The existence of homomorphisms to oriented cycles

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Abstract

We discuss the existence of homomorphisms of arbitrary digraphs to a fixed oriented cycle C. Our main result asserts that if the cycle C is unbalanced then a digraph G is homomorphic to C if and only if (1) every oriented path homomorphic to G is also homomorphic to C, and (2) the length of every cycle of G is a multiple of the length of C. This answers a conjecture from an earlier paper with H. Zhou, and generalizes a result proved there. We also show that this characterization does not hold for balanced cycles. We relate these results to work on the complexity of homomorphism problems.

1 Introduction

All the digraphs discussed in this paper are finite unless otherwise specified. A *homomorphism* $G \to H$ of a digraph G to a digraph H is a mapping of

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the vertex sets $f : V(G) \to V(H)$ which preserves the edges, i.e., such that $xy \in E(G)$ implies $f(x)f(y) \in E(H)$. If such a homomorphism exists, we say G is *homomorphic* to H and write $G \to H$. Otherwise we write $G \not\to H$. Note that these notions can also be applied to undirected graphs, by viewing them as symmetric digraphs. If K_n denotes the undirected complete graph on n vertices, then for an arbitrary undirected graph G a homomorphism $G \to K_n$ is just an n-colouring of G. Because of this fact, it is also common to call a homomorphism $G \to H$ (of general digraphs) an *H-colouring* of G.

Suppose $g_0 \in V(G)$ is a fixed vertex of G, called the root of G, and suppose $h_0 \in V(H)$ is the root of H. A *rooted homomorphism* of G to H is a homomorphism $h : G \to H$ such that $h(g_0) = h_0$. In this case we write $(G, g_0) \to (H, h_0)$ and say h is a homomorphism of (G, g_0) to (H, h_0). We observe that the composition of rooted homomorphisms $(G, g_0) \to (H, h_0)$ and $(H, h_0) \to (J, j_0)$ is a rooted homomorphism $(G, g_0) \to (J, j_0)$.

An *oriented path* P is a sequence of distinct vertices $[p_0, p_1, \dots, p_n]$, such that, for each $i \in \{0, 1, \dots, n-1\}$, either $p_i p_{i+1} \in E(P)$ (a *forward* edge of P), or $p_{i+1} p_i \in E(P)$ (a *backward* edge of P), and such that P has no other edges. The direction in which P is traversed is emphasized by saying that p_0 is the *initial vertex*, $i(P)$, of P, and p_n the *terminal vertex*, $t(P)$, of P, respectively.

An *oriented cycle* C is a digraph obtained from an oriented path P by identifying its initial and terminal vertices. Thus an oriented cycle C can be given by a circular sequence of vertices $[c_0, c_1, \dots, c_{m-1}, c_0]$, such that, for each $i \in \{0, 1, \dots, m-1\}$, either $c_i c_{i+1} \in E(C)$ (a *forward* edge of C), or $c_{i+1} c_i \in E(C)$ (a *backward* edge of C), and such that C has no other edges. (Subscript addition is taken modulo m.) The direction of C which agrees with the forward edges, i.e., $c_0, c_1, c_2, \dots$, is the *positive direction* of C, and the opposite direction, i.e., $c_0, c_{m-1}, c_{m-2}, \dots$, is the *negative direction* of C. We also use the term *path* (resp. *cycle*) to mean an oriented path (resp. an oriented cycle). Since we do not distinguish an initial vertex of an oriented cycle, $[c_0, c_1, \dots, c_{m-1}, c_0] = [c_1, c_2, \dots, c_{m-1}, c_0, c_1]$, and we usually choose a most convenient vertex to start listing C.

A *directed cycle* (resp. a *directed path*) is a cycle (resp. path) in which all edges are in the same direction; if they are all forward edges we speak of a forward directed cycle (resp. path), if they are all backward edges we speak of a backward directed cycle (resp. path).

Suppose G is a digraph. A *path* (resp. a *cycle*) in G is a subgraph of G which is an oriented path (resp. an oriented cycle). A digraph G is *connected* if any two vertices are joined by some path.

The *length* $l(X)$ of an oriented path or an oriented cycle X is the number of forward edges of X minus the number of backward edges of X. Note that the length can be negative. An oriented cycle C is *unbalanced* if $l(C) \neq 0$; otherwise C is *balanced*. A digraph G is balanced if each cycle of G is balanced. For an oriented path $P = [p_0, p_1, \ldots, p_n]$ and two vertices $p_i, p_j \in P$, we define the distance $d_P(p_i, p_j)$ from p_i to p_j to be the length of the subpath of P connecting p_i to p_j. The *level* of a vertex p_i of P is defined as $\lambda_P(p_i) = d_P(p_0, p_i)$. An oriented path P is *minimal* if P contains no proper subpath P' such that $l(P) = l(P')$.

Let H be a fixed graph or digraph. The *H-colouring problem*, or the *homomorphism problem with respect to the target H*, denoted by *H-col*, is the decision problem in which the instance is a graph or digraph G and the question is whether or not $G \to H$. The H-colouring problem has, from the algorithmic point of view, received much recent attention [1, 2, 3, 4, 8, 9, 14, 22, 30].

For undirected graphs, it was shown in [14] that H-col is polynomial for H bipartite and NP-complete for all other H. No such clear distinction is known for digraphs. We do know many cases of NP-complete problems and many nontrivial cases of polynomial problems, [1, 2, 3, 4, 8, 9, 22, 30]. One easy case is the existence of a homomorphism to a directed path. In fact, we have the following result, cf. [5, 10, 24]:

THEOREM 1.1 *Let P be a directed path of length k. A digraph G is homomorphic to P if and only if every oriented path homomorphic to G has length at most k.*

Theorem 1.1 asserts that paths of length greater than k are the only possible obstructions to a homomorphism to P. It is not hard to specify a polynomial algorithm to test the condition by a breadth first labeling of G, cf. [10].

A polynomial algorithm for the existence of a homomorphism to any oriented path was given in [9]. It did not depend on a theorem specifying the obstructions to homomorphisms, but such a theorem was later given in [18]:

THEOREM 1.2 *Let P be an oriented path. A digraph G is homomorphic to P if and only if every oriented path homomorphic to G is also homomorphic to P.*

This theorem generalizes Theorem 1.1 and it can be shown to also imply a polynomial algorithm for P-col, cf. [15].

The existence of homomorphisms to oriented cycles appears to be a harder problem. In particular, there exist oriented cycles C for which C-col is NP-complete, [8], cf. Section 4. It is still the case that for directed cycles there is a simple characterization in terms of obstructions, cf. [5, 10, 24]:

THEOREM 1.3 *Let C be a directed cycle of length k. A digraph G is homomorphic to C if and only if the length of every oriented cycle in G is divisible by k.*

This results also implies a polynomial algorithm via a breadth first labeling modulo k, cf. [10].

At this point one may wonder whether or not a general obstruction result, analogous to Theorem 1.2, holds for cycles. Let C be an oriented cycle and let D_C denote the statement

D_C: *A digraph G is homomorphic to C if and only if every oriented cycle homomorphic to G is also homomorphic to C.*

It is unlikely that D_C holds for all oriented cycles C, although we shall prove it for a large class of oriented cycles. We shall prove in the last section that if D_C holds then C-col is in $NP \cap coNP$. Thus it is not surprizing that when D_C holds we usually also find a polynomial algorithm for C-col. (It is of course a priori not clear how to find such an algorithm; but see the added remark at the end of the paper.) This also means that D_C is unlikely to hold for those cycles C for which C-col is NP-complete.

