The circular chromatic number of distance graphs with distance sets of cardinality 3

Xuding Zhu*
Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung, Taiwan 80424
Email: zhu@math.nsysu.edu.tw

Abstract

Suppose D is a subset of R^+. The distance graph $G(R, D)$ is the graph with vertex set R in which two vertices x, y are adjacent if $|x - y| \in D$. This paper investigates the circular chromatic number and the fractional chromatic number of distance graphs $G(R, D)$ with $|D| = 3$. As a consequence, we determine the chromatic numbers of all such distance graphs. This settles a conjecture proposed independently by Chen, Chang and Huang, [*Integral distance graphs*, J. Graph Theory, 1997] and Voigt [*Colouring of distance graphs*, Ars Combinatoria, 1999] in the affirmative.

1991 Mathematics Subject Classification. Primary 05C. Secondary 11B.

Keywords: distance graph, chromatic number, circular chromatic number, fractional chromatic number, Diophantine approximation, T-colouring.

*This research was partially supported by the National Science Counsil under grant NSC88-2115-M-110-001

1 Introduction

Let R be the real line. For a subset D of R, the distance graph $G(R, D)$ is the graph with vertex set R in which two vertices x, y are adjacent if $|x - y| \in D$. The set D is called the *distance set*. Distance graphs on the real line, first studied by Eggleton, Erdös and Skilton [11], were motivated by the plane colouring problem: What is the least number of colours needed to colour all points of the euclidean plane so that vertices of unit distance are coloured with distinct colours. This problem is equivalent to determining the chromatic number of the distance graph $G(R^2, \{1\})$ on the plane R^2 with distance set $D = \{1\}$. It is known that $4 \leq \chi(G(R^2, \{1\})) \leq 7$, but the exact value of $\chi(G(R^2, \{1\}))$ remains unknown. For distance graphs on the real line, the chromatic number of $G(R, D)$ is easy to determine if $|D| \leq 2$ [2, 16]. If $|D| = 1$ or D consists of two coprime odd integers then $\chi(G(Z, D)) = 2$; if D contains one odd integer and one even integer then $\chi(G(R, D)) = 3$. If D contains two non-integer numbers, then either it is equivalent to another distance set D' which contains two integers (if the two numbers are integrally dependent), or $\chi(G(R, D)) = 2$ (if the two numbers are integrally independent). If D contains three integrally independent real numbers, the chromatic number of $G(R, D)$ was determined in [20].

The case when D contains three integers is more complicated. There are several papers [2, 10, 11, 16, 19] discussing this problem, and many special cases have been settled. For instance, the case when D contains three primes was solved in [11], the case when $D = \{a, b, c\}$ and $a+1 = b < c$ was solved in [2], the case when $D = \{a, b, c\}$ and $a < b < c \leq 2a$ was solved in [2, 10, 16], the case when $D = \{a, b, c\}$ and $a < b < 2b \leq c$ was solved in [19], the case when $D = \{a, b, c\}$ and $a < b < c$ and $a|b$ was solved in [10]. There are other special cases, whose descriptions involve more complicated formulae, which were solved in [2, 10, 16]. However, the general problem for D containing three integers remained open.

In this paper, we investigate the circular chromatic number and the fractional chromatic number of distance graphs $G(R, D)$ whose distance sets D contain three integers. As a consequence, we shall determine the chromatic number of all such distance graphs.

The circular chromatic number of a graph is a refinement of the chromatic number of the graph. It has quite a few equivalent definitions. In this paper, we shall use the following definition. Suppose $r \geq 1$ is a real number. An *r-colouring* of a graph G is a mapping $f : V(G) \to [0, r)$ such that for every edge xy of G, $1 \leq |f(x) - f(y)| \leq r - 1$. We say G is r-colourable if there exists an r-colouring of G. In case $r = k$ is an integer, it is easy to verify that r-colourability coincides with ordinary vertex k-colourability. The *circular chromatic number* $\chi_c(G)$ of G is the infimum of those r for which G is r-colourable. It is known [22] that for any graph G, the infimum in the definition is always attained and hence can be replaced by minimum. Moreover for a finite graph G, the circular chromatic number of G is always rational.

