Lower bounds for the game colouring number of partial k-trees and planar graphs

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Abstract

This paper discusses the game colouring number of partial k-trees and planar graphs. Let $\mathrm{col}_g(\mathcal{PT}_k)$ and $\mathrm{col}_g(\mathcal{P})$ denote the maximum game colouring number of partial k trees and the maximum game colouring number of planar graphs, respectively. In this paper, we prove that $\mathrm{col}_g(\mathcal{PT}_k) = 3k + 2$ and $\mathrm{col}_g(\mathcal{P}) \geq 11$. We also prove that the game colouring number $\mathrm{col}_g(G)$ of a graph is a monotone parameter, i.e., if H is a subgraph of G, then $\mathrm{col}_g(H) \leq \mathrm{col}_g(G)$.

1 Introduction

Suppose $G = (V, E)$ is a graph. A *marking game* on G is played by two players, Alice and Bob, with Alice playing first. At the start of the game all vertices are unmarked. A play by either player consists of marking an unmarked vertex. The game ends when all vertices are marked. For each vertex v of G, let $s(v)$ denote the number of neighbours of v that are marked before v. The *score* of the game is

$$s = 1 + \max_{v \in V} s(v).$$

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Alice's goal is to minimize the score, while Bob's goal is to maximize it. The *game colouring number* $\mathrm{col}_g(G)$ of G is the least s such that Alice has a strategy that results in a score at most s.

The game colouring number of a graph was implicitly used in [5] and formally introduced in [12], as a tool in the study of the game chromatic number. The game chromatic number of a graph is also defined through a two person game. Let G be a finite graph and let X be a set of colours. Two players, say Alice and Bob, with Alice moving first, alternately colour the vertices of G with colours from the colour set X so that no two adjacent vertices receive the same colour. Alice wins the game if all the vertices of G are coloured. Bob wins the game if at any stage of the game, there is an uncoloured vertex which is adjacent to vertices of all colours. The game chromatic number $\chi_g(G)$ of G is the least number of colours in a colour set X for which Alice has a winning strategy.

It is easy to see that for any graph G, $\chi_g(G) \leq \mathrm{col}_g(G)$, as Alice can simply use an optimal strategy for playing the marking in choosing the vertex to be coloured, and then use the First-Fit to choose a colour. For many natural classes of graphs, the best known upper bounds for their game chromatic number are obtained by finding upper bounds for their game colouring number. On the other hand, the game colouring number of a graph is of independent interests, and has been studied extensively in the literature [2, 3, 6, 7, 8, 10, 11, 12, 13]. It seems to be easier to deal with the game colouring number than to deal with the game chromatic number. The parameter $\mathrm{col}_g(G)$ has some nice properties that are missing in the parameter $\chi_g(G)$. For example, the game chromatic number is not monotonic, i.e., a subgraph H of G may have larger game chromatic number. However, as will be shown in this paper, the game colouring number is monotonic.

For a family $\mathcal{H}$ of graphs, let

$$\chi_g(\mathcal{H}) = \max\{\chi_g(G) : G \in \mathcal{H}\},$$

and

$$\mathrm{col}_g(\mathcal{H}) = \max\{\mathrm{col}_g(G) : G \in \mathcal{H}\}.$$

We denote by $\mathcal{F}$ the family of forests, by $\mathcal{P}$ the family of planar graphs, by $\mathcal{Q}$ the family of outer planar graphs, by $\mathcal{PT}_k$ the family of partial k-trees, and by $\mathcal{I}_k$ the family of interval graphs with clique number $k+1$. It is proved by Faigle, Kern, Kierstead and Trotter [5] that $\chi_g(\mathcal{F}) = \mathrm{col}_g(\mathcal{F}) = 4$, by Guan and Zhu [6] and Kierstead and Yang [11] that $\mathrm{col}_g(\mathcal{Q}) = 7$ and by Faigle et al. [5] and Kierstead and Yang [11] that $\mathrm{col}_g(\mathcal{I}_k) = 3k+1$. Up to now, $\mathcal{F}$ is the only (non-trivial) family of graphs for which $\chi_g(\mathcal{F})$ is determined. $\mathcal{F}, \mathcal{Q}$ and $\mathcal{I}_k$ are the only (non-trivial) families $\mathcal{H}$ of graphs for which $\mathrm{col}_g(\mathcal{H})$ are determined.

