Chromatic numbers of distance graphs with distance sets missing multiples

Walter Deuber
Facultät für Mathematik
Universität Bielefeld
33501, Bielefeld
Germany

Xuding Zhu
Department of Applied Mathematics
National Sun Yat-sen University
Kaoshing, Taiwan 80424
Email: zhu@math.nsysu.edu.tw

March 5, 2003

Abstract

Given positive integers m, k and s with $m > ks$, let $D_{m,k,s}$ represents the set $\{1, 2, \cdots, m\} - \{k, 2k, \cdots, sk\}$. The distance graph $G(Z, D_{m,k,s})$ has as vertex set all integers Z and edges connecting i and j whenever $|i - j| \in D_{m,k,s}$. This paper investigates the chromatic numbers of the distance graphs $G(Z, D_{m,k,s})$. It is proved that if $m \geq (s+1)k$, then $\chi(G(Z, D_{m,k,s}) \leq \lceil (m+sk+1)/(s+1) \rceil + 1$, while the lower bound $\chi(G(Z, D_{m,k,s}) \geq \lceil (m + sk + 1)/(s + 1) \rceil$ was known. This upper bound improves previous known upper bounds.

1 Introduction

Given a set D of positive integers, the *distance graph* $G(Z, D)$ has all integers as vertices; and two vertices are adjacent if their difference falls within D, that is, the vertex set is Z and the edge set is $\{uv : |u - v| \in D\}$. We call D the *distance set*. The chromatic number of $G(Z, D)$ is denoted by $\chi(Z, D)$.

For different types of distance sets D, the problem of determining $\chi(Z, D)$ has been studied extensively (see [2, 3, 4, 5, 6, 7, 8, 11, 16, 14, 17].) For instance, suppose D is a subset of prime numbers and $\{2, 3\} \in D$, Eggleton, Erdös and Skilton [8] proved $\chi(Z, D)$ is either 3 or 4. The problem of classifying $G(Z, D)$ with distance sets D of primes into chromatic number 3 or 4 was studied by Eggleton, Erdös and Skilton [8], by Voigt [15], and by Voigt and Walther [16]. However, a complete classification is not obtained yet.

The case that D contains at most three integers were studied by Eggleton, Erdös and Skilton [5], Chen, Chang, and Huang [3], Voigt [14], and Zhu [17]. The chromatic number of such distance graphs has now been completely determined [17].

Given integers m, k and s with $m > ks$, let $D_{m,k,s}$ denote the distance set $D_{m,k,s} = \{1, 2, 3, \cdots, m\} - \{k, 2k, 3k, \cdots, sk\}$. This article studies the chromatic number $\chi(Z, D_{m,k,s})$ of $G(Z, D_{m,k,s})$.

For $s = 1$, the chromatic number of $G(Z, D_{m,k,1})$ was first studied by Eggleton, Erdös and Skilton [5], in which $\chi(Z, D_{m,k,1})$ was solved as $k = 1$ and partially solved as $k = 2$. The same results for the case $k = 1$ were also obtained in [11] by a different approach. If k is an odd number, or $k = 2$, or $k = 4$, then $\chi(Z, D_{m,k,1})$ were determined in [12]. Finally, the exact values of $\chi(Z, D_{m,k,1})$ for all m and k were determined in [2]. For $s = 2$, the chromatic number of $G(Z, D_{m,k,1})$ was recently determined in [13]. Some results concerning the problem for general s were also obtained in [13]. In this paper, we extend the study of $\chi(Z, D_{m,k,s})$ for general values of s.

Note that the chromatic number is easy to determine if $m < (s+1)k$: Define a coloring f of $G(Z, D_{m,k,s})$ by: for any $x \in Z$, $f(x) = y \pmod{k}$, $1 \le y \le k$. Since $D_{m,k,s}$ contains no multiples of k, it can be easily verified that f is a proper coloring. Thus, $\chi(Z, D_{m,k,s}) \le k$. As any consecutive k vertices in $G(Z, D_{m,k,s})$ form a complete graph, $\chi(Z, D_{m,k,s}) \ge k$. This implies $\chi(Z, D_{m,k,s}) = k$, if $m < (s+1)k$. Therefore, throughout the article, we shall assume $m \ge (s+1)k$.

