Relaxed game chromatic number of trees and outerplanar graphs

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Abstract

This paper studies the relaxed game chromatic number of trees and outerplanar graphs. It is proved that if G is a tree then for $d \geq 2$, $\chi_g^d(G) \leq 2$. If G is an outerplanar graph, then for $d \geq 2$, $\chi_g^d(G) \leq 5$; for $d \leq 4$, $\chi_g^d(G) \geq 3$.

Key words: Relaxed game chromatic number, tree, outerplanar graph.

1 Introduction

The relaxed game chromatic number of a graph is defined through a two person game. Let $G = (V, E)$ be a graph. Let k and d be positive integers. The (k, d)-*relaxed coloring game*

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is a game played by two persons, Alice and Bob, who alternately color the vertices of G with colors from a set X of k colors. A color $\alpha \in X$ is legal for an uncolored vertex $v \in V$ if the following two conditions hold:

1. v is adjacent to at most d vertices already colored α; and

2. if u is adjacent to v and u has already been colored with α, then u is adjacent to at most $d-1$ vertices already colored α.

In other words, by coloring v with α, the subgraph induced by all vertices of color α has maximum degree at most d. Each move of Alice or Bob colors an uncolored vertex with a legal color. Alice wins the game if all vertices of the graph are legally colored, Otherwise Bob wins. The *d-relaxed game chromatic number* $\chi_g^d(G)$ of G is the least number k for which Alice has a winning strategy in the above coloring game. If $d = 0$, then the paramter is called the *game chromatic number* of G and is also denoted by $\chi_g(G)$.

The game chromatic number of a graph was introduced by Bodlaender [2]. Bounds for $\chi_g(G)$ for various classes of graphs G have been obtained in the literature [11, 13, 15, 16, 20, 21]. In particular, forests have game chromatic number at most 4 [11], outerplanar graphs have game chromatic number at most 7 [13], planar graphs have game chromatic number at most 18 [15], partial k-trees have game chromatic number at most $3k+2$ [21]. The concept of relaxed game chromatic number was recently introduced by Chou, Wang and Zhu [4], who proved that $\chi_g^d(G) \leq 3$ if G is a forest and $d \geq 1$, and $\chi_g^d(G) \leq 6$ if G is an outerplanar graph and $d \geq 1$. It was also shown in [4] that there are trees T with $\chi_g^3(T) = 2$. Answering a question asked in [4], Dunn and Kierstead [9] proved that $\chi_g^d(G) \leq 2$ if G is a forest and $d \geq 3$. Moreover, they proved that $\chi_g^d(G) \leq k+1$ if G is a partial k-tree and $d \geq 4k-1$. As outerplanar graphs are partial 2-trees, so $\chi_g^d(G) \leq 3$ if G is an outerplanar graph and $d \geq 7$. Dunn and Kierstead [9] then asked the question whether $\chi_g^2(T) \leq 2$ for all forests T. This paper answers this question in affirmative. Then we prove that if G is an outer planar graph and $d \geq 2$, then $\chi_g^d(G) \leq 5$. It is interesting to note that for forests, the relaxed game chromatic number can be expressed in a uniform formula: if G is a forest and $0 \leq d \leq 2$ then $\chi_g^d(G) \leq 4-d$. These bounds for relaxed chromatic number of forests are known to be sharp. For outerplanar graphs, a uniform formula for the bounds of the relaxed game chromatic number is conjectured. Namely, for $2 \leq k \leq 7$, if $d \geq 7-k$ then $\chi_g^d(G) \leq k$. We prove that if the conjecture is true, then for the case $k=2$, the bound is sharp. However, for the other cases, it is unknown whether or not the bound is sharp. In particular, it is unknown if there is an outerplanar graph G with $\chi_g(G) = 7$.

2 Relaxed game chromatic number of forests

In this section, we will prove that if F is a forest and $d \geq 2$ then $\chi_g^d(F) \leq 2$.

The case $d \geq 3$ was proved in [9]. So it suffices to prove that $\chi_g^2(F) \leq 2$. In other words, it suffices to give a winning strategy for Alice for the $(2,2)$-relaxed coloring game.

