$, any two $X$-subgraphs in $S \cup \{X_u, X_v\}$ are mutually disjoint and not connected by edges in $R'$. Then any $k-2$ vertices in $K_{k-1}$ together with $u$ and $v$ would be a hyperedge of $H'$, a contradiction. Therefore $X_u$ and $X_v$ are connected by some edges in $R'$ and hence in $R$, implying $R[V(X_u) \cup V(X_v)]$ is not isomorphic to $2X$.

From the above argument, we have $C_{X,2X}(R) \cong G$ and hence $I_{X,2X}(R) \cong L(G)$. It follows from Corollary 4.3 that $(X, 2X)$ is an $L(G)$-generator. ■

It was proved in [1] that $(X, 2X)$ is an $L(G)$-generator whenever $X$ is a connected graph with no bipartite blocks. Therefore we can derive from Theorem 5.1 the following sufficient condition for $(X, kX)$ to be an $L(C_k)$-generator. Recall $I_{X,Y} = I_{X,Y}^*$

**Theorem 5.2** For any integer $k \geq 2$ and connected graph $X$ with no bipartite blocks, both $(X, kX)$ and $(\overline{X}, k\overline{X})$ are $L(C_k)$-generators.

On the other hand, combining Theorem 3.1, Theorem 3.3, and Theorem 5.1, we obtain the following necessary condition for $(X, kX)$ to be an $L(C_k)$-generator.

**Theorem 5.3** For any integer $k \geq 2$ and connected bipartite graph $X$ with at least two vertices, neither $(X, kX)$ nor $(\overline{X}, k\overline{X})$ is an $L(C_k)$-generator.

Therefore $(X, kX)$-type $L(C_k)$-generators are fully characterized when $X$ is 2-connected.

**Corollary 5.4** Let $X$ be a 2-connected graph and $k \geq 2$ an integer. Then $(X, kX)$ (likewise, $(\overline{X}, k\overline{X})$) is an $L(C_k)$-generator iff $X$ is not bipartite.

We leave the full characterization of $(X,kX)$-type $L(C_k)$-generators as an open problem for the reader to ponder.
References