We prove in this paper that for any integers m, k, s, $\chi(Z, D_{m,k,s}) \le \lceil (m + sk + 1)/(s+1) \rceil + 1$ hold for arbitrary s. Combined with the lower bound $\chi(Z, D_{m,k,s}) \ge \lceil (m + sk + 1)/(s+1) \rceil$ obtained in [12], we conclude that for any values of m, k, s, either $\chi(Z, D_{m,k,s}) = \lceil (m+sk+1)/(s+1) \rceil + 1$ or $\chi(Z, D_{m,k,s}) = \lceil (m+sk+1)/(s+1) \rceil$.

2 An upper bound

We shall use extensively the pre-coloring method introduced in [13] (a simpler version of this coloring method was used in [2]).

It is known and easy to verify, for any distance set D, $\chi(Z, D) = \chi(Z^+ \cup \{0\}, D)$, where $G(Z^+ \cup \{0\}, D)$ is the subgraph of $G(Z, D)$ induced by the set of non-negative integers $Z^+ \cup \{0\}$. Therefore, to color the graph $G(Z, D_{m,k,s})$, it suffices to color the subgraph of $G(Z, D_{m,k,s})$ induced by $Z^+ \cup \{0\}$.

There are two steps in the pre-coloring method. First, we partition the set of non-negative integers $Z^+ \cup \{0\}$ into $s+1$ parts by a mapping $c : Z^+ \cup \{0\} \to \{0, 1, 2, \cdots, s\}$. Second, for each non-negative integer x, according to the value of $c(x)$, we assign a color to x by the rule defined as follows.

Definition 1 *Suppose m, k, s are positive integers. For a given mapping $c : Z^+ \cup \{0\} \to \{0, 1, 2, \cdots, s\}$, define a coloring c' of $Z^+ \cup \{0\}$ recursively by:*

$$c'(j) = \begin{cases} j, & \text{if } j < k; \\ c(j-k), & \text{if } j \geq k \text{ and } c(j) \neq 0; \\ n, & \text{if } j \geq k \text{ and } c(j) = 0, \end{cases}$$

where n is the smallest non-negative integer (color) not been used in the m vertices preceding j, that is, $n = \min\{t \in Z^+ \cup \{0\} : c'(j-i) \neq t \text{ for } i = 1, 2, \cdots, m\}$.

Note that c' defined above is uniquely determined by c. We call c the pre-coloring, and c' the coloring induced by c. For any $x \in Z^+ \cup \{0\}$, $c(x)$ and $c'(x)$ are called the *pre-color* and the *color* of x, respectively.

The following Lemmas are proved in [13]:

Lemma 2 *Suppose c is a pre-coloring of $Z^+ \cup \{0\}$. If for any integer $j \geq sk$, $c(j), c(j-k), c(j-2k), \cdots,$ and $c(j-sk)$ are all distinct, then the induced coloring c' is a proper coloring for $G(Z, D_{m,k,s})$.*

Lemma 3 *Suppose c is a pre-coloring and c' is the induced coloring. Then the number of colors used by c' is $k+\ell$, where ℓ is the maximum number of vertices with pre-color 0 among any $m-k+1$ consecutive integers greater than k.*

Lemma 4 *Given integers m, k and s, $\chi(Z, D_{m,k,s}) \le n$ if there exists a pre-coloring c such that the following two conditions are satisfied:*

(1) *for any integer $j \ge sk$, $c(j), c(j+k), c(j+2k), \cdots, c(j+sk)$ are all distinct, and*

(2) *among any consecutive $m-k+1$ integers, there are at most $n-k$ vertices with pre-color 0.*

Theorem 5 *For any integers m, k, s with $m \ge (s+1)k$, we have $\chi(Z, D_{m,k,s}) \le \lceil (m+sk+1)/(s+1) \rceil + 1$. Moreover, if $m-k+1 = 1 \pmod{s+1}$ then $\chi(Z, D_{m,k,s}) = \lceil (m+sk+1)/(s+1) \rceil$.*