If C is a directed cycle then D_C holds, as can be easily seen using Theorem 1.3, since the condition concerning cycle lengths is weaker than the condition in D_C. If C is a cycle obtained from two copies of an oriented path by identifying their initial vertices and identifying their terminal vertices, then D_C can also be seen to hold, using Theorem 1.2. In this case we also have a weaker condition, namely that every oriented path homomorphic to G is also homomorphic to C. Indeed, if there is a path P homomorphic to G but not homomorphic to C then the cycle obtained from two copies of P by identifying their initial vertices and identifying their terminal verices is homomorphic to G but not to C. Another class of cycles C for which D_C holds is the class of B-cycles. A *B-cycle* is an oriented cycle obtained from a forward directed path I of length n and a minimal oriented path J of length $n-1$ by identifying their initial vertices and identifying their terminal verices. Thus B-cycles are particular cycles of length one. The following charaterization theorem was proved in [19]:

THEOREM 1.4 *Let C be a B-cycle. A digraph G is homomorphic to C if and only if every oriented path homomorphic to G is also homomorphic to C.*

Thus in this case we also know that D_C holds, and once again we have a weaker condition. These weaker conditions suggest the following modified statement D'_C:

D'_C: *A digraph G is homomorphic to C if and only if*

- *every oriented path homomorphic to G is also homomorphic to C, and*
- *the length of any cycle of G is a multiple of the length of C.*

Note that if C is a directed cycle then the first condition is vacuously satisfied and the statement D'_C becomes Theorem 1.3. Similarly, when C is a B-cycle then the second condition is vacuously satisfied and the statement D'_C becomes Theorem 1.4. Thus D'_C is a stronger statement than D_C, yet one which holds in all the cases in which we know D_C holds. We know that this stronger statement does not hold for all cycles. In Section 4 we will construct cycles C such that D'_C fails. In [19] we made the following conjecture:

Conjecture 1.5 *If C is an unbalanced oriented cycle, then a digraph G is homomorphic to C if and only if D_C holds.*

Here we shall prove this conjecture. Note that this implies both Theorem 1.3 and Theorem 1.4, whose proof in [19] is quite complex. Our proof also motivated a polynomial algorithm for C-col for any unbalanced C, [30]. In fact, such an algorithm has also been discovered independently by Gutjahr, [8]. More recently, it has been proved, [15], that whenever D_C holds there is a polynomial algorithm for C-col. (See the remark at the end of this paper.)

The characterization theorem proved here, Theorem 3.1, is of course interesting in its own right, regardless of any polynomial algorithms it may yield, or any implications regarding $NP \cap coNP$. It has already been applied, at least in the special case of B-cycles, to prove the multiplicativity of certain oriented cycles, [20], and so to complete the classification of multiplicative cycles, [20].

2 Auxiliary Results

We shall first prove some lemmas which are special cases of the conjecture and which are needed in the proof of the main theorem. In order not to have to repeat our assumptions everywhere we specify them explicitely here as follows:

Assumptions: Let C be an unbalanced oriented cycle which is not a directed cycle. Assume that $l(C) > 0$, and that n is the maximum length of a subpath of C. Let c_0 be a vertex of C such that $C = [c_0, c_1, \dots, c_{j_0}, \dots, c_{m-1}, c_0]$ where $[c_0, c_1, \dots, c_{j_0}]$ is a minimal path of length n.

These Assumptions result in no loss of generality, in view of the fact the result for directed cycles is known. In fact, when C is a directed cycle, $l(C) = n$, and $[c_0, c_1, \dots, c_{j_0}] = [c_0, c_1, \dots, c_0]$. It is easy to see that under our Assumptions we have $n > l(C)$.

LEMMA 2.1 *Let C be the cycle from our Assumptions. For any $1 \leq j \leq m-1$, we have $l([c_0, c_1, \dots, c_j]) > 0$; and for any $j_0 \leq j \leq m$, we have $l([c_0, c_{m-1}, c_{m-2}, \dots, c_j]) \geq 0$.*

Proof: Suppose $1 \leq j \leq j_0$. If $l([c_0, c_1, \dots, c_j]) \leq 0$ then $l([c_j, c_{j+1}, \dots, c_{j_0}]) \geq n$, contradicting either the minimality of $[c_0, c_1, \dots, c_{j_0}]$ or the maximality of n. Similarly, if $j_0 < j \leq m-1$ and if $l[c_0, c_1, \dots, c_j]) \leq 0$ then $l(c_j, c_{j+1}, \dots, c_{m-1}, c_0) > 0$ and $l([c_j, c_{j+1}, \dots, c_0, c_1, \dots, c_{j_0}]) > n$, contradicting the choice of n. On the other hand, if $l([c_0, c_{m-1}, c_{m-2}, \dots, c_j]) < 0$ for $j_0 \leq j \leq m-1$, then we also have $l(c_j, c_{j+1}, \dots, c_{m-1}, c_0) > 0$, and we obtain a contradiction as above.

Note that these inequalities imply that c_0 has indegree zero.

We sometimes view $[c_0, c_1, \dots, c_{m-1}, c_0]$ as an oriented path, and call it $R^1(C)$; formally we introduce a new vertex c_0^1 and let $R^1(C)$ be the path $[c_0, c_1, \dots, c_{m-1}, c_0^1]$, where $c_{m-1}c_0^1$ or $c_0^1 c_{m-1}$ is an edge of $R^1(C)$ just if $c_{m-1}c_0$ or $c_0 c_{m-1}$ is an edge of C. This is the path which wraps around C exactly once, in the positive direction, starting at c_0 and also ending at c_0. In a similar fashion we define $R^q(C)$ to be the path wrapping around C exactly q times, in the positive direction, i.e., $R^q(C) = [c_0, c_1, \dots, c_{m-1}, c_0^1, c_1^1, \dots, c_{m-1}^1, c_0^2, \cdots, c_0^{q-1}, c_1^{q-1}, \dots, c_{m-1}^{q-1}, c_0^q]$. For convenience we may also write $R^q(C) = [c_0^0, c_1^0, \dots, c_{m-1}^0, c_0^1, c_1^1, \dots, c_{m-1}^1, c_0^2, \cdots, c_0^{q-1}, c_1^{q-1}, \dots, c_{m-1}^{q-1}, c_0^q]$. We define the *index function* as $Ind(c_j^i) = im + j$. For $x, y \in R^q(C)$, we say $x \leq y$ if $Ind(x) \leq Ind(y)$ (and $x < y$ if $Ind(x) < Ind(y)$). Note that $R^q(C) \to C$ via the obvious homomorphism taking all c_j^i (for $i = 0, 1, \dots, q$) to c_j.

Frequently, we will need to consider homomorphisms h of an oriented path $P = [p_0, p_1, \dots, p_n]$ to the oriented cycle C. Each such h defines a sequence

$[h(p_0), h(p_1), \ldots, h(p_n)]$ of vertices of C, such that for each $1 \le i \le n$, either $h(p_{i-1})h(p_i)$ is an edge of C (if $p_{i-1}p_i$ is an edge of P) or $h(p_i)h(p_{i-1})$ is an edge of C (if p_ip_{i-1} is an edge of P). We call such a sequence a *walk* of C. If $P = [p_0, p_1, \ldots, p_n]$ is a minimal path and $h : P \to C$ is a homomorphism then we call $[h(p_0), h(p_1), \ldots, h(p_n)]$ a minimal walk.

Let $P = [p_0, p_1, \ldots, p_n]$ be an oriented path and let $h : P \to C$ be a homomorphism. We shall also view h, in a natural way, as a homomorphism $P \to R^{2q} = [c_0, c_1, \ldots, c_{m-1}, c_0^1, c_1^1, \ldots, c_{m-1}^1, c_0^2, \cdots, c_0^{2q-1}, c_1^{2q-1}, \ldots, c_{m-1}^{2q-1}, c_0^{2q}]$ for a large enough q. Formally, we define a homomorphism $h' : P \to R^{2q}(C)$ as follows:

We let $h'(p_0) = c_i^q$ if $h(p_0) = c_i$. Note that c_i^q is in the "middle range" of $R^{2q}(C)$. We will define h' so that $h(p_r) = c_j$ will always imply that $h'(p_r) = c_j^a$ for some a. If we have already defined $h'(p_r) = c_j^a$ and $h(p_{r+1}) = c_s$, then

$$h'(p_{r+1}) = c_s^{a+1} \text{ if } j = m-1 \text{ and } s = 0$$

$$h'(p_{r+1}) = c_s^{a-1} \text{ if } j = 0 \text{ and } s = m-1$$

$$h'(p_{r+1}) = c_s^a \text{ otherwise.}$$

It is easy to see that h' is well defined (as q is large enough), uniquely determined by h (for a fixed q), and is indeed a homomorphism. We call h' the *induced homomorphism* of h. We say $h(P)$ goes in the positive direction of C if $Ind(h'(p_0)) < Ind(h'(p_n))$, and $h(P)$ goes in the negative direction if $Ind(h'(p_0)) > Ind(h'(p_n))$. Observe that whether $h(P)$ goes in the positive direction or the negative direction of C is independent of the integer q, although in the definition of h' we need to choose a fixed integer q.