The strategy is a separation strategy. Suppose in the process of the game, the forest F is partially colored. We delete those edges whose two end vertices are colored by distinct colors. Each connected component of the resulting forest is called a *trunk*. It is easy to see that the colors of vertices of different trunks do not affect each other. So in the remaining of the game Alice only need to consider each trunk separately. Note that the trunks are dynamic. In the process of the game, the forest will break into smaller and smaller trunks. If T is a trunk and some more vertices of T are colored, then each resulting trunk is called a *descendant trunk* of T. Note that a descendant trunk of T could be smaller (as T may break into more than one trunks), and also could be of the same size (just with more colored vertices).

Let T be a trunk. Let $C(T)$ be the set of colored verticees in T and let $A(T)$ be the union of paths of T connecting vertices of $C(T)$. We call $A(T)$ the *active subtree* of T. Let $X(T) = \{x \in A(T) : d_{A(T)}(x) \geq 3\}$, where $d_{A(T)}(x)$ is the degree of x in the subtree $A(T)$.

A *monochromatic path* if a path all its vertices are colored the same color. Note that if $A(T) \subset C(T)$, i.e., every vertex of $A(T)$ is colored, then $A(T)$ is a monochromatic path. In this case, with an abuse of notation, we call T a monochromatic path.

For a vertex s of T, each connected component of $T \setminus \{s\}$ is called a *branch* of T with respect to s.

Two conventions: (1) Suppose B is a branch of a trunk T with respect to a colored vertex s. When we say Alice colors a vertex u of B, it is implied that u is an uncolored vertex of B which is nearest to s, i.e., $d_T(u,s) = \min\{d_T(v,s) : v \in B \text{ and } v \text{ is uncolored }\}$. Let w be the neighbour of u for which $d_T(w,s) = d_T(u,s) - 1$. Provided it is legal, Alice colors u with the opposite color of w (note that by the choice of u and because s is a colored vertex, w is a colored vertex).
(2) Unless specified otherwise, when Alice colors a vertex u, she uses a color which is least used among the colored neighbours of u, provided it is legal. In case both colors are legal and each used the same number of times among the colored neighbours of v, Alice arbitrarily picks a color. Otherwise, if u has only one legal color, then Alice uses that color.

A trunk T is called a *star trunk*, if the following hold:

- $|X(T)| \leq 1$ and $X(T) \subset C(T)$.
- If $X(T) = \{s\}$, then each branch of T with respect to s either has at most two colored vertices or the colored vertices form a monochromatic path starting from s (i.e., the vertex of the branch adjacent to s is colored).
- For any $u \in A(T) \setminus C(T)$, $c(u) \leq 4$, where $c(u) = |visinC(T) : 1 \leq d(u,v) \leq 2|$.

Note that a monochromatic path is a star trunk. In case T is a star trunk with $X(T) = \emptyset$, i.e., $A(T)$ is a path, we let s be a colored vertex in the middle of $A(T)$, so that each branch of T with respect to s has at most two colored vertices.

Let D_1, D_2, D_3, D_4, D_5 be trunks as depicted in Fig. 1 below, where a solid round vertex is a colored vertex, a solid square vertex is a colored vertex with no neighbour colored the same color, a solid triangle vertex is a colored vertex with at most one neighbour colored the same color. A hallowed round vertex is an uncolored vertex. A line indicates a path of length at least 1. Trunks as D_1, D_2, D_3, D_4, D_5 are called *D-type trunks.* The dotted ellipses in the figure indicates branches with respect to v, which are needed later in the description of Alice's strategy (so they can be ignored at this moment).