Proof. It was proved in [13] that $\chi(Z, D_{m,k,s}) \ge \lceil (m+sk+1)/(s+1) \rceil$. Therefore by Lemma 4, it suffices to find a pre-coloring c of $Z^+ \cup \{0\}$ such that for any integer $j \ge sk$, $c(j), c(j-k), c(j-2k), \cdots, c(j-sk)$ are all distinct, and that among any consecutive $m-k+1$ integers, there are at most $n-k$ vertices with pre-color 0, where $n = \lceil (m+sk+1)/(s+1) \rceil$ if $m-k+1 = 1 \pmod{s+1}$ and $n = \lceil (m+sk+1)/(s+1) \rceil + 1$ otherwise.

Let $d = (s+1, k)$, and let $q = (s+1)k/d$. For any integer i, we write i in the form $i = u(s+1)k + qx + (s+1)y + z$, where u, x, y, z are non-negative integers such that $x < d$, $y < k/d$ and $z < s+1$. Obviously u, x, y, z are uniquely determined by i. Let $c(i) = x + z \pmod{s+1}$. We shall prove that c satisfies the conditions above.

First for any j, we prove that $j, j+k, \cdots, j+sk$ have different pre-colors.

Assume to the contrary that $c(j+ak) = c(j+bk)$ for some $0 \le a < b \le s$. Suppose $j+ak = u(s+1)k + qx + (s+1)y + z$ and that $j+bk = u'(s+1)k + qx' +$

$(s+1)y' + z'$. Then

$$(b-a)k = (u'-u)(s+1)k + (x'-x)q + (y'-y)(s+1) + (z'-z).$$

Since d divides each of the terms $(b-a)k, (u'-u)(s+1)k, (x'-x)q, (y'-y)(s+1)$, it follows that d divides $z'-z$. Because $c(j+ak) = x+z \pmod{s+1} = c(j+bk) = x'+z' \pmod{s+1}$, it follows that $z'-z = x-x' \pmod{s+1}$. As d divides both $z'-z$ and $s+1$, we conclude that d divides $x-x'$. As $|x-x'| < d$, it follows that $x-x' = 0$. Therefore $z'-z = 0 \pmod{s+1}$, which implies that $z = z'$. Thus $(b-a)k - (u'-u)(s+1)k = (s+1)(y'-y)$, which implies that k divides $(s+1)(y'-y)$. Since $((s-1)/d, k) = 1$, we conclude that k divides $d(y'-y)$. However $|y'-y| < k/d$, i.e., $|k(y'-y)| < k$, therefore $y'-y = 0$. Hence $(b-a)k = (u'-u)(s+1)k$. This implies that $u'-u = 0 = b-a$ (as $0 \leq (b-a) \leq s$), contrary to the assumption that $b > a$.

Next we prove that among any consecutive $m-k+1$ integers, there are at most $n-k$ vertices with pre-color 0, where $n = \lceil (m+sk+1)/(s+1) \rceil$ if $m-k+1 = 1 \pmod{s+1}$, and $n = \lceil (m+sk+1)/(s+1) \rceil + 1$ otherwise.

Divide the set $Z^+ \cup \{0\}$ into segments $I_0, I_1, \cdots$, such that $I_j = \{j(s+1), j(s+1)+1, \cdots, (j+1)(s+1)-1\}$. It follows from the definition of the pre-coloring c that each segment I_j contains exactly one element of pre-color 0. Let X be a set of $m-k+1$ consecutive integers. If $m-k+1 = 1 \pmod{s+1}$, then X intersect at most $\lceil (m+sk+1)/(s+1) \rceil$ of the segments I_j, hence it contains at most $n = \lceil (m+sk+1)/(s+1) \rceil$ vertices of pre-color 0. In general, i.e., if $m-k+1 \neq 1 \pmod{s+1}$, X intersect at most $\lceil (m+sk+1)/(s+1) \rceil + 1$ of the segments I_j, hence it contains at most $\lceil (m+sk+1)/(s+1) \rceil + 1$ vertices of pre-color 0.