For balanced graphs G the second condition of Conjecture 1.5 is vacuously satisfied. Thus, when we restrict our attention to balanced graphs G, Conjecture 1.5 becomes the following statement:

LEMMA 2.2 *Let G be a balanced digraph and C an unbalanced cycle. Then $G \to C$ if and only if every oriented path homomorphic to G is also homomorphic to C.*

Proof. Clearly if $G \to C$ then any path P with $P \to G$ satisfies $P \to C$ by composition. For the converse, we shall use Theorem 1.2. Assume that G has q vertices. Then the absolute value of the length of any path in G is less than q. Since G is balanced, the same is true for any walk in G, and hence also for any path P homomorphic to G. Since the length of C is not zero, any homomorphism of such path P to C can wrap around C at most q times in either the positive or the negative direction. Assume now that every oriented path homomorphic to G is also homomorphic to C. Then any path homomorphic to G is also homomorphic to $R^{2q}(C)$. By Theorem 1.2, $G \to R^{2q}(C)$, and by composition with $R^{2q}(C) \to C$ we have $G \to C$.

COROLLARY 2.3 *Let T be an oriented tree and C an unbalanced cycle. Then $T \to C$ if and only if every oriented path homomorphic to T is also homomorphic to C.*

Lemma 2.2 has a corresponding rooted version.

LEMMA 2.4 *Let G be a balanced digraph and let g_0 be a fixed vertex of G. Let C and c_0 be as described in our Assumptions. Then $(G, g_0) \to (C, c_0)$ if and only if P, $(P, i(P)) \to (G, g_0)$ implies that $(P, i(P)) \to (C, c_0)$, for every oriented path P.*

Proof. The necessity of the condition is again clear by composition. Suppose the condition is satisfied. We shall construct a homomorphism $(G, g_0) \to (C, c_0)$. Assume again that G has q vertices and consider $R^{2q}(C)$. Let c^* be the "middle vertex" of R^{2q}, i.e., $c^* = c_0^q$. Recall that $(R^{2q}, c^*) \to (C, c_0)$. Note also that by Lemma 2.1 and the assumption that $l(C) > 0$ we have $d_{R^{2q}(C)}(c^*, x) > 0$ for any x with $x > c^*$.

We shall first define two mappings ϕ and ψ. For $x \in G$, let $\mathcal{P}_x$ be the set of all oriented paths P such that some homomorphism $h : P \to G$ has $h(i(P)) = g_0$ and $h(t(P)) = x$. Consider a path $P \in \mathcal{P}_x$. Since $(P, i(P)) \to (G, g_0)$, we also have $(P, i(P)) \to (C, c_0)$ by our assumption. This implies that $(P, i(P)) \to (R^{2q}(C), c^*)$ by an argument similar to the one given in the proof of the previous Lemma. Thus we may define

$$\phi(P) = \min\{h(t(P)) : h : (P, i(P)) \to (R^{2q}(C), c^*)\}.$$

Finally, we define $\psi : G \to R^{2q}(C)$ as

$$\psi(x) = \max\{\phi(P) : P \in \mathcal{P}_x\}.$$

Here the terms "max" and "min" are taken with respect to the order determined by the index function.

Next we prove that ψ is a homomorphism $(G, g_0) \to (R^{2q}(C), c^*)$. Let $(x, y) \in E(G)$ be an edge of G. We show that $(\psi(x), \psi(y)) \in E(R^{2q}(C))$. First $\psi(x) \neq \psi(y)$ because for any oriented paths $P_1 \in \mathcal{P}_x$ and $P_2 \in \mathcal{P}_y$ we must have $l(P_1) = l(P_2) - 1$ (as G is balanced), and therefore $\phi(P_1) \neq \phi(P_2)$.

Suppose $\psi(x) > \psi(y)$. Let $P \in \mathcal{P}_x$ be an oriented path such that $\phi(P) = \psi(x)$. Let P' be the path obtained from P by adding a new vertex $a = t(P')$ and an edge $t(P)a$. Then obviously $P' \in \mathcal{P}_y$. Therefore $\phi(P') \leq \psi(y)$. Let $h : (P', i(P')) \to (R^{2q}(C), c^*)$ be a homomorphism such that $h(a) = \phi(P')$. We have $h(t(P)) \geq \phi(P) = \psi(x)$ because h restricted to P is a homomorphism from $(P, i(P))$ to $(R^{2q}(C), c^*)$. Since $h(t(P))h(a) \in E(R^{2q}(C))$, we have $Ind(h(t(P))) \leq Ind(h(a)) + 1$. Hence $Ind(h(a)) \leq Ind(\psi(y)) < Ind(\psi(x)) \leq Ind(h(t(P))) \leq Ind(h(a)) + 1$. Therefore we must have $h(t(P)) = \psi(x)$ and $h(a) = \psi(y)$, and $\psi(x)\psi(y)$ is an edge of $R^{2q}(C)$.

A similar argument applies for the case $\psi(x) < \psi(y)$. Thus ψ is indeed a homomorphism.

It remains to check that $\psi(g_0) = c^*$. First we have $\psi(g_0) \geq c^*$ because the path P consists of a single vertex is in $\mathcal{P}_{g_0}$ and $\phi(P) = c^*$. If $\psi(g_0) > c^*$, let $P' \in \mathcal{P}_{g_0}$ be an oriented path such that $\phi(P') = \psi(g_0) > c^*$. Let $h : (P', i(P')) \to (R^{2q}(C), c^*)$ be a homomorphism such that $h(t(P')) = \phi(P')$. Since $l(P') = 0$ (because G is balanced), we must have $d_{R^{2q}(C)}(h(i(P')), h(t(P'))) = d_{R^{2q}(C)}(c^*, \psi(g_0)) = 0$. This is a contradiction with $d_{R^{2q}(C)}(c^*, \psi(g_0)) > 0$ implied by $\psi(g_0) > c^*$ (see the end of the first paragraph of this proof). Therefore $\psi(g_0) = c^*$.

COROLLARY 2.5 *Let T be an oriented tree and $t_0 \in T$. Let C and c_0 be as described in our Assumptions. Then $(T, t_0) \to (C, c_0)$ if and only if for any oriented path P, $(P, i(P)) \to (T, t_0)$ implies that $(P, i(P)) \to (C, c_0)$.*

It is easy to see (from Lemma 2.2 and Corollary 2.3) that both the above Lemma and Corollary remain true when C is a directed cycle and c_0 an arbitrary vertex of C.

3 The Main Theorem

The following theorem verifies Conjecture 1.5, and is the main result of this paper.

THEOREM 3.1 *Let C be an unbalanced cycle. A digraph G is homomorphic to C if and only if*

- *every oriented path homomorphic to G is also homomorphic to C, and*
- *the length of any cycle of G is a multiple of the length of C.*

The necessity of the condition is obvious. We shall prove the sufficiency from the following rooted version of the theorem, which is of independent interest.

THEOREM 3.2 *Let G be a digraph and g_0 a fixed vertex of G. Let C and c_0 be as described in our Assumptions. Then $(G, g_0) \to (C, c_0)$ if and only if*

- *for every oriented path P, $(P, i(P)) \to (G, g_0)$ implies that $(P, i(P)) \to (C, c_0)$, and*
- *the length of any cycle of G is a multiple of the length of C.*

It is again the case that the above theorem remains valid if C is a directed cycle and c_0 an arbitrary vertex of C. On the other hand we do not know whether or not Theorem 3.2 remains true if the choice of c_0 is unrestricted.

We first prove that Theorem 3.2 implies Theorem 3.1.