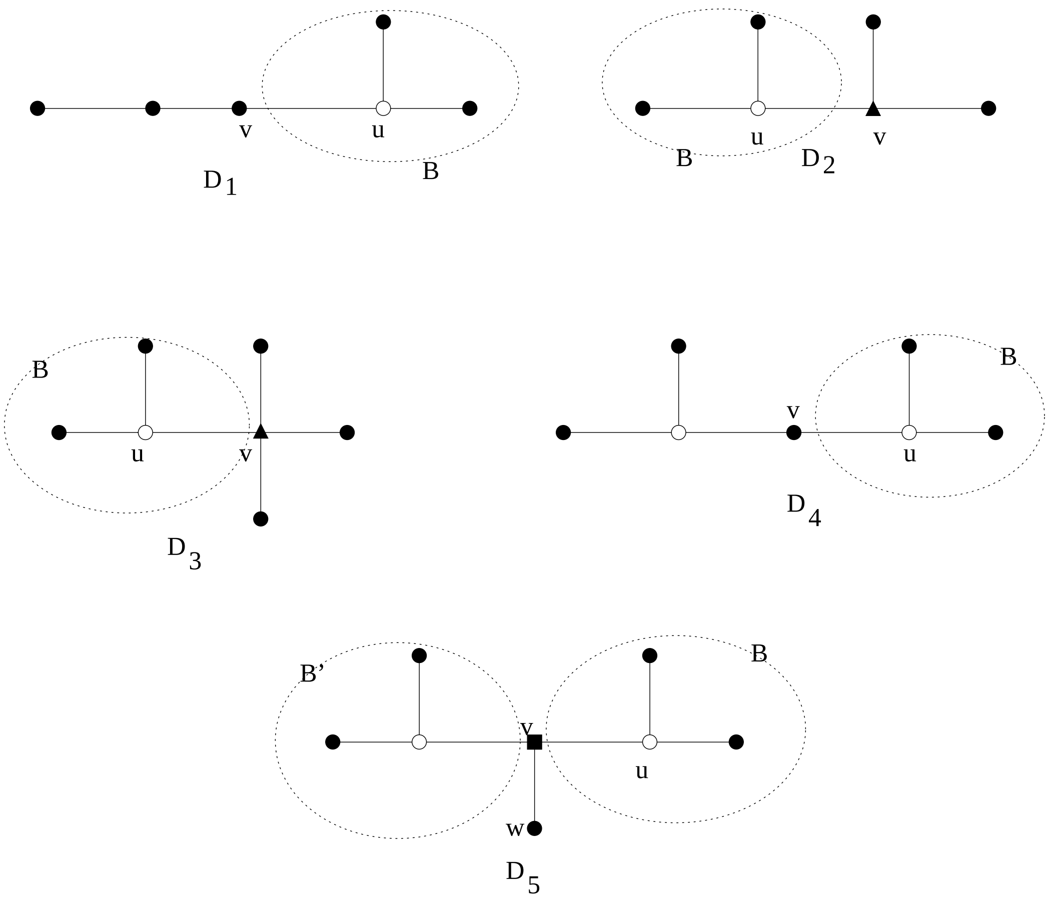

Figure 1: D-type trunks

Note that in the figure, we have only drawn the active subtree $A(T)$ of the trunk T. So T may have other vertices.

A trunk T is said to have *property (P)* if either T is a star trunk or a D-type trunk or has at most 4 colored vertices.

Theorem 2.1 *If F is a forest with at least one vertex of degree ≥ 3, then $\chi_g^2(F) = 2$.*

Proof. As F has a vertex of degree ≥ 3, so $\chi_g^2(F) \geq 2$. It remains to show that for the $(2,2)$-relaxed coloring game on F, Alice has a winning strategy.

We shall prove that Alice has a strategy to ensure that after each of her move, all the trunks have property (P). This is trivially true at the start of the game. Assume this is true after Alice's jth move. It is easy to see that after Bob's jth move, at most one trunk fails to have property (P). We leave it to the readers to verify that if after Bob's jth move, each trunk does have property (P), then Alice can take her $(j+1)$th move so that each descendant trunk still has property (P). In the following, we consider the case that after Bob's jth move, there is exactly one trunk, say T', which does not have property (P).

Assume that T' is a descendant trunk of T, where T is a trunk after Alice's jth move (and hence T has property (P)).