If $r' \geq r$ and G is r-colourable then G is r'-colourable. Since for an integer $r = k$, r-colourability coincides with ordinary k-colourability, for any graph G we have

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G).$$

Therefore $\chi(G) = \lceil \chi_c(G) \rceil$. So the circular chromatic number $\chi_c(G)$ is a refinement of the chromatic number $\chi(G)$, and $\chi(G)$ is an approximation of $\chi_c(G)$.

The circular chromatic numbers of distance graphs have been studied in [3, 4, 14, 21]. If D contains two integers, then the circular chromatic number of the distance graph $G(R, D)$ was determined in [3]. For distance sets D of the form $D_{m,k,s} = \{1, 2, \cdots, m\} - \{k, 2k, \cdots, sk\}$, the circular chromatic number of $G(R, D)$ was determined in [21].

In this paper, we shall estimate the circular chromatic numbers of distance graphs $G(R, D)$ where D contains three integers. In some cases, the exact values of $\chi_c(G(R, D))$ are determined. In the general case, the estimation of $\chi_c(G(R, D))$ is sharp enough to determine the chromatic numbers of these distance graphs. For finding an upper bound of the circular chromatic number of $G(R, D)$, we use the regular colouring method. This method relates the circular chromatic number of $G(R, D)$ to problems in the area of Diophantine approximations. Fortunately, there are some deep results in this area that match exactly what we need for a tight upper bound of $\chi_c(G(R, D))$. (This is also unfortunate, because this author, without knowing the existence of such results, independently proved some of these results with tremendous effort [23]).

The fractional chromatic number of a graph is another variation of the chromatic number. Let G be a graph and let $\mathcal{I}$ be the family of independent sets of G. A *fractional colouring* of G is a mapping $f : \mathcal{I} \to [0, \infty)$, which assigns a non-negative weight to each independent set such that for each vertex x,

$$\sum_{x \in X, X \in \mathcal{I}} f(X) \geq 1.$$

The *fractional chromatic number* $\chi_f(G)$ of G is the infimum of the total weight of a fractional colouring of G. Let $\alpha(G)$ be the independence number of G, i.e., $\alpha(G)$ is the size of a maximum independent set of G. It is well-known [22] that for any graph G,

$$|V(G)|/\alpha(G) \leq \chi_f(G) \leq \chi_c(G).$$

In this paper, we shall obtain the lower bound for $\chi_c(G(R, D))$ by considering lower bounds for the fractional chromatic number. In case the upper bound agrees with the lower bound, we not only determine the circular chromatic number but also the fractional chromatic number of the distance graphs $G(R, D)$. In other cases, the upper bound for the circular chromatic number is also an upper bound for the fractional chromatic number.

The fractional chromatic number of distance graphs is related to the so-called T-colouring problem (or the channel assignment problem). It has been studied in the literature [4, 8, 13, 15], however, under a different name. Let T be a subset of Z such that $0 \in T$. A T-colouring of a graph G is a colouring c of the vertices of G with colours $\{0, 1, 2, \cdots\}$ such that for any adjacent vertices x, y, we have $c(x) - c(y) \notin T$. An important parameter of the T-colouring problem is the minimum span (i.e., the difference between the minimum colour and the maximum colour) $sp_T(K_n)$ of a T-colouring of a complete graph K_n. The asymptotic ratio $R(T) = \lim_{n \to \infty} sp_T(K_n)/n$ was discussed in [4, 13, 15]. The parameter $R(T)$ was used to deduce an upper bound of the T-span $sp_T(G)$ of an arbitrary graph G in terms of its chromatic number $\chi(G)$, [8, 13, 15]. It was shown in [4] that $R(T)$ is equal to the fractional chromatic number of the distance graph with distance set $D = T - \{0\}$. The asymptotic ratio $R(T)$ for $|T| \leq 3$ has been determined in [15]. However, for $|T| \geq 4$, very little is known. As $R(T) = \chi_f(G(R, D))$ for $D = T - \{0\}$, our results determine the asymptotic ratio $R(T)$ for certain T with $|T| = 4$ and give the best estimation for the other T sets with $|T| = 4$.

2 The main results

The main result of this paper is the following theorem.

Theorem 2.1 *Suppose* $0 < a < b < c$ *are integers with* $gcd(a, b, c) = 1$, $D = \{a, b, c\}$ *and* $G = G(R, D)$.