LEMMA 3.3 *Let G be a digraph satisfying the two conditions of Theorem 3.1. Let C and c_0 be as described in our Assumptions. Then one of the following two situations must occur:*

- *there is a vertex $g_0 \in V(G)$ such that for any oriented path P, $(P, i(P)) \to (G, g_0)$ implies $(P, i(P)) \to (C, c_0)$, or*
- *every oriented path P homomorphic to G is also homomorphic to $C \setminus c_0$.*

Proof. Suppose the lemma is not true. Then for any vertex $g \in V(G)$ there is an oriented path P_g such that $(P_g, i(P_g)) \to (G, g)$ and $(P_g, i(P_g)) \not\to (C, c_0)$. Also there is an oriented path $P = [p_1, p_2, \ldots, p_n]$ which is homomorphic to G but not homomorphic to $C \setminus c_0$. Let $h : P \to G$ be a homomorphism. For each $1 \leq j \leq n$, let P_j be an oriented path such that $(P_j, i(P_j)) \to (G, h(p_j))$ and $(P_j, i(P_j)) \not\to (C, c_0)$ (i.e., we write P_j for $P_{h(p_j)}$). Let T be the oriented tree obtained from P by attaching to each vertex p_j of P the oriented path P_j, identifying $i(P_j)$ with p_j.

Obviously $T \to G$. Thus every oriented path homomorphic to T is also homomorphic to G and, by our assumption, also homomorphic to C. By Corollary 2.3 we have $T \to C$.

Let $f : T \to C$ be a homomorphism. Since $P \not\to C \setminus c_0$, there exists $p_j \in P \subset T$ such that $f(p_j) = c_0$. But then f restricted to $P_j \subset T$ is a homomorphism $(P_j, i(P_j)) \to (C, c_0)$, contradicting the assumption that $(P_j, i(P_j)) \not\to (C, c_0)$. This proves Lemma 3.3.

Suppose now that Theorem 3.2 is true and that G is a digraph satisfying the conditions of Theorem 3.1. By Lemma 3.3, either there is a vertex $g_0 \in V(G)$ such that (G, g_0) satisfies the conditions of Theorem 3.2 which implies $(G, g_0) \to (C, c_0)$, or every oriented path homomorphic to G is also homomorphic to $C \setminus c_0$, which again implies $G \to C \setminus c_0$ by Theorem 1.2. In both cases we have $G \to C$ and therefore Theorem 3.2 implies Theorem 3.1.

Next we proceed to prove Theorem 3.2. Since rooted homomorphisms can be composed, it is easy to see that the conditions are necessary for the existence of homomorphisms $(G, g_0) \to (C, c_0)$. Thus suppose the conditions are satisfied.

We first construct an auxiliary digraph D as follows: We take the subpath $c_0, c_1, c_2, \ldots, c_{m-1}, c_0^1, c_1^1, \ldots, c_{j_0-1}^1$ of $R^2(C)$ and identify the vertices c_0 and c_0^1, calling the new vertex c^*. Thus D is a copy of the cycle C with an additional oriented path $A = [c^*, c_1^1, \ldots, c_{j_0-1}^1]$ attached to it at vertex c_0. The path A is just another copy of the path $[c_0, c_1, c_2, \ldots, c_{j_0-1}]$. Note that c^* has indegree zero. We define the index function as $Ind(c_j) = j$ for $1 \leq j \leq m-1$ and $Ind(c_j^1) = m + j$ for $1 \leq j \leq j_0 - 1$ and $Ind(c^*) = m$. For $a, b \in V(D)$, we write $a \leq b$ if and only if $Ind(a) \leq Ind(b)$ (and $a < b$ if $Ind(a) < Ind(b)$). We again use the terms "min" and "max" with respect to this order.

Obviously (D, c^*) and (C, c_0) are homomorphically equivalent, i.e., $(D, c^*) \to (C, c_0)$ and $(C, c_0) \to (D, c^*)$. Therefore $(G, g_0) \to (C, c_0)$ if and only if $(G, g_0) \to (D, c^*)$. Instead of constructing a homomorphism of $(G, g_0) \to (C, c_0)$ we will construct a homomorphism of $(G, g_0) \to (D, c^*)$.

Given a vertex $x \in V(G)$, we let $\mathcal{T}_x$ be the set of triples (T, t_0, t) such that T is an oriented tree and t_0, t are vertices of T and there is a homomorphism $h : T \to G$ with $h(t_0) = g_0$ and $h(t) = x$.

REMARK 3.4 *Let (G, g_0) be a rooted digraph satisfying the conditions of Theorem 3.2. Let C and c_0 be as described in our Assumptions. Then for any rooted tree (T, t_0), $(T, t_0) \to (G, g_0)$ implies $(T, t_0) \to (C, c_0)$.*

Indeed, if $(T, t_0) \to (G, g_0)$, then for any oriented path P the existence of a homomorphism $(P, i(P)) \to (T, t_0)$ implies the existence of a homomorphism $(P, i(P)) \to (G, g_0)$, and therefore the existence of a homomorphism $(P, i(P)) \to (C, c_0)$ (according to one of the hypotheses). Thus $(T, t_0) \to (C, c_0)$ by Corollary 2.5.

Now we are ready to construct a homomorphism $(G, g_0) \to (D, c^*)$.

Define two mappings ϕ and ψ as follows:

For $(T, t_0, t) \in \cup_{x \in V(G)} \mathcal{T}_x$ let

$$\phi(T, t_0, t) = \max\{h(t) : h : (T, t_0) \to (D, c^*)\},$$

and for $x \in V(G)$ let

$$\psi(x) = \min\{\phi(T, t_0, t) : (T, t_0, t) \in \mathcal{T}_x\}.$$

Note that ϕ and ψ are well defined: For any x and any $(T, t_0, t) \in \mathcal{T}_x$ we have $(T, t_0) \to (G, g_0)$ and hence $(T, t_0) \to (C, c_0)$ by the above remark. Thus $(T, t_0) \to (D, c^*)$. It is also clear that for any $x \in V(G)$ the set $\mathcal{T}_x$ is not empty, since G is connected.

In the following we prove that ψ is a homomorphism $(G, g_0) \to (D, c^*)$.

First we need some lemmas which will help us restrict the possible images of a vertex of G under a homomorphism of G to D.

Let $l(C) = k$. For $x \in V(D)$, let $\lambda(x)$ to be the length of the path $[c^*, c_1, c_2, \ldots, x])$ if $x \in C$ and the length of the path $[c^*, c_1^1, \ldots, x])$ if $x \in A$. By the remarks at the begining of Section 2, we have that $\lambda(x) > 0$ for all $x \neq c^*$.

Consider any oriented path P and any homomorphism $h : (P, i(P)) \to (D, c^*)$. The image $h(P)$ of P under h is a walk of D. Since homomorphism of paths preserves distances we have $\lambda_P(x) = \lambda_{h(P)}(h(x))$ for any $x \in P$. The walk $h(P)$ may wind around C several times. Since $l(C) = k$, we have $\lambda_{h(P)}(h(x)) = \lambda(h(x)) + tk$ for some integer t (t can be positive, negative or zero). We state this important fact as a lemma.

LEMMA 3.5 *Let (D, c^*) be the rooted digraph defined above, and let $k = l(C)$. For any oriented path P and any homomorphism $h : (P, i(P)) \to (D, c^*)$, we have $\lambda_P(x) \equiv \lambda(h(x))(\mathrm{mod}(k))$ for all $x \in P$.*

COROLLARY 3.6 *Suppose that H is a connected digraph, and $h_0 \in V(H)$ a fixed vertex of H; suppose further that $h_1 : (H, h_0) \to (D, c^*)$ and $h_2 : (H, h_0) \to (D, c^*)$ are two homomorphisms. Then $\lambda(h_1(x)) \equiv \lambda(h_2(x))(\mathrm{mod}(k))$ for all $x \in H$.*

Note that the path $I = [c^*, c_1, \ldots, c_{j_0}]$ is the only minimal path of length n in D which starts at c^*.