Suppose T is a star trunk and Bob has just colored a vertex in branch B with respect to s (where s is the vertex as in the definition of star trunk and in the remark following the definition). If $A(T) \setminus C(T)$ has a vertex u with $c(u) = 5$, then Alice colors u. Otherwise, Alice colors a vertex of B (recall our convention of coloring such a vertex). Note that B must have an uncolored vertex, otherwise B is a monochromatic path and hence T' is still a star trunk, contrary to our assumption. It is easy to verify that after Alice's move, each descendant trunk of T' is either a star trunk, or has most 4 colored vertices, and hence has property (P).

If T is a D_i type trunk for some $i = 1, 2, 3, 4, 5$, then no matter which vertex Bob colored in his jth move (except in D_5, in which case, by symmetry, we may assume that the vertex colored by Bob in his jth move is not in B'), Alice colors a vertex of B, which is the branch of T with respect to v as shown in Fig. 1. Note that B contains uncolored vertices, for otherwise Bob must have colored u in his jth move. Moreover, all the vertices of $A(T') \cap B$ are colored, which implies that at least one neighbour of u is colored with different color as u, and hence T breaks into at least two trunks. But then T' is either a star trunk and D_1 type trunk, contrary to our assumption that T' does not have property (P).

Now we show that after Alice's move, each descendant trunk of T' has property (P). Most of the verifications are straightforward. In case of D_1, D_2, D_3, D_4, each descendant trunk of T' either has at most 4 colored vertices, or is a star trunk, or a D_2 type trunk. Observe that it is crucial that in D_2 and D_3, v has at most one neighbour colored the same color as v. This guarantees that the vertex chosen by Alice does have a legal color. If v has two colored neighbours in T colored the same color as v, say colored with color 1, then in the worst case, u and v have distance 2, and u has two neighbours colored with color 2. In Bob's jth move, he colors u with color 2. Then the common neighbour w of u and v (which is the vertex chosen by Alice according the strategy described above) has no legal color, and hence Bob wins the game.

In the D_5 case, after Alice's move, the descendant trunk of T' contained in B has at most 4 colored neighbours. The other descendant trunk is either a D_2 type trunk or a D_4 type trunk (note that in these cases v has at most one neighbour colored the same color, because in T, v had no neighbour colored the same color) or a D_1 type trunk. Observe that it is again crucial that v has no neighbour in T colored the same color, which implies that w is not adjacent to v. Otherwise, assume v and a neighbour are colored by color 1. Then Bob can color a neighbour w of v (w is not in $A(T)$ and so is not shown in the figure) with color 1. Then after Alice's $(j+1)$th move, we obtain a trunk similar to D_2, except that v has two neighbours colored the same color as v. As discussed in the previous paragraph, under such a situation, in the worst case, Bob wins the game.

Finally, we consider the case that T has at most 4 colored vertices. As T' is a descendant trunk of T (after one more move of Bob) and T' does not have property (P), we conclude that T' has exactly 5 colored vertices. Moreover $X(T') \neq \emptyset$, for otherwise T' is a star trunk. Assume $X(T') = \{s\}$. Then s is uncolored, for otherwise T' is again a star trunk. If every branch of T' with respect to s has at most 2 colored vertices, then Alice colors s, so each descendant trunk of T' is either a star trunk or has at most 4 colored vertices, and hence has property (P). If there is a branch of T' with respect to s which has 3 colored vertices, then Alice colors a vertex of that branch to break T' into two trunks so that these 3 colored vertices are separated from the other colored vertices of T'. It is easy to see that each descendant trunk has at most 4 colored vertices, and hence has property (P).

Assume next that $|X(T')| = 2$. If both vertices of $X(T)$ are uncolored, then T' is as depicted in Fig. 2(a) or Fig. 2(b) or Fig. 2(c). In Fig. 2(a) or 2(b), Alice colors the left neighbour of v, so each descendant trunk of T' has at most 4 colored neighbours and hence has property (P). In Fig. 2(c), Alice colors vertex v and obtain a D_3 type trunk (or a D_2 type trunk, if v has a colored neighbour; or a trunk with at most 4 colored vertices if v has two neighbours colored the same color). If one of the vertex of $X(T)$ is colored, then T' is as depicted in Fig. 2(d), and again Alice colors the left neighbour of v and each descendant trunk of T' has at most 4 colored neighbours and hence has property (P). Note that it cannot happen that both vertices of $X(T)$ are colored, as T' has only 5 colored vertices.