For some other families of graphs, we have upper and lower bounds for $\mathrm{col}_g(\mathcal{H})$. It is known [8, 9, 11, 14] that $10 \leq \mathrm{col}_g(\mathcal{P}) \leq 17$ and $3k+1 \leq \mathrm{col}_g(\mathcal{PT}_k) \leq 3k+2$.

In this paper, we prove that for any $k \geq 2$, there are partial k-trees G with $\mathrm{col}_g(G) = 3k+2$. Therefore $\mathrm{col}_g(\mathcal{PT}_k) = 3k+2$, and hence we have one more family of graphs whose game colouring number is determined. For planar graphs, we prove that $\mathrm{col}_g(\mathcal{P}) \geq 11$, improving the earlier known lower bound for $\mathrm{col}_g(\mathcal{P})$ by 1.

2 Monotonicity of game colouring number

In this section, we shall prove that the parameter of the game colouring number of a graph is monotonic. Namely, we shall prove:

Theorem 1 *If H is a (not necessarily spanning) subgraph of G, then* $\mathrm{col_g}(H) \leq \mathrm{col_g}(G)$.

Proof. If H is a spanning subgraph of G, this is certainly true. Alice shall just follow her strategy for playing the marking game on G. Thus we only need to consider the case that H is a proper induced subgraph of G. Also by an induction argument, without loss of generality, we may assume that $H = G - x$ for some vertex x of G.

Assume $\mathrm{col_g}(G) = s$. Then Alice has a strategy for playing the marking game on G that results in a score at most s. We shall now show that Alice also has a strategy for playing the marking game on $G - x$ that results in a score at most s. The strategy is as follows:

Alice would imagine that she is playing the marking game on G instead of on $G - x$. So she just use the strategy for playing the marking game on G. Suppose at a certain stage, according to that strategy, Alice should mark the vertex x (but the vertex x is not there). Then Alice would pretend that she has marked the vertex x, and then pretend that Bob has marked the vertex y_1 in the next move, where y_1 is any vertex among the unmarked vertices which has the minimum degree in G. Then Alice will continue the game, again as she is playing the game on G instead of on $G-x$. Then at a certain stage, Bob may mark the vertex y_1. (In Alice's imagination, the vertex y_1 has been marked some time ago, but in reality, it was not marked until now). Now Alice will pretend that the vertex marked by Bob is not y_1, but y_2, where y_2 is any vertex among the remaining unmarked vertices which has the minimum degree in G. Continue this way, whenever is her turn, Alice can always mark a vertex. It remains to show that the resulting score is at most s.

For any vertex v of $G - x$, we denote by $s(v)$ the number of neighbours in $G - x$ of v that are marked before v in the real game, and denote by $s'(v)$ the number of neighbours in G of v that are marked before v in the imagined game. By assumption $s'(v) \leq s - 1$ for all v. We need to show that $s(v) \leq s - 1$ for all v. Let v^* be the last marked vertex in the real game. Then either v^* is the last marked vertex in the imagined game, or $v^* = y_t$ is marked by Bob in his last move, and which was marked earlier in Alice's imagined game. In the former case, $s(v^*) \leq s'(v^*) \leq s - 1$. In the latter case, let u^* be the last marked vertex in the imagined game. By the choice of y_t, we have $s(v^*) = d_G(v^*) \leq d_G(u^*) = s'(u^*) \leq s - 1$. Assume v is not the last marked vertex. If $v \neq y_t$ for any t, it follows from the description of the strategy that any neighbour of v marked before v in the real game is also marked before v in the imagined game. So $s(v) \leq s'(v) \leq s-1$. Assume $v = y_t$ for some t. By the choice of y_t, we have $s(v) \leq s'(v^*) \leq s - 1$. This completes the proof of Theorem 1. ∎

We note that for the game chromatic number, the deletion of an edge or a vertex could increase the game chromatic number. The following is a folklore example: Let M be a perfect matching of the complete bipartite graph $K_{n,n}$, and let e be an edge of M. Let $G = (K_{n,n} - M) + e$. Then it is not difficult to verify that $\chi_\mathrm{g}(G) = 3$ but $\chi_\mathrm{g}(G - e) = n$. Let G be obtained from $K_{n,n} - M$ by adding a vertex v and connect v to all the vertices of one side of the bipartite graph $K_{n,n} - M$. Then it is not difficult to verify that $\chi_\mathrm{g}(G) = 3$ but $\chi_\mathrm{g}(G - v) = n$.