LEMMA 3.7 *Let (D, c^*) be the auxiliary digraph constructed from (C, c_0). Let n be the maximum length of a subpath of C, and k be the length of C. If $X = [x_0, x_1, \ldots, x_t]$ is a minimal walk of D of length n and $\lambda(x_0) \equiv 0(\mathrm{mod}(k))$, then $x_0 = c^*$ and $X \subset I$.*

Proof. Let $X = [x_0, x_1, \ldots, x_t]$ be a minimal walk of D of length n with $\lambda(x_0) = 0(\mathrm{mod}(k))$. First we show that $x_0 = c^*$. Otherwise suppose $x_0 \neq c^*$. It is easy to see (by Lemma 2.1) that $D \setminus c^*$ contains no path of length n, and thus we have $c^* = x_j$ for some $0 \leq j \leq t$. Since $c^* \neq x_0$, we have $\lambda(x_0) > 0$. Since X is minimal, $d_X(x_0, c^*) > 0$. However if X goes in the negative direction of C, then $d_X(x_0, c^*) = -\lambda(x_0) < 0$. Therefore X must go in the positive direction of C. Thus $0 < d_X(x_0, c^*) = k - \lambda(x_0)$, which implies $\lambda(x_0) < k$. Thus $0 < \lambda(x_0) < k$, contradicting the assumption that $\lambda(x_0) \equiv 0(\mathrm{mod}(k))$. Therefore $x_0 = c^*$. Since I is the only minimal path of length n in D which starts at c^*, we see that $X \subset I$.

COROLLARY 3.8 *Suppose that P is an oriented path and B is a minimal subpath of P of length n. If there is a homomorphism $h : (P, i(P)) \to (D, c^*)$ such that $h(B) = I$ then for any homomorphism $h' : (P, i(P)) \to (D, c^*)$ we must also have $h'(B) = I$.*

Proof. Let $h' : (P, i(P)) \to (D, c^*)$ be a homomorphism. Obviously $h'(B)$ is a minimal walk of D of length n. By Corollary 3.6, $\lambda(h'(i(B)) = 0(\mathrm{mod}(k))$. Therefore $h'(i(B)) = c^*$ and $h'(B) = I$ by Lemma 3.7.

In the following we assume that xy is an edge of G. We shall prove that $\psi(x)\psi(y)$ is an edge of D.

LEMMA 3.9 *Let $\psi : V(G) \to V(D)$ be the mapping defined just after Remark 3.4. If $xy \in E(G)$ then $\psi(x) \neq \psi(y)$.*

Proof. Otherwise suppose $\psi(x) = \psi(y) = a$. Let $(T', t'_0, t') \in \mathcal{T}_x$ be a triple such that $\phi(T', t'_0, t') = \psi(x)$, and let $(T'', t''_0, t'') \in \mathcal{T}_y$ be a triple such that $\phi(T'', t''_0, t'') = \psi(y)$.

Let T be the tree obtained from the disjoint union of T' and T'' by adding the edge from t' to t''. Then it is easy to see that $(T, t'_0, t'), (T, t''_0, t') \in \mathcal{T}_x$ and $(T, t'_0, t''), (T, t''_0, t'') \in \mathcal{T}_y$. By the definition of $\psi(y)$ we have $\phi(T, t''_0, t'') \geq \psi(y) = \phi(T'', t''_0, t'')$. Since any homomorphism $h : (T, t''_0) \to (D, c^*)$ restricted to T'' is a homomorphism from (T'', t''_0) to (D, c^*), we have $\phi(T, t''_0, t'') \leq \phi(T'', t''_0, t'')$. Therefore $\phi(T, t''_0, t'') = \phi(T'', t''_0, t'') = a$. Similarly $\phi(T, t'_0, t') = \phi(T', t'_0, t') = a$. Let $h : T \to D$ be a homomorphism such that $h(t'') = a$ and let $h' : T \to D$ be a homomorphism such that $h'(t') = a$. Suppose $h(t') = b$ and $h'(t'') = c$. Then $(b, a) \in E(D)$ and $(a, c) \in E(D)$ because $(t', t'') \in E(T)$ and h, h' are homomorphisms. Therefore a has positive in-degree and positive out-degree. In particular $a \neq c^*$, and therefore a has in-degree one and out-degree one.

To obtain the final contradiction, we consider two cases.

Case 1. Suppose that $a = c_1$. Let $h : (T, t''_0) \to (D, c^*)$ be a homomorphism such that $h(t'') = \phi(T, t''_0, t'') = c_1$. Then $h(t') = c^*$. Delete from T all the vertices t such that $h(t) = c^*$, and let B be the component which contains t'' after the deletion. If $c_{j_0} \notin h(B)$ then $h(B)$ is contained in $I \setminus c_{j_0}$. Now define a mapping $h' : T \to D$ as follows:

$h'(t) = h(t)$ if $t \notin B$, and

$h'(t) = c_j^1$ if $t \in B$ and $h(t) = c_j$.

It is easy to see that h' is a homomorphism and $h'(t''_0) = c^*, h'(t'') = c_1^1$. However $Ind(c_1^1) = m + 1 > Ind(c_1)$. This contradicts the fact that $\phi(T, t''_0, t'') = c_1$. Therefore $c_{j_0} \in h(B)$. Let $s \in B$ be a vertex such that $h(s) = c_{j_0}$ and for which the unique path P connecting t' and s in T has no other vertex u with $h(u) = c_{j_0}$. Thus P is a minimal path of length n.

Now let $h'' : (T, t_0') \to (D, c^*)$ be a homomorphism such that $h''(t') = \phi(T, t_0', t') = c_1$. By Corollary 3.8 $h''(P) = I$, which implies that $h''(t') = c^*$, a contradiction.

Case 2. Assume that $a \neq c_1$. Then for the two neighbours b and c of a we have $b < a < c$. As a has in-degree one and out-degree one, assume that $ba \in E(D)$ and $ac \in E(D)$. (A similar argument applies for the case $ab \in E(D)$ and $ca \in E(D)$).

By the definition of $\psi(x)$, we have $\phi(T, t_0'', t') \geq \psi(x) = a$. Let $h : (T, t_0'') \to (D, c^*)$ be a homomorphism such that $h(t') = \phi(T, t_0'', t')$. Then $(h(t'), h(t'')) \in E(D)$ implies that $h(t'') > a$. This contradicts the fact that $\phi(T, t_0'', t'') = a$, and proves that $\psi(x) \neq \psi(y)$.

LEMMA 3.10 *Let $\psi : V(G) \to V(D)$ be the mapping defined just after Remark 3.4. If $xy \in E(G)$ and $\psi(x) < \psi(y)$ then $\psi(x)\psi(y) \in E(D)$.*

Proof. We proceed as in the proof of Lemma 3.9, constructing (T', t_0', t'), (T'', t_0'', t'') and T; recall that $\phi(T, t_0', t') = \psi(x)$ and $\phi(T, t_0', t'') \geq \psi(y)$. Let $h : (T, t_0') \to (D, c^*)$ be a homomorphism such that $h(t'') = \phi(T, t_0', t'') \geq \psi(y)$. By the definition of ϕ we have $h(t') \leq \phi(T, t_0', t') = \psi(x)$. Therefore $h(t') \leq \psi(x) < \psi(y) \leq h(t'')$. However $h(t')h(t'') \in E(D)$, and hence $Ind(h(t'')) \leq Ind(h(t')) + 1$. Therefore we must have $h(t') = \psi(x)$, $h(t'') = \psi(y)$, and thus $\psi(x)\psi(y) \in E(D)$.