It was proved in [13] that for any integers m, k, s with $m \geq (s+1)k$, $\chi_f(Z, D_{m,k,s}) = (m+sk+1)/(s+1)$, which implies that $\chi(Z, D_{m,k,s}) \geq \lceil (m+sk+1)/(s+1) \rceil$. Therefore when $m \geq (s+1)k$, $\chi(Z, D_{m,k,s})$ is equal to either $\lceil (m+sk+1)/(s+1) \rceil$ or $\lceil (m+sk+1)/(s+1) \rceil + 1$. The results in [13], as well as the results in Section 3 of this paper, show that both the upper bound and lower bound are sharp. It remains an open problem to determine for which $D_{m,k,s}$ the lower bound is attained, and for which $D_{m,k,s}$ the upper bound is attained. For $s = 1$ and $s = 2$, the problem is completely solved in [2] and [13] respectively. The results in [13] shows that the problem is more difficult when $s + 1$ is not a prime. In the next section, we consider the case $s = 3$, and present some partial solutions.

3 s=3

In this section, we consider the case that $s = 3$. We shall divide the discussion into a few cases, according to value of $k \pmod 4$.

If k is odd, then it follows from a result (Theorem 13) in [13] that $\chi(Z, D_{m,k,3}) = \lceil (m + sk + 1)/(s + 1) \rceil$. In the following, we assume that k is even.

Our next two theorems give the answer for the case $m - k + 1 = 2 \pmod 4$.

Theorem 6 *If $k = 4t + 2$ for some integer t and $m - k + 1 \neq 0 \pmod 4$, then $\chi(Z, D_{m,k,3} = \lceil (m + 3k + 1)/4 \rceil$.*

Proof. Define a pre-coloring c as follows: for any integer i, write i in the form $i = 4ku + 2kx + 4y + z$, where u, x, y, z are non-negative integers such that $x \leq 1$, $y \leq 2t$ and $z \leq 3$. If $x = 0$, then let $c(i) = z \pmod 4$; if $x = 1$, then let $c(x) = 3 + z \pmod 4$.

In the following we show that for any integer $j \geq 3k$, $c(j), c(j+k), c(j+2k), c(j+3k)$ are all distinct, and that among any consecutive $m - k + 1$ integers, there are at most $n - k$ vertices with pre-color 0, where $n = \lceil (m + sk + 1)/(s + 1) \rceil$.

Assume to the contrary that there exist $j \geq 0$ and $0 \leq a < b \leq 3$ such that $c(j+ak) = c(j+bk)$. Suppose $j+ak = 4ku+2kx+4y+z$, $j+bk = 4ku'+2kx'+4y'+z'$. Then $j + bk - (j + ak) = (b - a)k = 4k(u' - u) + 2k(x' - x) + 4(y' - y) + (z' - z)$. As each of the terms $(b - a)k, 4k(u' - u), 2k(x' - x), 4(y' - y)$ is even, we conclude that

$z' - z$ is even, i.e., z, z' have the same parity. Because $c(j + ak) = c(j + bk)$, it follows from the definition of c that $x = x'$, and hence $z = z'$. This implies that $k = 4t + 2$ divides $2(y' - y)$, which implies that $y' - y = 0$ (because $|y' - y| < k/2$). Therefore $b - a = 4(u' - u)$. This implies that $u' - u = 0$ (because $0 \leq b - a \leq 3$), and hence $b = a$, contrary to our assumption.

Next we show that among any consecutive $m - k + 1$ integers, there are at most $n - k$ vertices with pre-color 0, where $n = \lceil (m + sk + 1)/(s + 1) \rceil$. Let X be a set of $m - k + 1$ consecutive integers. We divide the set $Z^+ \cup \{0\}$ into segments I_j, where $I_j = \{4j, 4j + 1, 4j + 2, 4j + 3\}$. Then each I_j has exactly one vertex of pre-color 0. Indeed, the pre-colors of the elements of I_j are either 0123 or 3012 (in that order). The set X intersects either with $n - k$ of the segments I_j, or with $n - k + 1$ of the segments I_j. In the former case, of course X contains at most $n - k$ elements of pre-color 0. In the latter case, assume the $n - k + 1$ segments I_j that intersects X are $I_q, I_{q+1}, \cdots, I_{q+n-k}$. Since $|X| = m - k + 1 \neq 0 \pmod 4$, we conclude that $|X \cap I_q| \leq 2$. Note that the pre-colors of the elements of segment I_q are either 0123 or 3012, hence none of the last two elements of I_q are of pre-color 0. Therefore $X \cap I_q$ contains no vertex of pre-color 0. So X contains at most $n - k$ vertices of pre-color 0.