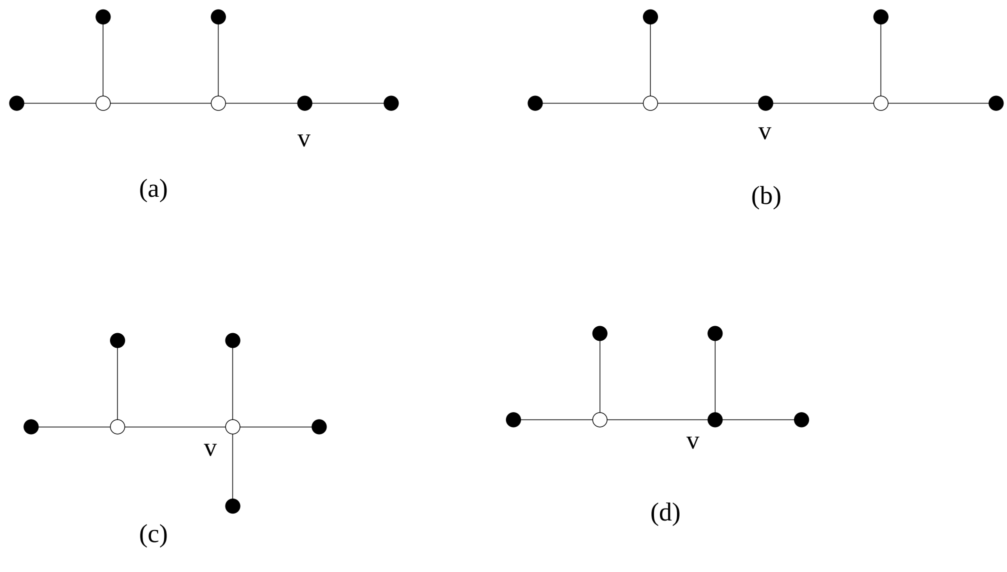

Figure 2: Possible configurations when $|X(T)| = 2$

If $|X(T)| = 3$, then T is as depicted in Fig. 3 and Alice colors vertex v, obtain a D_5 type trunk (or a D_4 type trunk if the colored vertex below v is adjacent to v).

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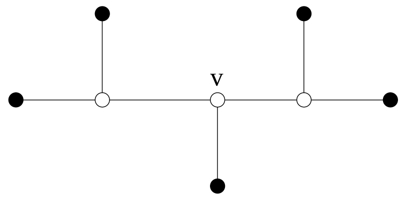

Figure 3: The case that $|X(T)| = 3$

3 Outerplanar graphs

It was proved in [4] that if G is an outerplanar graph then for any $d \geq 1$, Alice has a winning strategy for the $(6, d)$-relaxed coloring game for G. In this section, we proves that for $d \geq 2$, Alice has a winning strategy for the $(5, d)$-relaxed coloring game for G.

Let G be a 2-connected triangulated outerplanar graph, which is an outerplanar graph each of whose inner faces is a triangle. We produce an ordering of the vertices of G as follows: start from an edge incident to the infinite face, label the two vertices v_1, v_2. Let v_3 be the other vertex of the triangle (inner face) incident to the edge v_1v_2. Suppose we have labeled vertices $v_1, v_2, \cdots, v_i$, and there are unlabeled vertices. Then choose a triangle which contains only one unlabeled vertex and label it v_{i+1}. This method produces a labeling $v_1, v_2, \cdots, v_n$ of $V(G)$ such that for each j $(3 \leq j \leq n)$, v_j is adjacent to two labeled vertices v_{j_1}, v_{j_2} with $j_1 < j_2 < j$. We call v_j a *parent* of v_i if $v_i \sim v_j$ and $j < i$. The ordering constructed above has the following properties:

1. For $i \geq 3$, the vertex v_i has exactly two parents and these two parents are adjacent.
2. If $i \neq j$, then v_i and v_j cannot have the same two parents.