LEMMA 3.11 *Let $\psi : V(G) \to V(D)$ be the mapping defined just after Remark 3.4. If $xy \in E(G)$ and $\psi(x) > \psi(y)$ then $\psi(x)\psi(y)) \in E(D)$.*

Proof. Let $(T', t_0', t'), (T'', t_0'', t'')$ and T be again defined as in the proof of Lemma 3.9; thus we again have $\phi(T, t_0'', t'') = \psi(y)$ and $\phi(T, t_0'', t') \geq \psi(x)$. Let $h : (T, t_0'') \to (D, c^*)$ be a homomorphism such that $h(t') = \phi(T, t_0'', t') \geq \psi(x)$. As in the proof above, we have $h(t'') \leq \psi(y) < \psi(x) \leq h(t')$ and $h(t')h(t'') \in E(D)$. If $Ind(h(t')) \leq Ind(h(t'')) + 1$ then the same argument shows that $\psi(x)\psi(y) = h(t')h(t'') \in E(D)$. Otherwise we must have $h(t') = c^*$ and $h(t'') = c_1$ and $c_1 \leq \psi(y) < \psi(x) \leq c^*$. In the following we prove that in this case we must have $\psi(y) = c_1$ and $\psi(x) = c^*$ and therefore $\psi(x)\psi(y) \in E(D)$.

Let $K = \{t \in T : h(t) = c^*\}$ and let B be the component of $T \setminus K$ which contains t''. With the same argument as in Case 1 of the proof of Lemma 3.9, we find a vertex $s \in B$ such that the unique path P in T joining t' and s is a minimal path of length n. Let $h' : (T, t'_0) \to (D, c^*)$ be a homomorphism such that $h'(t') = \phi(T, t'_0, t') = \psi(x)$. Since $h(t') = c^*$, we have $h(P) = I$. By Corollary 3.8, we have $h'(P) = I$ as well. Therefore $h'(t') = c^* = \psi(x)$.

To show that $\psi(y) = c_1$, we let $h'' : (T, t''_0) \to (D, c^*)$ be a homomorphism such that $h''(t'') = \phi(T, t''_0, t'') = \psi(y)$. Again by Corollary 3.8, $h''(P) = I$ and $h''(t') = c^*$. Observe that t'' is the vertex adjacent to t' in P, we have $h''(t'') = c_1$. Now Lemma 3.11 is proved.

This completes the proof that $\psi : G \to D$ is a homomorphism. To complete the proof of Theorem 3.2, we still need the following:

LEMMA 3.12 *The homomorphism ψ satisfies $\psi(g_0) = c^*$.*

Proof. First we observe that $\psi(g_0) \leq c^*$ because for the tree T^* consisting of a single vertex t_0 we have $(T^*, t_0, t_0) \in \mathcal{T}_{g_0}$ and $\phi(T^*, t_0, t_0) = c^*$.

Assume that $\psi(g_0) < c^*$ and let $(T, t_0, t) \in \mathcal{T}_{g_0}$ be a triple such that $\phi(T, t_0, t) = \psi(g_0)$. Thus there exists a homomorphism $h_1 : T \to G$ with $h(t_0) = h(t) = g_0$ and a homomorphism $h_2 : T \to D$ with $h_2(t_0) = c^*$ and $h_2(t) = \phi(T, t_0, t) = \psi(g_0)$. By identifying c_i^1 with c_i, we view h_2 as a homomorphism of T to C with $h_2(t_0) = c_0$. We shall proceed to construct a homomorphism $h : T \to C$ with $h(t_0) = h(t) = c_0$, which can be viewed as a homomorphism of T to D with $h(t_0) = h(t) = c^*$, in contradiction to $\phi(T, t_0, t) < c^*$.

To construct h we shall use h_1, h_2, and a third homomorphism $h_3 : T \to C$ with $h_3(t) = c_0$. Such a homomorphism exists by Remark 3.4, since $(T, t) \to (G, g_0)$, via h_1. We shall construct h by letting it equal to h_2 on part of the tree T and equal to h_3 on the rest of the tree T. For this purpose we need the following claim:

Claim. Let P be the unique path of T connecting t_0 to t. Then there is a vertex t^* of P such that $h_2(t^*) = h_3(t^*)$.

Since $h_1(P)$ is a closed walk of G, we have $l(P) = l(h_1(P)) = sk$ for an integer q. Now we consider four cases.

Case 1. $s \leq -1$. Since $h_2(t_0) = c_0$ and $l(h_2(P)) \leq -k$, it is easy to see from Lemma 2.1 and the minimality of I that $h_2(P)$ must wind around C in the negative direction at least once. Therefore there is a minimal subpath X of P such that $h_2(X) = I$. By Corollary 3.8, $h_3(X) = I$ and $h_2(i(X)) = h_3(i(X)) = c_0$. Thus in this case we let $t^* = i(X)$.

This argument also shows that the claim follows whenever there exists a vertex $v \in P$ which has level less than or equal to $-k$.

Case 2. $s \geq 2$. The composition $\xi = \psi \circ h_1$ of the two homomorphisms ψ and h_1 is a homomorphism of T to D. Since $\xi(t_0) = \xi(t) = \psi(g_0)$ and C is a cycle of length k, we see that $\xi(P)$ is a closed walk of C which wind around C at least twice. This implies that there is a minimal subpath X such that $\xi(X) = I$, and again by Corollary 3.8, $h(i(X)) = h'(i(X))$. In this case we also let t^* to be $i(X)$.

Case 3. $s = 0$. We choose a large integer q and consider the induced homomorphisms of h_2 and h_3: $h'_2, h'_3 : P \to R^{2q}(C)$ (cf. Section 2). As $h_2(t_0) = h_3(t) = c_0$, we have $h'_2(t_0) = c_0^q$ and $h'_3(t) = c_0^a$ for some a. Let $h''_3 : P \to R^{2q}(C)$ be the homomorphism defined as $h''_3(x) = c_j^{r+q-a}$ if $h'_3(x) = c_j^r$. In other words, h''_3 is obtained from h'_3 by shifting the image so that t is sent to c_0^q. It is obvious that if $h'_2(t^*) = h''_3(t^*)$ for some $t^* \in P$ then $h_2(t^*) = h_3(t^*)$.

Note that Lemma 2.1 implies that $d_{R^{2q}(C)}(c_0^q, v) > 0$ for any $v \in R^{2q}(C)$ with $v > c_0^q$ (recall that the order is defined by the index function). Since $h'_2(P)$ and $h''_3(P)$ are walks of $R^{2q}(C)$ of length zero, and, we must have $h'_2(t) < c_0^q$ and $h''_3(t_0) < c_0^q$. Thus we have $h'_2(t_0) > h''_3(t_0)$ and $h'_2(t) < h''_3(t)$. Let x be the last vertex of P such that $h'_2(x) \geq h''_3(x)$, and let y be the next vertex of P. Thus we have $h'_2(y) < h''_3(y)$. If $h'_2(x) = h''_3(x)$ then we let $t^* = x$ and the claim follows. Assume that $h'_2(x) > h''_3(x)$. Observe that either xy is an edge of P which implies that $h'_2(x)h'_2(y)$ and $h''_3(x)h''_3(y)$ are edges of

$R^q(C)$, or yx is an edge of P which implies that $h_2'(y)h_2'(x)$ and $h_3''(y)h_3''(x)$ are edges of $R^q(C)$. In any case we have $Ind(h_2'(x)) \leq Ind(h_2'(y)) + 1$ and $Ind(h_3''(y)) \leq Ind(h_3''(x)) + 1$. Therefore we must have $h_2'(x) = h_3''(y)$ and $h_2'(y) = h_3''(x)$. This is a contradiction, as $R^{2q}(C)$ has no digons.

Case 4. $s = 1$. Again we let $\xi = \psi \circ h_1$ be the composition of the two homomorphisms ψ and h_1. Since $\xi(P)$ is a closed walk of C of length k, it winds around C exactly once in the positive direction of C. Therefore the induced homomorphism $\xi' : P \to R^{2q}(C)$ satisfies $\xi'(t_0) = c_r^q$ and $\xi'(t) = c_r^{q+1}$, where $c_r = \xi(t_0) = \psi(g_0) = h_2(t)$. In particular there is a vertex $v \in P$ such that $\xi'(v) = c_0^{q+1}$.