Theorem 7 *Suppose $k = 4t + 2$ for some integer t, and that $m - k + 1 = 4p$ for some integer p. Then $\chi(Z, D_{m,k,3} = \lceil (m + 3k + 1)/4 \rceil$ if p is even, and $\chi(Z, D_{m,k,3} = \lceil (m + 3k + 1)/4 \rceil + 1$ if p is odd.*

Proof. First consider the case that p is odd. Assume to the contrary that $\chi(Z, D_{m,k,3}) = (m+3k+1)/4$. For any two integers i and j, let $G[i, j]$ be the subgraph of $G(Z, D_{m,k,3})$ induced by vertices $\{i + 1, i + 2, \cdots, j\}$. Then for any integer i, the graph $G[i, i + m + 3k + 1]$ has $m + 3k + 1$ vertices and maximum independent set of size 4. Since f is an $(m + 3k + 1)/4$-coloring, exactly 4 vertices of $G[i, i + m + 3k + 1]$

are colored by the same color. It follows that $f(i) = f(i+m+3k+1)$ for any integer i.

Define a circulant graph G on the set $\{0, 1, \cdots, m+3k\}$ with generating set $D_{m,k,3}$, that is, ij is an edge of G if and only if $j - i \pmod{m+3k+1} \in D_{m,k,3}$ or $i - j \pmod{m+3k+1} \in D_{m,k,3}$. The argument in the previous paragraph shows that f induces a proper n-coloring of G. Moreover, each color class consists of 4 vertices in G. It is not difficult to verify that all 4-independent sets of G are of the form $\{i, i+k, \cdots, i+3k\}$ (here each number is calculated by modulo $m+3k+1$.)

Let $d = (k, m+3k+1)$ and $u = (m+3k+1)/d$. Divide the vertex set of G into d subsets of the form $\{i, i+k, i+2k, \cdots, i+(u-1)k\} \pmod 4$, each of size u. Then each of these d subsets is the union of some color classes of size 4, so 4 divides u. However, this would implies that $m-k+1$ is a multiple of 8, because d is certainly even.

Suppose p is even. Let d, u be as defined in the previous section. Then 4 divides u. Indeed, as $k = 4t+2$ and $m+3k+1 = m-k+1+4k = 4(p+k)$, we know that $d = 2a$ for some odd integer a. Hence $u = (m+3k+1)/d = 4(p+k)/d$ is a multiple of 4. One can easily define a proper u-coloring f on G by using $u/4$ colors to each of the subsets $\{i, i+k, i+2k, \cdots, i+(u-1)k\} \pmod 4$: the first 4 vertices in a subset use one color and the next 4 vertices use the next, and continue the process until all vertices are colored. It is easy to check that f is a proper coloring of G. Furthermore, f can be extended to a proper coloring of $G(Z, D_{m,k,3})$ by letting $f'(y) = f(x)$, $x = y \pmod{m+3k+1}$. Therefore, $G(Z, D_{m,k,3})$ is u-colorable, where $u = (m+3k+1)/4$. This completes the proof of Theorem 7.

In the following we consider the case that $k \equiv 0 \pmod 4$. Suppose $k = 4^a k'$, where $a \geq 1$ and $k' \not\equiv 0 \pmod 4$. Suppose $m+3k+1 = 4^b q$, where $b \geq 0$ and $q \not\equiv 0 \pmod 4$. The following theorem is a special case of Theorem 3 of [13]:

Theorem 8 *If $a < b$ and k' is odd, then $\chi(Z, D_{m,k,3}) = (m+3k+1)/4$. If $0 < b \leq a$,* *then* $\chi(Z, D_{m,k,3}) = (m + 3k + 1)/4 + 1$.

By this theorem, it remains to consider the following two cases:

(1) $a < b$ and k' is even;

(2) $b = 0$.

