The game chromatic index and game coloring index of graphs

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May, 1998.†

Abstract

This paper discusses the game chromatic index and game coloring index of graphs. We prove that if a graph G is k-degenerate, then its game coloring index is at most $\Delta + 3k - 1$. As a consequence, the game coloring index of a forest is at most $\Delta + 2$, and the game coloring index of a planar graph is at most $\Delta + 14$. The upper bound $\Delta + 2$ for the game coloring index of a forest is sharp, provided $\Delta \geq 3$. However, we shall prove that the if F is a forest of maximum degree 3, then the game chromatic index of F is at most 4. This bound is also sharp.

*This research was partially supported by the National Science under grant NSC87-2115-M-110-004.

†*1991 Mathematics Subject Classification.* 05C20, 05C35, 05C15

Key words and phrases. Game chromatic number, game coloring number, game chromatic index, game coloring index,

This copy was printed on March 6, 2003

1 Introduction

Let G be a graph and let k be a positive integer. The following two-person game is played on G: Alice and Bob alternate turns, with Alice having the first move. A move consisting of selecting an uncolored vertex x of G and assigning it a color from $\{1, 2, \cdots, k\}$ distinct from the colors assigned previously (by either players) to the neighbours of x. Alice wins if all the vertices of G are successfully colored. Bob wins if an impass is reached before all vertices in the graph are colored, i.e., there is an uncolored vertex which is adjacent to vertices of all the colors. The *game chromatic number* of a graph $G = (V, E)$, denoted by $\chi_g(G)$, is the least k for which Alice has a winning strategy. This parameter is well-defined, since Alice always wins if $k = |V|$.

The game coloring number is a variation of the game chromatic number of graphs. Instead of playing a game coloring the vertices of G, consider the following two-person game: Alice and Bob alternate turns, with Alice having the first move. A move consisting of selecting a vertex (among the remaining vertices) and put it at the end of the linear order formed by the previously selected vertices. Thus the outcome of the game is a linear order, say L, on the vertex set of G. Alice's goal is to minimize the back degree of L (i.e., the maximum back degree of a vertex relative to L), while Bob's goal is to maximize the back degree of L. This is a zero-sum two person game. Therefore each player has an optimal strategy. The *game coloring number* $\mathrm{col}_g(G)$ of G is defined to be $1 + k$, where k is the back degree of a linear order L, which is produced by playing the game with both players use their optimal strategies.

It is obvious that for any graph G, $\chi_g(G) \leq \mathrm{col}_g(G)$ (cf. [11]). On the other hand, complete bipartite graphs $K_{n,n}$ have game chromatic number 3 and game coloring number $n + 1$. The game chromatic number and the game coloring number has been studied in [1, 2, 4, 5, 7, 10, 11]. Upper and lower bounds for the game chromatic number and game coloring number are obtained for a few classes of graphs, including forests, planar graphs, outer-planar graphs, partial k-trees, chordal graphs, etc. Interestingly, for all these classes of graphs, the best known upper bounds for the game coloring number are also the best known upper bounds for the game chromatic number. The following table lists the presently known upper and lower bounds for the maximum game chromatic number and game coloring number of these classes of graphs.

Class of graphs	Upper bound for maximum χ_g and col_g	Lower bound for maximum χ_g
Forests	4 [4]	4 [4]
Planar graphs	19 [11]	8 [7]
Outer-planar graphs	7	6 [7]
Interval graphs	$3\omega(G)-2$ [4]	
Chordal graphs	$3\omega(G)-1$ [10]	
Partial k-trees	$3k+2$ [10]	

In this paper, we investigate the game coloring number and the game chromatic number of line graphs. We call the game coloring number (resp. the game chromatic number) of the line graph $L(G)$ of G the *game coloring index* (resp. *game chromatic index*) of G, and denote it by $\mathrm{col}'_g(G)$ (resp. $\chi'_g(G)$). We shall prove that if a graph G is k-degenerate, then $\mathrm{col}'_g(G) \leq \Delta(G)+3k-1$. As a consequence the game coloring index of a forest is at most $\Delta+2$. This upper bound is sharp when $\Delta \geq 3$. Then we prove that the game chromatic index of a forest of maximum degree 3 is at most 4. This upper bound is also sharp. For forests with $\Delta \geq 4$, we know that there are forest F with game chromatic index $\Delta+1$, and each forest F has game chromatic index at most $\Delta+2$. However, we do not know if there are forests F with whose game chromatic index are equal to $\Delta+2$.