3 Partial k-trees

The family of *k-trees* is defined recursively as follows: K_k is a k-tree. If G is a k-tree and $X \subseteq V(G)$ induces a copy of K_k, then by adding a vertex v and connect v to each vertex of X by an edge, the resulting graph is a k-tree. Equivalently, a graph G is a k-tree if and only if there is a linear order, say $v_1, v_2, \cdots, v_n$, on the vertex set V, such that (i) $v_1, v_2, \cdots, v_k$ induces a K_k, and (ii) for each

$i \geq k+1$, the set $\{v_j : j < i, v_i v_j \in E\}$ induces a K_k. A *partial k-tree* is a subgraph of a k-tree. Let $\mathcal{PT}_k$ denote the set of all partial k-trees and let $\mathcal{T}_k$ denote the set of k-trees.

Lemma 1 *For any positive integer k,* $\mathrm{col_g}(\mathcal{T}_k) = \mathrm{col_g}(\mathcal{PT}_k)$.

Proof. By definition, each k-tree is a partial k-tree, i.e., $\mathcal{T}_k \subseteq \mathcal{PK}_k$. Therefore $\mathrm{col_g}(\mathcal{T}_k) \leq \mathrm{col_g}(\mathcal{PT}_k)$. Assume G is a partial k-tree with $\mathrm{col_g}(G) = \mathrm{col_g}(\mathcal{PT}_k)$. Then G is a subgraph of a k-tree G'. By Theorem 1, $\mathrm{col_g}(G) \leq \mathrm{col_g}(G')$. Therefore $\mathrm{col_g}(\mathcal{T}_k) \geq \mathrm{col_g}(G') \geq \mathrm{col_g}(\mathcal{PT}_k)$. ∎

It is unknown if $\chi_g(\mathcal{T}_k) = \chi_g(\mathcal{PT}_k)$ for $k \geq 2$.

The following result is proved in [13]:

Theorem 2 *If G is a partial k-tree, then* $\mathrm{col_g}(G) \leq 3k+2$.

Now we prove that the bound in Theorem 2 is sharp.

Theorem 3 *If $k \geq 2$, then there is a partial k-tree G such that* $\mathrm{col_g}(G) = 3k+2$.

Combining Theorems 2 and 3, we have

Corollary 1 *For any integer $k \geq 2$,* $\mathrm{col_g}(\mathcal{PT}_k) = 3k+2$.

Proof of Theorem 3 Suppose $k \geq 2$ is an integer. We build a partial k-tree G as follows: Let P_n^k be the kth power of the path P_n, i.e., P_n^k has vertex set $a_1, a_2, \cdots, a_n$, in which $a_i \sim a_j$ if and only if $|i-j| \leq k$. For each $k+1 \leq i \leq n$ which is not a multiple of k, add a vertex b_i and connect b_i to each of $a_i, a_{i-1}, \cdots, a_{i-k+1}$ by an edge. For $1 \leq i < j \leq i+k \leq n$ and $m = 1, 2$, add a vertex $c_{i,j,m}$ and add edges connect $c_{i,j,m}$ to a_i and a_j. The resulting graph is G. It is obvious that G is a partial k-tree. We shall prove that if n is large enough, then $\mathrm{col_g}(G) \geq 3k+2$, and hence $\mathrm{col_g}(G) = 3k+2$ by Theorem 2.

For convenience, we call a vertex a_j an A-vertex, a vertex b_j a B-vertex, and a vertex $c_{i,j,m}$ a C-vertex. The number n will be chosen to be a multiple of k. Then we have n A-vertices, $\frac{k-1}{k}(n-k)$ B-vertices. Suppose $x \sim y$. If x is an A-vertex (respectively, a B-vertex or a C-vertex), then we say x is an A-neighbour (respectively, a B-neighbour or a C-neighbour) of y.

Observe that if $k+1 \leq j \leq n-k$, then a_j has $2k$ A-neighbours, $k-1$ B-neighbours, and $4k$ C-neighbours. To prove Theorem 3, it suffices to show that Bob has a strategy to ensure that at a certain step, a vertex a_j $(k+1 \leq k \leq n-k)$ is not marked yet, but all its A-neighbours and B-neighbours are marked, and moreover, at least 2 of its C-neighbours are marked. If this is the case, then the unmarked vertex a_j will have $3k+1$ marked neighbours.