Recall that $h_2(t_0) = c_0$. Suppose $h_2(P)$ goes in the negative direction of C. Then the induced homomorphism (for some large q) $h_2' : P \to R^{2q}(C)$ satisfies $h_2'(t_0) = c_0^q$ and $h_2'(t) = c_r^s$ for some $s \leq q-1$. Thus the distance $d_{R^{2q}(C)}(c_0^q, c_r^s) = l(P) = k$. However $d_{R^{2q}(C)}(c_0^q, c_0^{s+1}) = (-k)\cdot(q-(s+1)) \leq 0$. Therefore $d_{R^{2q}(C)}(c_0^{s+1}, c_r^s) \geq k$. This implies that $d_{R^{2q}(C)}(c_r^q, c_0^{q+1}) \leq -k$, and hence the level of the vertex v in P is less than or equal to $-k$. We have already shown that in this case the claim is true (see the remark at the end of the proof of Case 1).

Thus we may assume that $h_2(P)$ goes in the positive direction of C, i.e., $h_2'(t_0) = c_0^q$ and $h_2'(t) > c_0^q$. If there is a vertex v of the tree T such that $h_2'(v) = c_{j_0}^q$, then the path P' of T connecting t_0 to v has length n. Thus it contains a minimal subpath B of length n such that $h_2'(B) = [c_0^q, c_1^1, \cdots, c_{j_0}^q]$. This implies that $h_2(B) = I$ and therfore $h_3(B) = I$ by Corollary 3.8. In this case we let $t^* = i(B)$. On the other hand, suppose that there is no vertex v of T such that $h_2'(v) = c_{j_0}^q$. We delete all the vertices x of T such that $h_2(x) = c_0$. Let B be the component which contains t (recall that $h_2(t) = \psi(g_0) \neq c^*$). Then there is no vertex $x \in B$ such that $h_2(x) = c_{j_0}$, and hence $h_2(B)$ is contained in $I \setminus c_{j_0}$. As in the proof of Lemma 3.9 we can shift the image of B to A (recall that A is just another copy of $I \setminus \{c_{j_0}\}$ attached to c^* in the auxiliary digraph D). The new homomorphism shows that $\phi(T, t_0.t) \geq c^*$, contradicting our assumption.

Now the claim is proved and we can define a homomorphism $h'' : (T, t_0) \to (D, c^*)$ as follows:

Let B be the component of $T \setminus t^*$ which contains t. Let

$h''(x) = h'(x)$ if $x \in B$,

$h''(x) = h(x)$ if $x \notin B$.

Then h'' is obviously a homomorphism and $h''(t_0) = h'(t) = c^*$. Therefore $\psi(g_0) = c^*$ and $(G, g_0) \to (D, c^*)$. This completes the proof of Lemma 3.12, as well as that of Theorem 3.2.

4 General Oriented Cycles

We first construct an example to show that D'_C does not always hold.

Let P_1 and P_2 be two minimal paths of the same length such that $P_1 \not\to P_2$ and $P_2 \not\to P_1$. Such paths are easy to construct; Figure 1 gives one such pair of paths.

Let G be obtained by identifying $t(P_1)$ with $t(P_2)$ and $i(P_1)$ with $i(P_2)$ (cf. the left graph in Figure 2, where the directed edge labeled P_i represents the path P_i). Let C be obtained in a similar way from two copies of P_1 and two copies of P_2, as depicted in the right graph of Figure 2.

It is easy to see that $G \not\to C$ and yet that G satisfies the hypotheses of D'_C. Thus D'_C does not hold for this cycle C. Note that this does not show that D_C fails for C, as G does not satisfy the hypotheses of D_C. However Theorem 4.1 below suggests that D_C may fail for any cycle C such that C-col is NP-complete, cf. Figure 3.

THEOREM 4.1 *If D_C holds for an oriented cycle C then C-col is in $NP \cap$ coNP.*

Proof. Obviously each problem C-col is in the class NP. Let C be a fixed oriented cycle for which D_C holds, i.e., such that a digraph G is homomorphic to C if and only if every cycle homomorphic to G is also homomorphic to C.

In order to prove that C-col is also in coNP we shall prove the following two statements (in Lemmas 4.2 and 4.3).

(1): There is an algorithm which decides whether or not $X \to H$ in time $O(|E(X)| \cdot |E(H)| \cdot |V(H)|)$, for any oriented cycle X and any digraph H.

(2): Let H be a fixed digraph. If there is a cycle X which is homomorphic to a digraph G but not to H, then there is such a cycle X' with $O(|V(G)|)$ edges.

It is not difficult to see that these two statements imply that C-col is in the class coNP. Indeed, if G is a digraph with $G \not\to C$, then there is a cycle X

which is homomorphic to G but not to C. Thus by (2), there is such a cycle X' with $O(|V(G)|)$ edges. By (1), it can be verified in time $O(|V(G)| \cdot |E(G)|^2)$ that X' is indeed homomorphic to G and not to C (observe that the size of C is a constant).

Let $W = [w_0, w_1, \ldots w_{m-1}, w_m]$ be an oriented path, H any digraph and $h_0 \in V(H)$ a fixed vertex of H. The *cannonical labeling* of W by (H, h_0) is the unique mapping λ of W to the subsets of $V(H)$ for which

$\Lambda(w_0) = \{h_0\}$
$\Lambda(w_{i+1}) = \{v \in V(H) : uv \in E(H)$ for some $u \in \Lambda(w_i)\}$ if $w_i w_{i+1} \in E(W)\}$
$\Lambda(w_{i+1}) = \{v \in V(H) : vu \in E(H)$ for some $u \in \Lambda(w_i)\}$ if $w_{i+1} w_i \in E(W)\}$

LEMMA 4.2 *Let $W = [w_0, w_1, \ldots w_{m-1}, w_m]$ be an oriented path and X the oriented cycle obtained from W by identifying w_0 with w_m. Let H be any digraph and let h_0 be a fixed vertex of H. Then $(X, w_0) \to (H, h_0)$ if and only if $h_0 \in \Lambda(w_m)$ in the cannonical labeling of W by H.*

Proof. Suppose $h : (X, w_0) \to (H, h_0)$ is a homomorphism. Then it is easy to show by induction on j that $h(w_j) \in \Lambda(w_j)$ for all $0 \leq j \leq m$. Since $h(w_m) = h(w_0) = h_0$ we have $h_0 \in \Lambda(w_m)$.

On the other hand, suppose $h_0 \in \Lambda(w_m)$ in the cannonical labeling of W by (H, h_0). A homomorphism $h : (X, w_0) \to (H, h_0)$ can be constructed as follows: Let $h(w_m) = h(w_0) = h_0$. If $h(w_j) = v_j \in \Lambda(w_j)$ has been chosen, then let $h(w_{j-1}) = v_{j-1}$, where v_{j-1} is an element of $\Lambda(w_{j-1})$ such that either $(v_{j-1}, v_j) \in E(H)$ or $(v_j, v_{j-1}) \in E(H)$, according to whether $(w_{j-1}, w_j) \in E(W)$ or $(w_j, w_{j-1}) \in E(W)$. Such an element exists by the definition of the cannonical labeling. It is clear that the mapping h is a homomorphism.

The cannonical labeling of W by (H, h_0) can be found in time $O(|E(W)| \cdot |E(H)|)$. Thus it can be determined in time $O(|E(W)| \cdot |E(H)|)$ whether or not $(X, w_0) \to (H, h_0)$. In order to determine whether or not $W \to H$, it is enough to determine whether or not $(W, w_0) \to (H, h)$ for some $h \in V(H)$. Therefore it suffices to find the cannonical labeling of W by (H, h) for each of the vertices $h \in V(H)$. Thus it can be determined in time $O(|E(X)| \cdot |E(H)| \cdot |V(H)|)$ whether or not $X \to H$.