For each $i \geq 3$, suppose v_{i_1}, v_{i_2} are the two parents of v_i. If $i_1 < i_2$, we call v_{i_1} the *major parent* of v_i, and call v_{i_2} the *minor parent* of v_i. The vertex v_i is called a *major child* of v_{i_1} and a *minor child* of v_{i_2}. For each vertex x of G, we shall denote by $f(x)$ the major parent of x, and denote by $m(x)$ the minor parent of x. For convenience, we let $f(v_1) = f(v_2) = v_1 = m(v_1) = m(v_2)$.

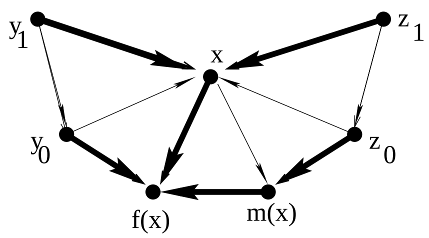

Figure 4: The parents and children of x

Note that if two vertices of G are joined by an edge, then one is a parent of the other. If w is a minor child of x, then $f(w)$ is a parent of x. Since any vertex can have at most two parents, a vertex x can have at most two minor children, one (say y_0) with major parent $f(x)$ and the other (say z_0) with major parent $m(x)$. The children of x belong to at most two paths $\cdots y_1 y_0 x$ and $\cdots z_1 z_0 x$ in which each vertex is a minor child of the next – see Fig. 4, in which an arrow from a to b denotes that a is a child of b, and thick lines denote major children. (The line from $m(x)$ to $f(x)$ could be either thick or thin.)

Theorem 3.1 *Suppose $G = (V, E)$ is a 2-connected triangulated outerplanar graph and $d \geq 2$. For the $(5, d)$-coloring game on G, Alice has a winning strategy.*

Proof. Let G be a 2-connected triangulated outerplanar graph and let $v_1, v_2, \cdots, v_n$ be an ordering of the vertices of G which has the properties listed above. We say v_i is less than v_j if $i < j$.

We shall first describe the strategy for Alice to pick the vertex to be colored. Let U denote the set of uncolored vertices. Alice maintains a subset $A \subseteq V$ of *active* vertices. Initially $A = \emptyset$. When a new vertex is put into A, we say x is activated. Once a vertex is activated, it remains active forever. Initially, Alice colors v_1 and activates v_1. Now suppose that Bob has colored the vertex b. Alice updates A and chooses the next vertex x to be colored by using the following strategy:

Alice will jump from vertex to vertex until she finds the vertex she wants to color. The so called "jumps" are done by applying the following rules successively:

First Alice jumps to b. Assume Alice has jumped to a vertex x.

Rule 1. If x is active and uncolored, then she colors x;

Rule 2. If x is inactive, uncolored, and both $f(x)$ and $m(x)$ are colored, then she activates x and colors x;

Rule 3. If $x = v_1$, then Alice colors the least uncolored vertex (and activates it if it is not active yet);

Rule 4. If none of the above is true, then Alice activates x (if x is inactive), and jumps to either $f(x)$ or $m(x)$ (by following the Jumping Rule below) and returns to Rule 1.

Jumping Rule: *If $f(x)$ is uncolored, or $f(x)$ and x are colored the same color, then jump to $f(x)$; otherwise, jump to $m(x)$.*

After choosing the vertex x to be colored, Alice finds a legal color for x as follows: if the colored neighbours of x use at most 4 colors, then she colors x with any color not used by its colored neighbours; if the colored neighbours of x use 5 colors, then we shall prove that one of the colors not used by the parents of x is legal for x. Alice will color x with such a legal color.