2 Game coloring number of k-degenerate graphs

Let $G=(V,E)$ be a finite graph and let L be a linear order on the vertex set V. For a vertex $x \in V$, the *back degree of x relative to L* is defined as $|\{y \in V : xy \in E \text{ and } x > y \text{ in } L\}|$. The *back degree of* L is then the maximum back degree of vertices relative to L. The graph $G=(V,E)$ is said to be k-*degenerate* if there is a linear order L on V that has back degree k.

In this section, we prove the following theorem:

Theorem 1 *If G is k-degenerate, then* $\mathrm{col}_g(G) \leq \Delta(G)+3k-1$.

Proof. First we assume that G is connected. It suffices to give a strategy for Alice so that no matter how Bob plays the game, the outcome of the game, which is a linear order L of $E(G)$, has back degree at most $\Delta(G)+3k-2$, i.e., each edge e is adjacent to at most $\Delta(G)+3k-2$ edges proceding e in the linear order L.

Since G is connected and k-degenerate, there is an acyclic orientation of G that has a single sink and each vertex has out-degree at most k. We denote by $\vec{G} = (V, \vec{E})$ such an orientation of G. Let r be the single sink. For each edge e, define the *level* of e to be the length of a longest directed path starting from e.

For each vertex x, we call the edges directed from x the *out-edges* of x, and call those edge directed towards x the *in-edges* of x. Thus each vertex of $\vec{G}$ has at most k out-edges. We denote by $d^+(x)$ the out-degree (i.e., the number of out-edges) of x in $\vec{G}$.

Let T be any fixed directed spanning subtree of $\vec{G}$, i.e., all the paths of T are directed toward the root r. Hence all the vertices of T has out-degree 1, except the root, which has out-degree 0. In the process of the game, Alice will keep track of a subset of V, which is called the *active set*, and denoted by T_a. The set T_a contains the root r, and will always induce a connected subgraph of T, i.e., T_a induces a subtree. The vertices of T_a are called *active vertices*. Alice will also keep track of a subset of the edges of $\vec{G}$, denoted by E_a, which has the property that all the edges of E_a have both end vertices in T_a. The edges in E_a are called *active edges*. Note that elements of E_a need not to be edges of T_a.

Suppose at certain stage, the active vertex set is T_a and the active edge set of E_a. Suppose $P = (v_1, v_2, \cdots, v_k)$ is a directed path of $\vec{G}$, i.e., for each $1 \leq j \leq k-1$, $v_j v_{j+1} \in \vec{E}$. Let P' be the unique directed path of T connecting v_k to T_a, i.e., the first vertex of P' is v_k and the last vertex of P' is a vertex of T_a, and all the inner vertices (if any) are not in T_a. Recall that T_a induces a subtree of T and contains the root r. Therefore the path P' exists and is unique. The concatenation PP' of P and P' is called the *extension of* P. Since $\vec{G}$ is acyclic, it follows that PP' is a directed path of $\vec{G}$. Note that given a directed path P of $\vec{G}$, its extension is unique (as T_a is fixed). Also note that if the last vertex of P is in T_a, then its extension is itself. By the definition, P is allowed to intersect T_a.

At any stage of the game, we say an edge is a *selected edge* if it has been selected by either player before that stage. Otherwise, the edge is *a free edge* at that stage.

Now we describe Alice's strategy.

Initially, Alice select an edge of minimum level, i.e., an edge of the form $e = r'r$, which has level 1. Let $T_a = \{r', r\}$ and let $E_a = \{e\}$.

Suppose at certain stage of the game, Bob has selected the edge $e = uv$. Then Alice select the next edge by the following rule:

Let $P_1 = uv$, and let P_2 be the extension of P_1. Alice will repeat the following procedure until she found the edge to be selected.