1. *If all the integers* a, b, c *are odd, then*

$$\chi_f(G) = \chi_c(G) = 2.$$

2. *If* $a = 1, b = 2$ *and* $c = 3k$ *where* k *is a positive integer, then*

$$\chi_f(G) = \chi_c(G) = 3 + \frac{1}{k}.$$

3. *If* $c = a + b$ *and* $b - a = 3k$ *where* k *is a positive integer, then*

$$\chi_f(G) = \chi_c(G) = 3.$$

4. *If* $c = a + b$ *and* $b - a = 3k + 1$ *where* k *is a nonnegative integer, then*

$$3 + \frac{1}{a + 2k} \leq \chi_f(G) \leq \chi_c(G) \leq 3 + \frac{1}{a + k}.$$

5. *If* $c = a + b$ *and* $b - a = 3k + 2$ *where* k *is a nonnegative integer, then*

$$3 + \frac{1}{b - k} \leq \chi_f(G) \leq \chi_c(G) \leq 3 + \frac{1}{b - k - 1}.$$

6. *If a, b, c are not all odd, $c \neq a + b$ and $(a, b, c) \neq (1, 2, 3k)$ for any integer k, then*

$$2 < \chi_f(G) \leq \chi_c(G) \leq 3.$$

7. *If a, b, c are not all odd, $c \neq 2b$, $b \neq 2a$, $c \neq 2a$ and $c \neq a + b$ then*

$$2 < \chi_f(G) \leq \chi_c(G) \leq 8/3$$

with finite many exceptional triples (a, b, c). (See Section 3 for a list $\mathcal{T}$ of these exceptional triples.)

Although for most of triples (a, b, c), the exact values of $\chi_c(G)$ are not determined, the above estimations of $\chi_c(G)$ and $\chi_f(G)$ are sharp enough to determine the chromatic numbers of all distance graphs $G(R, D)$ whose distance sets D contain three integers. This is stated in Corollary 2.1. This corollary confirms a conjecture of Chen, Chang and Huang [2] and Voigt [16].

Corollary 2.1 *Suppose $a < b < c$ are integers with $gcd(a, b, c) = 1$, and that $D = \{a, b, c\}$ and $G = G(R, D)$. Then*

- *$\chi(G) = 2$ if all the integers a, b, c are odd;*
- *$\chi(G) = 4$ if $a = 1, b = 2$ and $c \equiv 0 (\mathrm{mod} 3)$;*
- *$\chi(G) = 4$ if $a = c - b$ and $a \not\equiv b (\mathrm{mod} 3)$;*
- *$\chi(G) = 3$ for all the other cases.*

3 The proof of Theorem 2.1

Proving the upper bounds stated in Theorem 2.1 amounts to finding a proper circular colouring. The method of colouring we use here is very simple. Suppose we want to find an r-colouring f for a distance graph $G(R, D)$. We find a real number t and colour the real line R as follows:

$$f(x) = tx \bmod r.$$

Here "$tx \bmod r$" is the unique number $s \in [0, r)$ for which $tx - s$ is a multiple of r. We call such a colouring of the real line an *r-regular colouring of R* (with multiplier t). Theorem 3.1 below follows from the definition.

Theorem 3.1 *The r-regular colouring of R with multiplier t is an r-colouring of the distance graph $G(R, D)$ if and only if for every $d \in D$ we have $1 \leq |td \bmod r| \leq r - 1$.* ∎

For a real number x, we denote by $||x||$ the distance from x to the nearest integer. For a set X of real numbers, let

$$||X|| = \inf\{||x|| : x \in X\},$$

and for a real number t let

$$||tX|| = \inf\{||tx|| : x \in X\}.$$

Let

$$\kappa(X) = \sup_{t \in R} ||tX||.$$

Then we have the following result:

Theorem 3.2 *Suppose $G(R, D)$ is a distance graph on the real line. Then*

$$\chi_c(G(R, D)) \leq \frac{1}{\kappa(D)}.$$

Proof. For every $r > \frac{1}{\kappa(D)}$, there exists a real number t such that $||tD|| \geq \frac{1}{r}$, i.e., for any $d \in D$ we have $\frac{1}{r} \leq ||td||$, which implies that

$$\frac{1}{r} \leq td \bmod 1 \leq \frac{r-1}{r}.$$

(Recall that $tx \bmod 1$ is the unique number $s \in [0,1)$ for which $tx - s$ is an integer.) Let $t' = tr$. Then for any $d \in D$ we have $1 \leq t'd \bmod r \leq r-1$. By Theorem 3.1, $G(R,D)$ is r-colourable. As $\chi_c(G(R,D))$ is the infimum of those r for which $G(R,D)$ is r-colourable, we conclude that $\chi_c(G(R,D)) \leq \frac{1}{\kappa(D)}$. ∎

The following two results were proved in [5, 6] (with some notation modified).