To prove the existence of such a legal color for x, we construct a directed graph D with vertex set $V(D) = V(G)$ as follows: Consider all Alice's moves before the current move (i.e., before the move in which she chooses x). Put a directed edge from v to v' if in a certain step, Alice jumped from v to v' and v' is not colored before this jump. Parallel directed edges are allowed, i.e., if Alice jumped twice from v to v' before v' being colored, then there are two directed edges from v to v'.

By Alice's strategy, if Alice jumps once to a vertex v' then v' is activated, and if she jumps twice to v' then v' is colored. So in the directed graph D defined above, each vertex has in-degree at most 2, and each uncolored vertex has in-degree at most 1. Since x is uncolored, x has in-degree at most 1 in D.

By our strategy, each time a major child w of x is activated, Alice will jump from w to x, which will result in a directed edge from w to x. Since x has in-degree at most 1, so x has at most 1 active major child. Hence x has at most 5 active neighbours: two parents $f(x)$ and $m(x)$, two minor children, say u_1, u_2, and one major child, say u_3. Each colored vertex is either active, or has just been colored by Bob. Therefore x has at most 6 colored neighbours. Assume that the colored neighbours used 5 colors. Then no color is used by 3 neighbours of x. So if a color i is not legal for x, the only reason is that x has a neighbour y which is colored by color i and is adjacent to 2 other vertices colored by color i.

There are at least 3 colors, say $1, 2, 3$, not used by the parents of x. Assume that none of the three colors is legal for x. Then x has three children, say u_1, u_2, u_3, such that u_i is colored by color i and is adjacent to two other vertices, say $w_{i,1}, w_{i,2}$ of color i.

We shall prove that for each i, when all the three vertices $u_i, w_{i,1}, w_{i,2}$ are colored, there are at least two directed edges in D from the set $\{u_i, w_{i,1}, w_{i,2}\}$ to $\{f(x), m(x), x\}$.

First of all, whether u_i is a minor or major child of x, when u_i is first activated, Alice jumps from u_i to $\{f(x), m(x), x\}$, resulting in a directed edge from u_i to $\{f(x), m(x), x\}$. If one of $w_{i,1}, w_{i,2}$, say $w_{i,1}$, is also a child of x, then when $w_{i,1}$ is first activated, Alice jumps from $w_{i,1}$ to $\{f(x), m(x), x\}$, resulting another directed edge from the set $\{u_i, w_{i,1}, w_{i,2}\}$ to $\{f(x), m(x), x\}$.

Assume that none of $w_{i,1}, w_{i,2}$ is a child of x. Then both $w_{i,1}, w_{i,2}$ are children of u_i. If both of them are minor children of u_i, then at least one of them is a child of x, contrary to our previous assumption. So at least one of them is a major child of u_i.

Lemma 3.1 *The last colored vertex of* $u_i, w_{i,1}, w_{i,2}$ *is colored by Bob.*

Proof. If the last colored vertex is $w_{i,j}$ for some $j = 1, 2$, then it is colored by Bob, because Alice never colors a vertex the same color as a parent of that vertex. If the last colored vertex is u_i, then u_i has an uncolored parent x, so u_i has at most 5 colored neighbours. Moreover, u_i has two children colored the same color, so the colored neighbours of u_i use at most 4 colors. By our strategy, if Alice colors u_i, she colors it with a color not used by its colored neighbours. ∎

If both $w_{i,1}, w_{i,2}$ are major children of u_i, then when Bob colors the last vertex of $u_i, w_{i,1}, w_{i,2}$, Alice jumps again from u_i to $\{f(x), m(x), x\}$, resulting another directed edge from the set $\{u_i, w_{i,1}, w_{i,2}\}$ to $\{f(x), m(x), x\}$. Assume that one of $w_{i,1}, w_{i,2}$, say $w_{i,1}$, is a major child of u_i and the other, $w_{i,2}$, is a minor child of u_i. If the last colored vertex among $u_i, w_{i,1}, w_{i,2}$ is not $w_{i,2}$, then the argument above shows that there is another directed edge from u_i to $\{f(x), m(x), x\}$. Assume that the last colored vertex among $u_i, w_{i,1}, w_{i,2}$ is $w_{i,2}$.