Bob's strategy is as follows: Let $A' = \{a_{k+1}, a_{k+2}, \cdots, a_{n-k}\}$. First Bob marks all the B-vertices and all the A-vertices not in A'. After all these vertices are marked, Bob has made at most $\frac{k-1}{k}(n-k)+2k$ moves (if Alice have marked some of these vertices, then Bob will have made less number of moves). This implies that Alice has marked at most $\frac{k-1}{k}(n-k)+2k+1$ vertices of A' before Bob's next move. As $|A'| = n-2k$, at least $n-2k-(\frac{k-1}{k}(n-k)+2k+1) = \frac{n}{k}-3k-2$ vertices of A' are unmarked yet. We denote by U the set of unmarked vertices of A' at this step. Straightforward calculation shows that if $n \geq 3k^2+4k+3$, then $|U| \geq 1+|A'|/(k+1)$. This implies that U contains two vertices $a_j, a_{j'}$ such that $|j-j'| \leq k$, i.e., the subgraph $G[U]$ of G induced by U contains an edge. Let W be a connected component of $G[U]$ which contains at least one edge.

Claim 1 *Suppose all the B-vertices and all the A-vertices not in A' have been marked and W is a connected component of $G[U]$ which contains at least one edge. If it is Bob's turn, then he has a strategy to ensure that at a certain step, there will be one vertex a_l of W which is unmarked, and it has at least $3k+1$ marked neighbours.*

We prove Claim 1 by induction on the number of vertices of W. If W contains only one edge, say $a_i a_j$, then Bob marks vertices in the following order of preference

$$c_{i,j,1}, c_{i,j,2}, a_i.$$

We consider the moment that both $c_{i,j,1}, c_{i,j,2}$ are marked. At this moment, at least one of a_i, a_j is unmarked (because we start with Bob's turn).

If a_i is marked and a_j is not marked, then a_j has $2k$ marked A-neighbours, $k-1$ marked B-neighbours and at least 2 marked C-neighbours. So we are done. If none of a_i, a_j is marked, then Bob's next move is to mark a_i. After Bob marked a_i, then a_j is an unmarked vertex with $3k+1$ marked neighbours.

Assume W contains at least two edges. We consider two cases.

Case 1 There is a vertex of W which has degree 1.

Assume a_i has degree 1 in W. Let $a_i a_j$ be the edge of W incident to a_i. Bob marks vertices in the following order of preference

$$c_{i,j,1}, c_{i,j,2}, a_j,$$

until Alice marks one vertex of W.

If after Bob marked all the vertices $c_{i,j,1}, c_{i,j,2}, a_j$, the vertex a_i is still unmarked, then a_i has at least $3k+1$ marked neighbours (as calculated above).

Assume before Bob mark all the vertices $c_{i,j,1}, c_{i,j,2}, a_j$, Alice marked a vertex of W. If the vertex marked by Alice is a_j, then after Bob's second move, a_i remain unmarked and $c_{i,j,1}, c_{i,j,2}, a_j$ are all marked. So a_i has at least $3k+1$ marked vertices. If the vertex marked by Alice is not a_j, then W becomes a smaller component W' which still contains at least one edge. By induction hypothesis, Bob has a strategy to ensure that at certain step, there will be one vertex a_j of W' (and hence a vertex of W) which is unmarked, and it has at least $3k+1$ marked neighbours.

Case 2 Each vertex of W has degree at least 2. We divide this case further into two subcases.

Case 2a For each edge $a_i a_j$ of W, both $c_{i,j,1}$ and $c_{i,j,2}$ are marked.

In this case, each vertex of W has $k-1$ marked B-neighbours and at least 4 marked C-neighbours (as each vertex of W has degree at least 2). Bob arbitrarily marks a vertex a_j of W. The last unmarked vertex of W will have at least $3k+3$ marked neighbours (as it has $2k$ marked A-neighbours).

Case 2b There is an edge $a_i a_j$ of W such that $c_{i,j,1}$ or $c_{i,j,2}$ is unmarked.