Theorem 3.3 [5] *Suppose $a < b < c$ are positive integers and $gcd(a,b,c) = 1$. Let $D = \{a,b,c\}$.*

1. *If $c \neq a+b$ and $(a,b,c) \neq (1,2,3k)$ for any integer k, then $\kappa(D) \geq 1/3$.*

2. *If $(a,b,c) = (1,2,3k)$ for some integer k, then $\kappa(D) = \frac{k}{3k+1}$.*

3. *If $c = a+b$ and $b-a = 3k$ for some integer k, then $\kappa(D) = 1/3$.*

4. *If $c = a+b$ and $b-a = 3k+1$ for some integer k, then $\kappa(D) = \frac{a+k}{3(a+k)+1}$.*

5. *If $c = a+b$ and $b-a = 3k+2$ for some integer k, then $\kappa(D) = \frac{b-k-1}{3(b-k)-2}$.*

Let $\mathcal{T}$ be the following set of triples:

$$\begin{aligned}\mathcal{T} = \{ \; & (1,3,8),\ (1,4,10),\ (1,4,13),\ (1,4,18),\ (1,5,8), (1,5,12), \\ & (1,5,14),\ (1,7,10),\ (1,7,16),\ (1,7,18),\ (1,8,11),\ (1,8,14), \\ & (1,10,13),\ (1,10,18),\ (1,12,26),\ (1,15,18),\ (2,3,9),\ (2,3,10), \\ & (2,3,11),\ (2,3,15),\ (2,3,16),\ (2,3,20),\ (2,5,6),\ (2,5,17),\end{aligned}$$

$(2,7,10)$, $(2,8,9)$, $(2,9,10)$, $(2,9,14)$, $(2,9,15)$, $(2,11,17)$,
$(3,4,14)$, $(3,4,15)$, $(3,5,14)$, $(3,7,16)$, $(3,7,20)$, $(3,8,10)$,
$(3,11,16)$, $(4,5,7)$, $(4,5,18)$, $(4,6,7)$, $(4,6,13)$, $(5,6,13)$,
$(5,6,22)$, $(6,8,11)$, $(6,13,14)$ $\}$.

Theorem 3.4 [6] *Suppose* $a < b < c$ *are positive integers such that* $gcd(a,b,c) = 1$, *and that* $D = \{a,b,c\}$. *If* $c \neq 2b$, $b \neq 2a$, $c \neq a+b$ *and* $(a,b,c) \notin \mathcal{T}$ *then*

$$\kappa(D) \geq 3/8.$$

Proof of Theorem 2.1:
(1): If a,b,c are all odd then the graph $G(R,D)$ is bipartite and contains K_2. Therefore $\chi_f(G) = \chi_c(G) = 2$.

For the other triples, the upper bounds for the circular chromatic number of $G(R,D)$ follow from Theorems 3.3, 3.4 and 3.2. It remains to prove the lower bounds.

(2): Assume $a = 1, b = 2$ and $c = 3k$ where k is a positive integer. Consider the subgraph H of $G(R,D)$ induced by the set $\{0,1,\cdots,3k\}$. We shall prove that $\alpha(H) \leq k$. (Indeed, $\alpha(H) = k$, but we do not need that.) Let I be an independent set of H. Since 0 is adjacent to $3k$, one of them is not in I. Assume that $0 \notin I$. Then $I \subset \{1,2,\cdots,3k\}$. But the vertices $\{1,2,\cdots,3k\}$ are covered by k copies of K_3: $\{1,2,3,\}$, $\{4,5,6\}$, $\cdots$, $\{3k-2, 3k-1, 3k\}$. The set I contains at most one vertex of each copy of K_3. Therefore $|I| \leq k$. If $3k \notin I$ then $I \subset \{0,1,\cdots,3k-1\}$. The vertices $\{0,1,\cdots,3k-1\}$ are also covered by k copies of K_3. So

$$\chi_f(G(R,D)) \geq \chi_f(H) \geq |V(H)|/\alpha(H) \geq 3 + \frac{1}{k}.$$

(3): If $c = a+b$ and $b-a = 3k$, then the set $\{0, a, a+b\}$ induces a copy of K_3. Hence $\chi_f(G) \geq 3$.