Theorem 9 *Suppose* $a < b$ *and* k' *is even. If* $b \geq a+2$ *or* q *is odd, then* $\chi(Z, D_{m,k,3}) = (m + 3k + 1)/4$. *Otherwise* $\chi(Z, D_{m,k,3}) = (m + 3k + 1)/4 + 1$.

Proof. The proof of this result is parallel to the proof of Theorem 7. First we consider the case that $b = a + 1$ and q is even. Let $n = (m + 3k + 1)/4$. We shall prove that $G(Z, D_{m,k,3})$ is not n-colorable. Assume to the contrary, there exits an n-coloring f of $G(Z, D_{m,k,3})$.

For any two integers i and j, let $G[i, j]$ be the subgraph of $G(Z, D_{m,k,3})$ induced by vertices $\{i+1, i+2, \cdots, j\}$. Then for any integer i, the graph $G[i, i+m+3k+1]$ has $m + 3k + 1$ vertices and maximum independent set of size 4. Since f is an $(m + 3k + 1)/4$-coloring, exactly 4 vertices of $G[i, i + m + 3k + 1]$ are colored by the same color. It follows that $f(i) = f(i + m + 3k + 1)$ for any integer i.

Define a circulant graph G on the set $\{0, 1, \cdots, m + 3k\}$ with generating set $D_{m,k,3}$, that is, ij is an edge of G if and only if $j - i \pmod{m + 3k + 1} \in D_{m,k,3}$ or $i - j \pmod{m + 3k + 1} \in D_{m,k,3}$. The argument in the previous paragraph shows that f induces a proper n-coloring of G. Moreover, each color class consists of 4 vertices in G. It is not difficult to verify that all 4-independent sets of G are of the form $\{i, i + k, \cdots, i + 3k\}$ (here each number is calculated by modulo $m + 3k + 1$.)

Let $d = (k, m + 3k + 1)$ and $u = (m + 3k + 1)/d$. Since k', q are both even, it follows that $d = 2 \cdot 4^a d'$ for some odd integer d'. As $b = a + 1$, this implies that u is not a multiple of 4.

Divide the vertex set of G into d subsets of the form $\{i, i + k, i + 2k, \cdots, i + (u-1)k\} \pmod 4$, each of size u. Then each of these d subsets is the union of some color classes of size 4. However this is impossible because u is not a multiple of 4.

On the other hand, if $b \geq a+2$ or q is odd, then the integer u defined as above is a multiple of 4. In this case, one can easily partition each of the d sets $\{i, i+k, i+2k, \cdots, i+(u-1)k\} \pmod 4$ into independent sets of size 4. This implies that the circulant graph G defined as above is indeed n-colorable, and hence $G(Z, D_{m,k,3})$ is n-colorable.

For the remaining part of the paper, we assume that $b = 0$. If $m + 3k + 1 \equiv 1 \pmod 4$, then it follows from Theorem 5 that $\chi(Z, D_{m,k,3}) = (m + 3k + 1)/4$.

Theorem 10 *If* $m + 3k + 1 \equiv 2 \pmod 4$*, then* $\chi(Z, D_{m,k,3}) = \lceil (m + 3k + 1)/4 \rceil$.

Proof. Let $n = \lceil (m + 3k + 1)/4 \rceil$. Suppose $m + 3k + 1 = 4ck + d$, where c, d are integers such that $0 < d < 4k$ (since $b = 0$, we know that $d \neq 0$). If $d \leq 2k$, then define a pre-coloring c of $Z^+ \cup \{0\}$ as follows:

For any non-negative integer i, write i in the form $i = kx + y$, where x, y are non-negative integers such that $y \leq k-1$. Let $c(i) = x + y \pmod 4$. We show that for any integer j, the vertices $j, j+k, j+2k, j+3k$ have distinct pre-colors, and that any $m - k + 1$ consecutive integers contains at most $n - k$ integers of pre-color 0.

Assume to the contrary that $c(j + uk) = c(j + vk)$ for some integers j, u, v such that $0 \leq u < v \leq 3$. Assume that $j + uk = kx + y$ and $j + vk = kx' + y'$. Then $(v - u)k = (x' - x)k + (y' - y)$. It follows that k divides $y' - y$, which implies that $y' - y = 0$ (because $|y' - y| \leq k - 1$). As $c(j + uk) = c(j + vk)$, it follows that $x' \equiv x \pmod 4$. This implies that $v - u \equiv 0 \pmod 4$, contrary to the assumption that $0 \leq u < v \leq 3$.