Suppose the presently found directed path is P_{2t} for some $t \geq 1$, the last vertex is x.

If all the out-edges of x are selected, then select any free edge of minimum level.

If all the out-edges of x are active, and some of them are free, then select any free out-edge of x.

If there is an out-edge, say xy, which is not active, then let P_{2t+1} be the concatenation of P and xy, and let P_{2t+2} be the extension of P_{2t+1}. Go back to repeat the procedure, with P_{2t} replaced by P_{2t+2}.

It is obvious that the procedure will stop in $O(V)$ steps, and hence Alice will eventually select an edge.

After Alice selected the next edge, say e, add the edges of the directed path P_{2t} and the edge e to E_a, and add the end vertices of these edges to T_a, where P_{2t} is the last directed path found in the procedure above. It is obvious that T_a induces a subtree of T after adding these vertices.

Theorem 2 *If Alice uses the strategy described above, then the back degree of the linear order on $\vec{E}(G)$ produced in the game is at most $\Delta + 3k - 2$.*

We shall prove that at any stage of the game, after Alice finished her move and before Bob takes his next move, any free edge has at most $\Delta + 3k - 3$ selected neighbours. First we make a few observations.

Observation 1 *All the selected edges are active.*

This follows from the strategy.

Observation 2 *Suppose x is a vertex. Let $a_i(x)$ be the number of active in-edges of x, let $a_o(x)$ be the number of active out-edges of x and let $s_o(x)$ be the number of selected out-edges of x. Then*

$$\min\{d^+x, a_i(x)\} \leq a_o(x),$$

and

$$\min\{d^+(x), a_i(x) - d^+(x)\} \leq s_o(x).$$

Let ux be the first in-edge of x that become active. At that stage, the out-edge of x in T becomes active (if it is not active before) after Alice finish that step. Also x become an active vertex. Whenever another in-edge of x becomes active, then the path P_{2t} constructed above will terminate at x at certain time (because x is active), and then either one more out-edge of x become active or one of the out-edges of x become a selected edge, until all the out-edges of x are selected. Therefore the two inequalities of the observation follows.

With these two observations, we are ready to count the number of active edges incident to a free edge. Suppose Alice has finished her move, and that $e = xy$ is still a free edge.

Since y has degree at most Δ, so there are at most $\Delta - 1$ active edges of the form uy or yv which are distinct from xy.

Since x has out-degree $d^+(x) \leq k$, so there are at most $k - 1$ active edges of the form xz which are distinct from xy.

Since the selected out-edges of x is at most $d^+(x) - 1$ (because xy is still free), by Observation 2, x has at most $2d^+(x) - 1 \leq 2k - 1$ active in-edges, i.e., active edges of the form ux.

Therefore, the total number of active edges incident to e is at most $\Delta + 3k - 3$. By Observation 1, the total number of selected edges incident to e is at most $\Delta + 3k - 3$.

When Bob select an edge, the selected neighbours of a free edge increases at most by 1. Thus after Bob's move, each free edge is incident to at most $\Delta + 3k - 2$ selected edges.

We observe that in the strategy described above, it is not necessary that Alice having the first move. In other words, even if Bob moves first, the corresponding game coloring number of G is still at most $\Delta + 3k - 1$. Therefore for disconnected graph, we may play the game on connected components separately, and hence the conclusion is true for disconnected graphs as well. This completes the proof of Theorem 2, and hence the proof of Theorem 1 ∎

Corollary 1 *For any planar graph G, $\chi'_g(G) \leq \mathrm{col}'_g(G) \leq \Delta + 14$.*

Proof. Because planar graphs are 5-degenerate. ∎

The *arboricity* $\mathrm{Arb}(G)$ of a graph G is the minimum number of forests that can cover the edges of G. It was proved by Nash-William that $\mathrm{Arb}(G) = \max\{e(H)/(v(H)-1) : H \subset G\}$. Therefore if $\mathrm{Arb}(G) = k$, then for any subgraph H of G, the average degree of the vertices of H is less than $2k$, and hence contains a vertex of degree $\leq 2k - 1$. It follows that G is $(2k - 1)$-degenerate. Therefore we have the following corollary:

Corollary 2 *If G has arboricity k, then $\chi'_g(G) \leq \mathrm{col}'_g(G) \leq \Delta + 6k - 4$.*

Corollary 3 *For any forest,* $\chi'_g(G) \leq \text{col}'_g(G) \leq \Delta + 2$.