(4): Assume $c = a+b$ and $b-a = 3k+1$ where k is a nonnegative integer. For each integer i, the set $\{i, i-a, i+b, i+b-a\}$ induces a copy of $K_4 - e$, where the missing edge is the pair $i(i+b-a)$. The set $\{i, i+a, i+a+b, i+2a+b\}$ also induces a copy of $K_4 - e$, where the missing edge is the pair $i(i+2a+b)$.

We denote by A_i the copy of $K_4 - e$ induced by $\{i, i-a, i+b, i+b-a\}$, and by B_i the copy of $K_4 - e$ induced by $\{i, i+a, i+a+b, i+2a+b\}$. The vertex i is called the *head* of A_i and B_i, and the vertices $i+b-a$ and $i+2a+b$ are called the tail of A_i and B_i, respectively. Each A_i is called an *A-subgraph* of $G(R,D)$ and each B_i is called a *B-subgraph* of $G(R,D)$.

Suppose $s,t \geq 1$ are integers. We call a subgraph H' of $G(R,D)$ an *(s,t)-chain* if the following is true:

- H' consists of s A-subgraphs, A_{s_0}, $A_{s_0+(b-a)}$, $A_{s_0+2(b-a)}$, $\cdots$, $A_{s_0+(s-1)(b-a)}$, which are joined together head to tail, and t B-subgraphs B_{t_0}, $B_{t_0+(2a+b)}$, $\cdots$, $B_{t_0+(t-1)(2a+b)}$, which are also joined head to tail.

- $t_0 + t(2a+b) = s_0 + s(b-a)$ (i.e., the tail of the last A-subgraph is equal to the tail of the last B-subgraph) and s_0 is adjacent to t_0 (i.e., the head of the first A-subgraph is adjacent to the head of the first B-subgraph).

It is straightforward to verify that the union of

$$A_0, A_{b-a}, A_{2(b-a)}, \cdots, A_{(a+k-1)(b-a)}$$

and

$$B_a, B_{a+(2a+b)}, B_{a+2(2a+b)}, \cdots, B_{a+(k-1)(2a+b)}$$

is an $(a+k,k)$-chain.

Claim 1 *If H' is an (s,t)-chain of $G(R,D)$ then $\chi_f(H') \geq 3 + \frac{1}{s+t}$.*

Since $G(R,D)$ contains an $(a+k,k)$-chain, it remains to prove Claim 1. Claim 1 is also used in the next case, i.e., (5). We prove Claim 1 by induction on $s+t$.

If $b = 2a$ or $3a$, then since $\gcd(a,b,c) = 1$ we conclude that $a = 1, b = 2$ or 3. The case $(a,b,c) = (1,2,3)$ is included in Case 2. For the case $(a,b,c) = (1,3,4)$, it is easy to verify that the subgraph H of $G(R,D)$ induced by $\{-1,0,1,2,3,4,5\}$ has independence number 2. Hence $\chi_f(G(R,D)) \geq \chi_f(H) \geq 7/2$. This proves Claim 1 for the case when $s = t = 1$. In the following, we assume that $b \neq 2a, 3a$, which implies that the vertices of different A-subgraphs in H' are distinct, except the head of $A_{s_0+i(b-a)}$ is the tail of $A_{s_0+(i-1)(b-a)}$. Of course the vertices of different B-subgraphs in H' are distinct, except the head of $B_{t_0+i(2a+b)}$ is the tail of $B_{t_0+(i-1)(2a+b)}$.