LEMMA 4.3 *Let $V(H) = k$. If there exists an oriented cycle X homomorphic to G but not to H, then there exists such a cycle X with $|V(X)| \leq 2^{k^2} \cdot |V(G)|$.*

Proof. Suppose that X is obtained from the oriented path $W = [w_0, w_1, \ldots, w_m]$ by identifying w_0 with w_m, and that $X \to G$ and $X \not\to H$. Let $f : X \to G$ be a homomorphism. For each vertex $h \in V(H)$ let Λ_h be the cannonical labeling of W by (H, h). By the previous lemma, $h \notin \Lambda_h(w_m)$ for any $h \in V(H)$ (for otherwise we would have $(X, w_0) \to (H, h)$, and hence $X \to H$). If $m > 2^{k^2} \cdot |V(G)|$ then by the pigeon hole principle there two vertices w_i, w_j of W $(i < j)$ such that $f(w_i) = f(w_j)$ and $\Lambda_h(w_i) = \Lambda_h(w_j)$ for all $h \in V(H)$. (The mappings f and $\{\Lambda_h : h \in V(H)\}$ can be viewed as a single mapping of W into the set $V(G) \times 2^{V(H)} \times 2^{V(H)} \cdots \times 2^{V(H)}$ of size $2^{k^2} \cdot |V(G)|$.) Let $W' = [w_0, w_1, \ldots, w_i, w_{j+1}, \ldots, w_m]$ (i.e., W' is obtained from W by deleting all the vertices $w_{i+1}, \cdots, w_{j-1}$ and identifying w_i with w_j), and let X' be the cycle obtained from W' by identifying w_0 with w_m. Then obviously $W' \to G$ and for the cannonical labeling Λ'_h of W' by (H, h) we have $\Lambda'_h(w_t) = \Lambda_h(w_t)$ for all $w_t \in W'$ and all $h \in V(H)$. Therefore $h \notin \Lambda'_h(w_m)$ for all $h \in V(H)$ and $X' \not\to H$.

COROLLARY 4.4 *Let G be a digraph and let C be an oriented cycle with k edges. If there exists a cycle homomorphic to G but not to C, then there exists such a cycle X with $|V(X)| \leq 2^{k^2} \cdot |V(G)|$.*

As C is fixed, 2^{k^2} is a constant. So the size of X is $O(|V(G)|)$. Thus we have proved statements (1) and (2), as well as Theorem 4.1.

It follows from Corollary 4.4 and Theorem 3.1 that C-col is in $NP \cap coNP$ whenever C is unbalanced. In fact, it follows from [8, 30] that C-col is polynomial for unbalanced cycles C. This can also be derived from Theorem 3.1, using a technique explained in [15]. On the other hand, suppose that P_i ($i = 1, 2, 3, 4, 5, 6$) are minimal oriented paths of length n such that $P_i \not\to P_j$ whenever $i \neq j$. (Such paths are easy to construct using the technique apparent in Fig. 1.) Gutjahr proved that C-col is NP-complete for the balanced cycle C depicted in Figure 3, [8].

Remark on new results. In [15], we shall argue that statements like D_C can be viewed as "duality" properties of graph homomorphisms. In the terminology of [15], our main result here asserts that unbalanced cycles have *cycle duality*, and Theorem 1.2 asserts that oriented paths have *path duality*. We have recently considered more general duality statements: A digraph H is said to have *treewidth-k duality* just if a digraph G is homomorphic to H if and only if every oriented partial k-tree homomorphic to G is also homomorphic to H. Since oriented paths are partial 1-trees and oriented cycles are partial 2-trees, we have many examples of graphs with treewidth-k duality. It is proved in [15] that if H has treewidth-k duality (for any k), then H-col is polynomial. This allows us to conclude, from our main theorem, that C-col is polynomial for each unbalanced cycle C. Polynomial algorithms for this problem have previously been proposed by X. Zhu ([30], motivated by the main technique of this paper), and independently by W. Gutjahr, [8]. (It also allows us to conclude from Theorem 1.2 that P-col is polynomial for each oriented path P; this was first proved in [9].) T. Feder has shown that the class of graphs with treewidth-k duality corresponds exactly with H-col problems he calls *of bounded width*, which admit polynomial Datalog algorithms. Most recently, Feder has shown that for all oriented cycles C, the problem C-col is either polynomial or NP-complete, [7]. J. Nesetril and X. Zhu, [25], have shown that there exist balanced oriented cycles which have no treewidth-k duality for any integer k. This implies, in particular, that D_C does not hold for these cycles.

References

[1] J. Bang-Jensen and P. Hell, *On the effect of two cycles on the complexity of colouring,* Discrete Applied Math. 26 (1990), 1-23.

[2] J. Bang-Jensen, P. Hell and G. MacGillivray, *The complexity of colouring by semicomplete digraphs,* SIAM J. on Discrete Math. 1 (1988), 281-298.

[3] J. Bang-Jensen, P. Hell and G. MacGillivray, *On the complexity of colouring by superdigraphs of bipartite graphs,* Discrete Math. 110 (1992), in print.

[4] J. Bang-Jensen, P. Hell and G. MacGillivray, *Hereditarily hard colouring problems,* submitted to J. Comput. Systems Science.

[5] S. Bloom and S. Burr, *On unavoidaable digraphs in orientation of graphs*, J. Graph Theory, 11(1987), 453-462.

[6] T. Feder and M. Vardi, *Monotone monadic SNP and constraint satisfaction: an approach via Datalog and group theory*, manuscript, 1993.

[7] T. Feder, *Classification of homomorphisms to oriented cycles*, manuscript 1993.

[8] W. Gutjahr, *Graph colorings*, Ph. D. Thesis, Free University, Berlin, 1991.

[9] W. Gutjahr, E. Welzl, and G. Woeginger, *Polynomial graph colourings*, Disc. Applied Maths., 35 (1992), 29-46.

[10] R. Häggkvist, P. Hell, D.J. Miller and V. Neumann-Lara, *On multiplicative graphs and the product conjecture,* Combinatorica 8 (1988), 71-81.

[11] R. Häggkvist and P. Hell, *On A-mote universal graphs,* European J. of Combinatorics 13 (1992), in print.

[12] F. Harary, **Graph Theory**, Addison Wesley, 1969.

[13] P. Hell, *An introduction to the category of graphs,* Annals of the N.Y. Acad. Sc. 328 (1979), 120-136.

[14] P. Hell and J. Nešetřil, *On the complexity of H-colouring,* J. Combin. Theory B 48 (1990), 92-110.

[15] P. Hell, J. Nešetřil and X. Zhu, *Duality theorems and polynomial tree-coloring,* Manuscript, 1992.

[16] P. Hell, J. Nešetřil and X. Zhu, *NP-complete tree-coloring and forbidden subtrees,* Manuscript, 1992.

[17] P. Hell and J. Nešetřil, *Homomorphisms of graphs and their orientations,* Monatshefte fűr Math. 85 (1978), 39-48.

[18] P. Hell and X. Zhu, *Homomorphisms to oriented paths*, Discete Math., in print.

[19] P. Hell, H. Zhou and X. Zhu, *Homomorphisms to oriented cycles*, Combinatorica, in print.

[20] P. Hell, H. Zhou and X. Zhu, *Multiplicativity of oriented cycles*, J. Combin. Th. (B), in print.

[21] P. Komárek, *Some new good characterizations of directed graphs*, Časopis Pěst. Mat. 51 (1984), 348-354.

[22] G. MacGillivray, *On the complexity of colouring by vertex-transitive and arc-transitive digraphs*, SIAM J. Discrete Math. 4 (1991) 397 - 408.

[23] H.A. Maurer, J.H. Sudborough and E. Welzl, *On the complexity of the general coloring problem* Information and Control 51 (1981), 123-145.

[24] J. Nešetřil and A. Pultr, *On classes of relations and graphs determined by subobjects and factorobjects*, Discrete Math. 22 (1978), 287-300.

[25] J. Nešetřil and X. Zhu, *On bounded treewidth duality of graphs*, manuscript, 1994.

[26] E. Welzl, *Symmetric graphs and interpretations*, J. Combin. Th.(B), 37(1984), 235-244.

[27] H. Zhou, *Homomorphism properties of graph products*, Ph.D. Thesis, Simon Fraser University, 1988.

[28] H. Zhou, *Characterization of the homomorphic preimages of certain oriented cycles*, SIAM J. Discrete Math. 6 (1993) 87 - 99.

[29] X. Zhu, *Multiplicative structures*, Ph.D. Thesis, The University of Calgary, 1990.

[30] X. Zhu, *A polynomial algorithm for homomorphisms to oriented cycles*, Manuscript, 1991.