This upper bound for the game coloring index of forests is sharp if $\Delta \geq 3$. Examples are easy to construct, and we leave it as an excercise for the readers. However, for the game chromatic number of forest of maximum degree 3, we shall derive a better upper bound in the next section.

3 Game chromatic index of forests of maximum degree 3

In this section, we shall prove the following theorem:

Theorem 3 *If F is a forest of maximum degree* 3*, then* $\chi'_g(F) \leq 4$.

Proof. Let F be a forest of maximum degree 3. For convenience, we shall assume that each vertex of F either has degree 1 or has degree 3. The strategy we give here for Alice can be easily adopted to apply to those forests with degree 2 vertices. We shall prove that if the color set is $\{1, 2, 3, 4\}$, then Alice has a winning strategy.

We need to define a few terms. Let F be a forest with some edges (properly)-colored by colors from $\{1, 2, 3, 4\}$. We say an edge e of H is *safe* if either e is colored, or at least three of the edges incident to e are colored, and two of them are colored with the same color. We define a *c-block* B of H to be a maximal subtree of F such that each safe edge of B is a leaf-edge of B. Intuitively, the c-blocks of a partially colored forest could be obtained by cutting the safe edges in the middle, so that they become leaves. After this cutting, each connected component becomes a c-block. Note that each safe edge belongs to two c-blocks, and each other edge belongs to one c-block.

In the process of the game, the edges of F are successively colored. At each stage, the forest F is a partially colored forest. For that partially colored forest, we have a family of c-blocks. When another edge is colored, the family of c-blocks is changed. It is obvious that each time a new edge is colored, some old c-block may break into two or three new c-blocks. So the c-blocks become smaller and smaller, and eventually, when all the edges are colored, each c-block is a star.

An important observation is that when an edge e is being colored, it only affect the edges in the c-block at the previous stage that contains e. In other words, we may regard different c-blocks of the forest F as disjoint subtrees, and consider each of the c-blocks separately.

Another observation is that if an uncolored edge is a safe edge, then it belongs to two c-blocks. However, one of the c-blocks is a star $K_{1,3}$.

We shall prove by induction that Alice has a strategy for the game so that at any stage, after her move (and before Bob takes his next move), each c-block B of F has the following property:

($*$): B contains at most 3 colored edges, and that no uncolored edge is incident to three edges colored with distinct colors.

Initially, this is certainly true.

Suppose this is the case at a certain stage, i.e., after Alice finished her move, the partially colored forest has property ($*$). Then Bob color an edge, say e, with color j. We shall show that Alice can choose an uncolored edge and a suitable color for that edge so that after she colors the chosen edge with that color, each c-block of the resulting partially colored forest has property ($*$). We shall only describe Alice's strategy, and leave to the readers to verify that the resulting partially colored forest does have the the required property.

If e was not a safe edge before Bob colors it, then e belonged to a single c-block. If the edge e was a safe edge before Bob colors it, then e belonged to two c-blocks, however, one of the c-block is a star, and hence Alice need not to consider that c-block. In any case, Alice only need to consider one c-block, say B, that contains the edge e. If B contains only one colored edge. Then after Bob colors e, B breaks into two c-blocks, and each of the two c-blocks has at most two colored edges. In this case, Alice can easily find a suitable edge and color it with a suitable color, so that the resulting new c-blocks have property ($*$). We omit the details.

Assume that B has two colored edges. After Bob colors e, B breaks into two or three new c-blocks (B breaks into three new c-blocks only if an uncolored edge becomes a safe edge), say $B'_1, B'_2, \cdots$. It is easy to see that at most one new c-block contains three colored edges.

If each new c-block contains at most two colored edges, then again it is easy for Alice to find a suitable edge and color it with a suitable color, so that the resulting new c-blocks have property ($*$).