Then Bob marks $c_{i,j,1}$ or $c_{i,j,2}$. After Alice's next move, W becomes W' which still contains at least one edge. Observe that W' could be the same as W, if Alice does not mark any vertex of W. So we cannot use induction at this step. However, Case 2b cannot repeat forever. It will eventually becomes Case 2a or Case 1. So eventually induction can be used. This completes the proof of Claim 1, as well as the proof of Theorem 3. ∎

4 Planar graphs

This section proves the following result:

Theorem 4 *There is a planar graph G with game colouring number* 11.

Proof. The planar graph we shall construct is a partial 3-tree. Hence it has game colouring number at most 11. The partial 3-tree is very much like the one constructed in the previous section. However, we need to do a small modification, as the partial 3-tree constructed in the previous section is not planar.

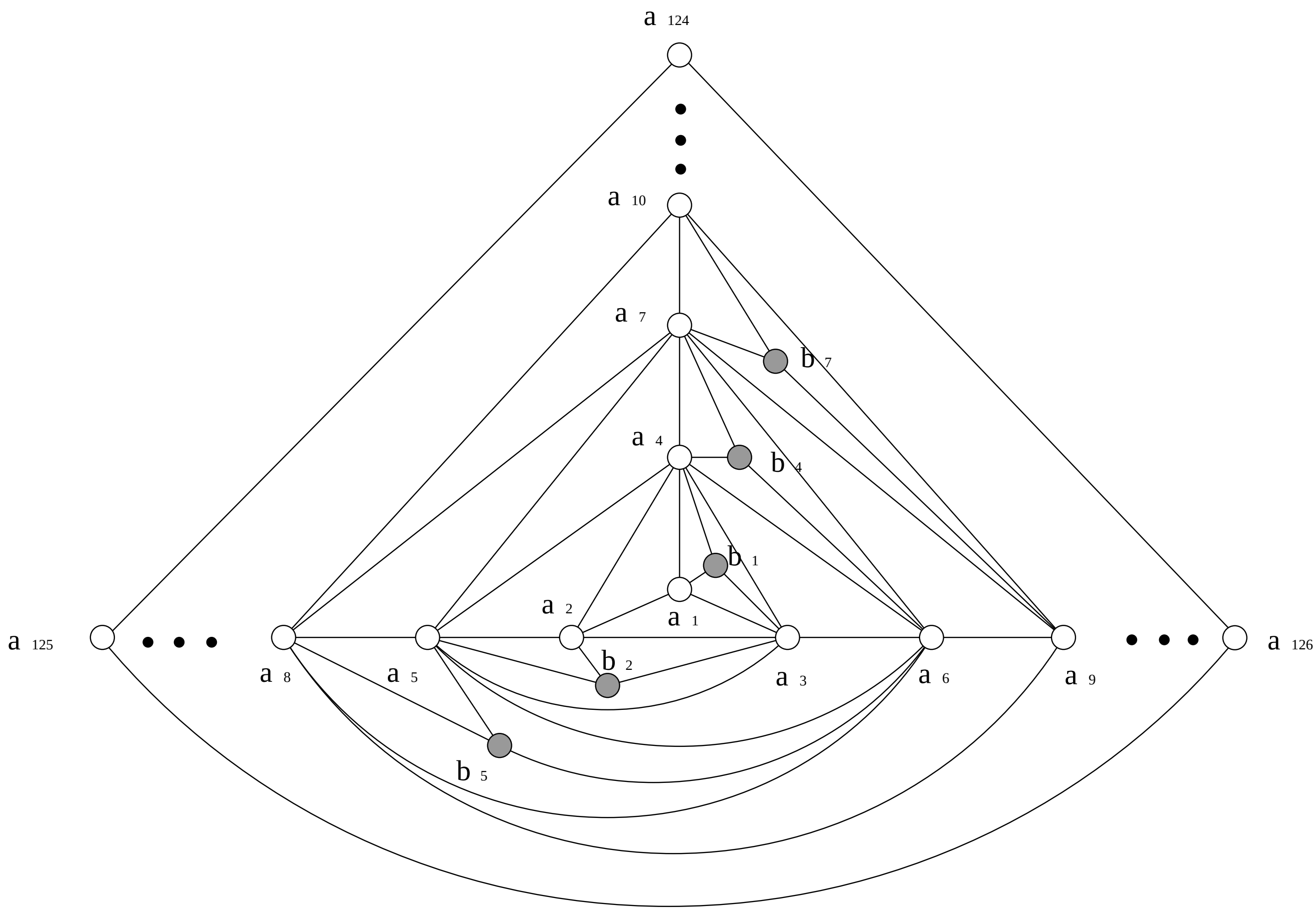

Figure 1:

Our graph G is obtained from P^3_{126} by adding some vertices and edges. Let $A = \{a_1, a_2, \cdots, a_{126}\}$, in which $a_i \sim a_j$ if and only if $|i - j| \leq 3$. Now for $i = 0, 1, 2, \cdots, 40$, we add vertices b_{3i+1} and b_{3i+2}, and add edges $a_{3i+1}b_{3i+1}$, $a_{3i+3}b_{3i+1}$, $a_{3i+4}b_{3i+1}$, $a_{3i+2}b_{3i+2}$, $a_{3i+3}b_{3i+2}$, $a_{3i+5}b_{3i+2}$. Then for each pair i, j such that $1 \leq i < j \leq i + 3 \leq 126$, add vertices $c_{i,j,1}$ and $c_{i,j,2}$, and add edges $a_i c_{i,j,m}, a_j c_{i,j,m}$ for $m = 1, 2$.

The graph G constructed here is planar. Figure 1 is a plane embedding of the graph G. For simplicity, we do not draw the vertices $c_{i,j,m}$. Each of such a vertex comes with an edge of the form $a_i a_j$. So they can easily be drawn in the plane.

On the other hand, the graph G has similar properties as the partial 3-tree constructed in the previous section. In particular, let $A' = \{a_j : 4 \leq j \leq 123\}$, then each $a_j \in A'$ has 2 B-neighbours, 6 A-neighbours and 12 C-neighbours. For any $U \subseteq A'$ with $|U| \geq 31$, the subgraph $G[U]$ of G induced by U contains an edge, and $B \cup \{a_1, a_2, a_3, a_{124}, a_{125}, a_{126}\}$ has cardinality 88. So after all the vertices in $B \cup \{a_1, a_2, a_3, a_{124}, a_{125}, a_{126}\}$ are marked, the set U of unmarked vertices of A' has size $|U| \geq 31$, and hence contains an edge. The same argument as in the proof of Theorem 3 can be carried out for the graph G constructed here to prove that $\mathrm{col_g}(G) = 11$. ∎

References

[1] H. L. Bodlaender, *On the complexity of some colouring games*, International Journal of Foundations of Computer Science 2(1991), 133-148.

[2] L. Cai, K. Lih and W.Wang, *Game coloring number of planar graphs without 4-cycles*, preprint, 2001.

[3] L. Cai and X. Zhu, *Game chromatic index of k-degenerate graphs*, J. Graph Theory 36 (2001), no. 3, 144–155.

[4] T. Dinski and X. Zhu, *A bound for the game chromatic number of graphs*, Discrete Mathematics 196(1999), 109-115.

[5] U. Faigle, U. Kern, H. A. Kierstead and W. T. Trotter, *On the game chromatic number of some classes of graphs*, Ars Combin. 35 (1993), 143–150.

[6] D. Guan and X. Zhu, *The game chromatic number of outerplanar graphs*, J. Graph Theory 30(1999), 67-70.

[7] W.He, X.Hou, K. Lih, J. Shao, W. Wang and X. Zhu, *Edge-partitions of planar graphs and their game coloring numbers*, Journal of Graph Theory, 41(2002), 307-317.

[8] H. A. Kierstead, *A simple competitive graph coloring algorithm*, J. Combinatorial Theory (B) 78(2000), 57-68.

[9] H. A. Kierstead and W. T. Trotter, *Planar graph coloring with an uncooperative partner*, J. Graph Theory 18 (1994), no. 6, 569–584.

[10] H. A. Kierstead and Zs. Tuza, *Marking games and the oriented game chromatic number of partial k-trees*, Graphs and Combinatorics, 19(2003), 121-129.

[11] H. A. Kierstead and D. Yang, *Very assymetric marking games*, manuscript, 2002.

[12] X. Zhu, *The game coloring number of planar graphs*, J. Combinatorial Theory (B) 75(1999), 245-258.

[13] X. Zhu, *Game coloring number of pseudo partial k-trees*, Discrete Mathematics 215(2000), 245-262.

[14] X. Zhu, *Refined activation strategy for the marking game*, Journal of Combinatorial Theory Ser. B, to appear.