If none of the A-subgraphs intersects a B-subgraph, except the last A-subgraph and the last B-subgraph have the same tail, then easy counting shows that $|V(H')| = 3(s+t)+1$. On the other hand, it is easy to prove that $\alpha(H') \leq s+t$. Indeed, if I is an independent set of H', then either s_0 or t_0 does not belong to I, as they are adjacent. Assume that $s_0 \notin I$. Then $I \subset V(H') - \{s_0\}$. However, $V(H') - \{s_0\}$ can be covered by $s+t$ copies of K_3, namely, each of the sets $A_{s_0+i(b-a)} - \{s_0 + i(b-a)\}$ induces a K_3 and each of the sets $B_{t_0+j(2a+b)} - \{t_0 + (j+1)(2a+b)\}$ induces a K_3. Therefore $|I| \leq s+t$. The case $t_0 \notin I$ is symmetric, implying the set $V(H') - \{t_0\}$ can also be covered by $s+t$ copies of K_3. Thus $\alpha(H') \leq s+t$, and it follows that

$$\chi_f(H') \geq \frac{|V(H')|}{\alpha(H')} \geq 3 + \frac{1}{s+t}.$$

Next we consider the case that A_{s_0} intersects B_{t_0} and no other A-subgraphs intersects a B-subgraph. Because s_0 is adjacent to t_0, and $b \neq 2a, 3a$, the only possibilities are $t_0 = s_0 - a$ or $t_0 = s_0 + b$ or $t_0 + a + b = s_0$. If $t_0 = s_0 + b$, then the union of

$$A_{s_0+(b-a)}, A_{s_0+2(b-a)}, \cdots, A_{s_0+(s-1)(b-a)}$$

and

$$B_{t_0}, B_{t_0+(2a+b)}, \cdots, B_{t_0+(t-1)(2a+b)}$$

is an $(s-1,t)$-chain. If $t_0 + a + b = s_0$, then the union of

$$A_{s_0}, A_{s_0+(b-a)}, \cdots, A_{s_0+(s-1)(b-a)}$$

and

$$B_{t_0+2a+b}, B_{t_0+2(2a+b)}, \cdots, B_{t_0+(t-1)(2a+b)}$$

is an $(s,t-1)$-chain. If $t_0 = s_0 - a$, then A_{s_0} and B_{t_0} share a triangle. As no other A-subgraph intersects a B-subgraph, we have $|V(H')| = 3(s+t) - 2$. On the other hand, $\alpha(H') \leq s+t-1$. Indeed, because $s_0 \sim t_0$, so for any independent set I either $s_0 \notin I$ or $t_0 \notin I$. Assume $s_0 \notin I$. Then

$|I| \leq s+t-1$, because $V(H')-\{s_0\}$ is covered by $s+t-1$ copies of K_3 (for $i=0,1,\cdots,s-1$) each of the sets $A_{s_0+i(b-a)}-\{s_0+i(b-a)\}$ induces a K_3 and for $j=1,2,\cdots,t-1$, each of the sets $B_{t_0+j(2a+b)}-\{t_0+(j+1)(2a+b)\}$ induces a K_3). The case $t_0 \notin I$ is similar. Therefore

$$\chi_f(H') \geq \frac{|V(H')|}{\alpha(H')} \geq 3+\frac{1}{s+t-1}.$$

Note that $A_{s_0+(s-1)(b-a)}$ shares the same tail with $B_{t_0+(t-1)(2a+b)}$. So $A_{s_0+(s-1)(b-a)}$ cannot intersect any B-subgraphs (other than the tail). Assume that there is a $1 \leq i \leq s-2$ such that the A-subgraph $A_{s_0+i(b-a)}$ intersects a B-subgraph $B_{t_0+j(2a+b)}$. We shall find an (s',t')-chain H'' contained in H' for which $s'+t' < s+t$. If this is so, then by induction hypothesis we have

$$\chi_f(H') \geq \chi_f(H'') \geq 3+\frac{1}{s'+t'} > 3+\frac{1}{s+t}.$$

It is not difficult to find such an (s',t')-chain H'', but the description is tedious. We shall do two cases, and leave the other cases for the readers to check.

Recall that the vertices of $A_{s_0+i(b-a)}$ are

$$s_0+i(b-a), s_0+i(b-a)-a, s_0+i(b-a)+b, s_0+(i+1)(b-a),$$

and the vertices of $B_{t_0+j(2a+b)}$ are

$$t_0+j(2a+b), t_0+a+j(2a+b), t_0+a+j(2a+b)+b, t_0+(j+1)(2a+b).$$

If $s_0+(i+1)(b-a)=t_0+j(2a+b)$, then the union of

$$A_{s_0}, A_{s_0+b-a}, \cdots, A_{s_0+i(b-a)}$$

and

$$B_{t_0}, B_{t_0+(2a+b)}, \cdots, B_{t_0+(j-1)(2a+b)}$$

is an $(i+1,j)$-chain.