Assume there is one c-block, say B'_1, contains three colored edges. If the three colored edges have a common end vertex, then it is again trivial. If two of the colored edges have a common end vertex, then Alice colors the third edge incident to that vertex with any legal color. Thus we assume that no two of the three colored edges have a common end vertex. If the three colored egdes are as shown in Fig. 1(a), where the thick edges denote the colored edges, and x, y denote colors, then Alice color egde e' with color y. In case $x = y$, then Alice color e'

with any legal color. Otherwise the three colored egdes are as shown in Fig. 1(b), where e_3 is not adjacent to e', and either e_1 or e_2 (or both) is not adjacent to e'. In this case, Alice color the indicated edge e' with the color of e_3, or e_1 if the color of e_3 is not legal for e'.

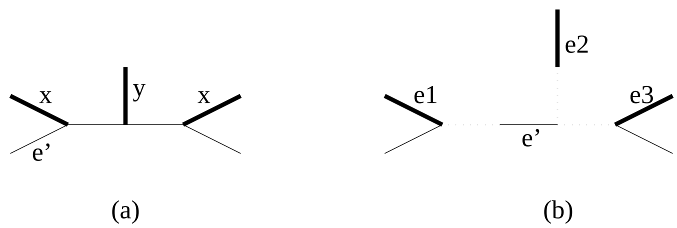

Figure 1:

Finally we conisder the case that B contains three colored edge. After Bob colors the edge e, B breaks into two or three new c-blocks.Again, at most one new c-block contains more than two colored edges. If each of the new blocks contains at most three colored edges, then Alice use the same strategy as described in the previous case. Assume now that one of the new c-blocks contains four colored edges. If there is one colored edge which has distance ≥ 2 to each of the other three colored edges, then Alice simply consider the other three colored edges, and use the rules described in the previous paragraph. However, in Fig. 1(b), there are three edges which could be the edge e'. By carefully choosing that edge (among the three possible choices), Alice can make sure that in the resulting partially colored forest, each new c-block has at most three colored edges.

Assume that none of the four colored edges has distance ≥ 2 to every other colored edges in B'_1.

If there is an edge which has distance ≥ 1 to each of the other three colored edges, then it must be as depicted in Fig. 2(a). If the three edges e_3, e_4 and e^* have a common end vertex and e_2, e_3, e_4 are colored with three distinct colors, then Alice color the edge e'' with the color of e_3 or e_4, whichever is legal. In this case, after coloring e'', the edge e^* becomes a safe edge, and will separate the colored edges into two different blocks. Otherwise, Alice color the edge e' with the color of e_2 (or in case e_1 and e_2 are colored the same color, then color e' with any legal color.) In this case, after coloring e', the edge between e_1 and e_2 become a safe edge, and will separate the colored edges into two different blocks.

Assume now that each of the colored edges is adjacent to another colored edge. Then the colored edges are as depicted in Fig. 2(b) or 2(c). In Fig. 2(c), Alice colors the indicated edge e' with color x. In Fig. 2(b), we note that the indicated edge e' is not adjacent to e_3 and e_4, because if so, before Bob's move, e' is adjacent to three colored edges, and by the induction hypothesis, two of the

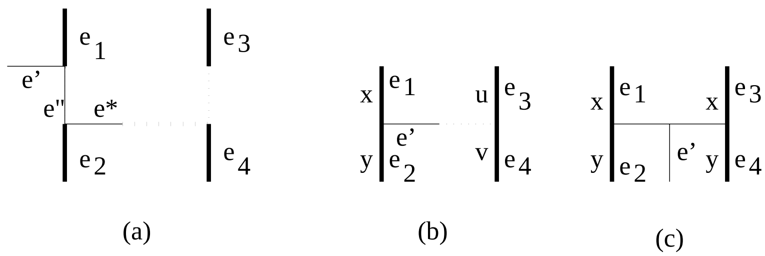

Figure 2:

edges are colored the same color. Therefore e' would have been a safe edge before Bob's move. Now Alice colors e' with color u or v, whichever is different from x and y. This completes the proof of Theorem 3. ∎

We leave it as an exercise to the readers to construct a tree of maximum degree 3 whose game chromatic index is 4.

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