Lemma 3.2 *If $w_{i,1}$ is a major child of u_i, $w_{i,2}$ is a minor child of u_i, and $w_{i,2}$ is the last colored vertex among $u_i, w_{i,1}, w_{i,2}$, then the last colored vertex among $u_i, w_{i,1}$ is colored by Bob.*

Proof. If the last colored vertex is $w_{i,1}$, then it is colored by Bob, because Alice never colors a vertex the same color as a parent of that vertex. If the last colored vertex is u_i, then u_i has an uncolored parent x, an uncolored minor child $w_{i,2}$. Hence by our strategy, when Alice colors u_i, it has at most 4 colored neighbours. Therefore Alice will not color u_i the same color as a colored neighbour. ∎

When Bob colors the last vertex of $u_i, w_{i,1}$, Alice jumps from u_i to $\{f(x), m(x), x\}$, resulting another directed edge from u_i to $\{f(x), m(x), x\}$. Therefore there are two directed edges from $\{u_i, w_{i,1}, w_{i,2}\}$ to $\{f(x), m(x), x\}$ for each $i \in \{1, 2, 3\}$. Hence there are at least 6 directed edges directed towards $\{f(x), m(x), x\}$. In this calculation, the move of Alice following Bob's last move is counted. Do not count that move (which may results in at most one directed edge towards $\{f(x), m(x), x\}$, there are at least 5 directed edges directed from outside towards $\{f(x), m(x), x\}$. Moreover, when x is first activated, there will be a directed edge from x to $\{f(x), m(x)\}$. So altogether, the in-degree sum of $\{f(x), m(x), x\}$ is at least 6, which is a contradiction. ∎

The following is probably true:

Conjecture 3.1 *Suppose G is an outerplanar graph. For every $k = 2, 3, 4$, for $d \geq 7 - k$, Alice has a winning strategy for the (k, d)-relaxed game for G.*

The following result shows that for the case $k = 2$, if the above conjecture is true, then it is sharp.

Theorem 3.2 *There is an outerplanar graph G for which Bob has a winning strategy for the $(2, 4)$-relaxed game for G.*

Proof.

Let G be an outerplanar graph, part of which is as drawn in Fig. 5 below.

Bob can easily force the above partially colored outerplanar graph: The vertex v in the center has large degree and a large number of its neighbours induce a path. Bob colors v by color 1, then color 4 neighbours of v by color 1 (v could also be colored by Alice in her first move). So all the other neighbours of v cannot be colored by color 1. As v has a large number of neighbours induce a path, so after Bob colored 4 neighbours of v by color 1 (and after Alice's responses) there are three neighbours of v that are consecutive on the path

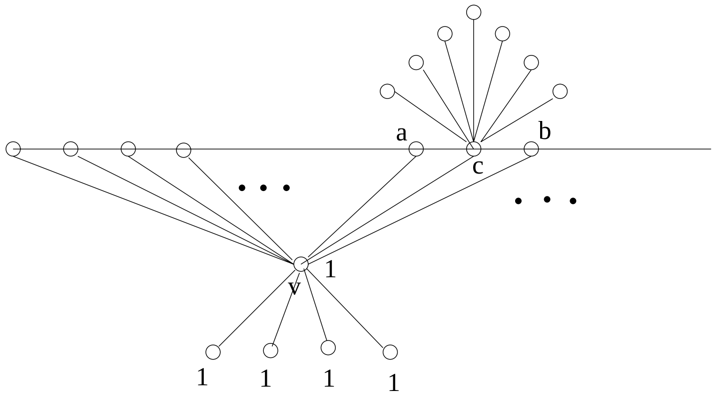

Figure 5: A partially colored outerplanar graph

and that are not colored yet (i.e., the vertices a, b, c shown in the Fig. 4). Now Bob keeps coloring neighbours of c other than a and b by color 2. Alice will eventually need to color each vertex of a, b, c by color 2. However, before Alice can finish the job of coloring these 3 vertices by color 2, Bob has colored 3 neighbours of c by color 2. So the last uncolored vertex of a, b, c has no legal color. ∎

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