If $s_0+i(b-a)+b=t_0+j(2a+b)$, then the union of

$$A_{s_0+(i+1)(b-a)}, A_{s_0+(i+2)(b-a)}, \cdots, A_{s_0+(s-1)(b-a)}$$

and

$$B_{t_0+j(2a+b)}, B_{t_0+(j+1)(2a+b)}, \cdots, B_{t_0+(t-1)(2a+b)}$$

induces an $(s-i-1, t-j)$-chain.

The other cases are similar and omitted. This completes the proof of Claim 1 as well as (4).

(5): Assume $c = a+b$ and $b-a = 3k+2$ where k is a nonnegative integer. The union of

$$A_0, A_{b-a}, A_{2(b-a)}, \cdots, A_{(a+k)(b-a)}$$

and

$$B_{-a}, B_{-a+(2a+b)}, B_{-a+2(2a+b)}, \cdots, B_{-a+k(2a+b)}$$

is an $(a+k+1, k+1)$-chain. By Claim 1, we have $\chi_f(G(R,D) \geq 3+\frac{1}{a+2k+2} = 3+\frac{1}{b-k}$.

(6) and (7): If a, b, c are not all odd, then $G(R,D)$ contains odd cycles, hence $\chi_f(G(R,D)) > 2$.

4 Some remarks

If $D = \{a, b\}$ then it is known [3] that

$$\chi_f(G(R,D)) = \chi_c(G(R,D)) = \frac{1}{\kappa(D)}. \quad (*)$$

If $D = \{a, b, c\}$ then we only have estimations for each of the values $\chi_c(G(R,D)), \chi_f(G(R,D))$ and $\kappa(D)$. However, for all those triples (a, b, c) for which the exact values of $\chi_c(G(R,D)), \chi_f(G(R,D))$ and $\kappa(D)$ are known, equality $(*)$ also holds. It is natural to ask if equality $(*)$ holds for all distance sets D. However, this is not the case.

Theorem 4.1 *If $D = \{1, 3, 4, 7\}$ then $\kappa(D) = \frac{1}{5}$ and $\chi_f(G(R,D)) = \chi_c(G(R,D)) = 4$.*

Proof. The value of $\kappa(D)$ was determined in [12]. Since $G(R,D)$ contains a K_4 (induced by $0, 3, 4, 7$ we have $\chi_f(G(R,D)) \geq 4$. On the other hand the subgraph $G(Z,D)$ of $G(R,D)$ induced by the integers can be 4-coloured periodically with period 8 and colour sequence 12343412, i.e., colour the vertices $0,1,2,3,4,5,6,7$ with colours $1,2,3,4,3,4,1,2$ respectively and colour vertex i with the same colour as vertex $i-8$. This shows that $\chi(G(R,D)) \leq 4$ (each connected component of $G(R,D)$ is a copy of $G(Z,D)$). Hence $\chi_f(G(R,D)) = \chi_c(G(R,D)) = 4$. ∎

It is unknown if there are triples (a,b,c) for which $\chi_c(G(R,D)) \neq \kappa(D)$, where $D = \{a,b,c\}$.

The function $\kappa(D)$ is related to the view-obstruction problem in geometry [7, 9], to the flow problem in graph theory [1], and has been studied extensively. It was conjecture by Wills [17] that for any finite set D which does not contain 0, $\kappa(D) \geq \frac{1}{|D|+1}$. This conjecture seems to be difficult. It has attracted considerable attention, has been verified for $|D| \leq 4$, but remains open for $|D| \geq 5$. The readers are referred to [1, 5, 6, 7, 9, 17] for more details of research concerning this conjecture. In contrast to the difficulty of Wills' conjecture, it is easy to see that $\chi_c(G(R,D)) \leq |D|+1$.

There are examples to show that the upper bound $|D|+1$ for the circular chromatic number of $G(R,D)$ and the lower bound $\frac{1}{|D|+1}$ for $\kappa(D)$ conjectured by Wills are tight. However, the results in this paper suggest that there are very few sets D for which $\chi_c(G(R,D)) = |D|+1$. Similarly, for very few sets we have $\kappa(D) = \frac{1}{|D|+1}$. We have the following conjecture.

Conjecture 4.1 *Suppose D is a finite set and $|D| = n \geq 3$. If $G(R,D)$ is triangle-free, i.e., there are no elements a,b,c of D such that $a+b=c$ (where a,b,c need not be distinct), then $\chi(G(R,D)) \leq n$.*

The following conjecture is stronger:

Conjecture 4.2 *Suppose D is a finite set and $|D| = n \geq 3$. If there are no elements a,b,c of D such that $a+b=c$ (where a,b,c need not be distinct), then $\kappa(D) \geq \frac{1}{n}$.*

If $n=3$, then these two conjectures are true (as shown in [5, 6]).

Finally we remark that the regular colouring method can also be used to colour circulant graphs. In many cases, it gives a good upper bound for the circular chromatic numbers and chromatic numbers of circulant graphs [18].

Acknowledgement

I thank T. W. Cusick for bringing to my attention references [5, 6]. I also thank the referees for a careful reading of the manuscript.

References

[1] W. Bienia, L. Goddyn, P. Gvozdjak, A. Sebő and M. Tarsi, *Flows, view obstructions, and the lonely runner*, J. Combinatorial Theory (B), 72(1998), 1-9.

[2] J. Chen, G. Chang and K. Huang, *Integral distance graphs*, J. Graph Theory, 25(1997), 281-287.

[3] G. J. Chang, L. Huang and X. Zhu, *The circular chromatic numbers and the fractional chromatic numbers of distance graphs*, European Journal of Combinatorics, 19 (1998), pp. 423-431.

[4] G. Chang, D. Liu and X. Zhu, *Distance graphs and T-colouring*, J. Combinatorial Theory (B), 75(1999), 159-169.

[5] Y. G. Chen, *On a conjecture about diophantine approximations, II*, J. Number Theory 37(1991), 181-198.

[6] Y. G. Chen, *On a conjecture about diophantine approximations, IV*, J. Number Theory 43(1993), 186-197.

[7] Y. G. Chen and T. W. Cusick, *The View-obstruction problem for n-dimensional cubes*, J. Number Theory 74(1999), 126-133.

[8] M. B. Cozzens and F. S. Roberts, *T-colorings of graphs and the channel assignment problems*, Congr. Numer. 35(1982), 191-208.

[9] T. W. Cusick, *View-obstruction problems in n-dimensional geometry*, J. Combinatorial Theory (A), 16(1974), 1-11.

[10] W. Deuber and X. Zhu, *The chromatic number of distance graphs*, Disc. Math. 165/166(1997), 195-204.

[11] R. B. Eggleton, P. Erdös and D. K. Skilton, *Colouring the real line*, J. Combinatorial Theory, Ser. (B) 39(1985) 86-100.

[12] L. Goddyn and E. Wong, *Tight Instances of the Lonely Runner*, Technical report, Laboratoire Leibniz, IMAG, 1996.

[13] J. Griggs and D. Liu, *The channel assignment problem for mutually adjacent sites*, J. Combinatorial Theory, Ser. (A) 68(1994), 169-183.

[14] L. Huang and G. Chang, *Circular chromatic number of distance graphs with missing multiples in the distance set*, European Journal of Combinatorics, .

[15] J. H. Rabinowitz and V. K. Proulx, *An asymptotic approach to the channel assignment problem*, SIAM J. Algebraic Discrete Methods 6(1985), 507-518.

[16] M. Voigt, *Colouring of distance graphs*, Ars Combinatoria, 52(1999), 3-12.

[17] J. M. Wills, *Zwei Sätze über inhomogene diophantische appromixation von irrationlzahlen*, Monatsch. Math. 71(1967), 263-269.

[18] H. Yeh and X. Zhu, *Colouring some circulant graphs* manuscript, 1999.

[19] X. Zhu, *Colouring the distance graphs*, manuscript, 1995.

[20] X. Zhu, *Non-integral distance graphs on the real line*, manuscript, 1996.

[21] X. Zhu, *The circular chromatic number of a class of distance graphs*, manuscript, 1998.

[22] X. Zhu, *Circular chromatic number, a survey*, Discrete Mathematics, 229 (1-3) (2001), 371-410.

[23] X. Zhu, *Diophantine approximations and its applications to graph colouring*, manuscript, 1998.