Next, let X be a set of $m-k+1$ consecutive integers. We divide the set $Z^+ \cup \{0\}$ into segments I_j, where $I_j = \{4j, 4j+1, 4j+2, 4j+3\}$. Each of the segments I_j contains exactly one integer of per-color 0. Since $\lceil (m-k+1)/4 \rceil = n$, we know that X either intersects with $n-k$ of the segments I_j, or $n-k+1$ of the segments I_j. If X intersect with $n-k$ of the segments I_j, then of course X contains at most $n-k$ integers of pre-color 0. Assume that X intersect with $n-k+1$ segments I_j. Since $m+3k+1 \equiv 2 \pmod 4$, it follows that $m-k+1 \equiv 2 \pmod 4$. Thus $m-k+1 = 4(n-k)-2$. Let $I_i, I_{i+1}, \cdots, I_{i+n-k}$ be the segments that intersect X. Then $|X \cap I_i| = |X \cap I_{i+n-k}| = 1$. If the last element of I_i does not have pre-color 0, or the first element of I_{i+n-k} does not have pre-color 0, then X contains at most $n-k$ elements of pre-color 0. Assume that the last element of I_i has pre-color 0, and the first element of I_{i+n-k} also have pre-color 0. Then by the definition of c, $4i = kx+y$ for some $x \equiv 1 \pmod 4$ and $y \leq k-1$, and that $4(i+(n-k)) = kx'+y'$ for some $x' \equiv 0 \pmod 4$ and $y' \leq k-1$. Now $m-k+1 = 4(i+(n-k)) - 4i + (y'-y) - 2 = k(x'-x) + (y'-y) - 2$, where $x'-x \equiv 3 \pmod 4$. Hence if $m-k+1 = 4(c-1)k+d$ for some integers c, d such that $d \leq 4k$, then $d > 2k$ (because $|y'-y| \leq k-2$, and that $m-k+1$ is not a multiple of k), contrary to the assumption that $d \leq 2k$.

If $d > 2k$, then define a pre-coloring c of $Z^+ \cup \{0\}$ as follows:

For any non-negative integer i, write i in the form $i = kx+y$, where x, y are non-negative integers such that $y \leq k-1$.

1. If $x \equiv 0 \pmod 4$, then let $c(i) = y \pmod 4$.

2. If $x \equiv 1 \pmod 4$, then let $c(i) = 3+y \pmod 4$.

3. If $x \equiv 2 \pmod 4$, then let $c(i) = 2+y \pmod 4$.

4. If $x \equiv 3 \pmod 4$, then let $c(i) = 1+y \pmod 4$.

We show that for any integer j, the vertices $j, j+k, j+2k, j+3k$ have distinct pre-colors, and that a set of any $m-k+1$ consecutive integers contains at most $n-k$

integers of pre-color 0. The proof is similar to the proof for the case that $d \le 2k$, and we omit the details.

Theorem 11 *Suppose* $m + 3k + 1 \equiv 3 \pmod 4$ *and that* $m + 3k + 1 = 4kc + d$*, where* $d \le 4k$*. If* $d \le k$ *or* $d \ge 3k$*, then* $\chi(Z, D_{m,k,3}) = \lceil (m + 3k + 1)/4 \rceil$*.*

The proof of Theorem 11 is parallel to the proof of Theorem 10. We omit the details.

The case that $s = 3$, $m + 3k + 1 \equiv 3 \pmod 4$ and that $m + 3k + 1 = 4kc + d$ for some $k < d < 3k$ remains unsolved.

Remark Since the circulation of this manuscript, the problem of determining the chromatic number and circular chromatic number of the distance graphs $G(Z, D_{m,k,s})$ has been completely solved in [10] and [19].

References

[1] G. J. Chang, L. Huang and X. Zhu, *Circular chromatic numbers of distance graphs*, Discrete Mathematics, (revised).

[2] G. J. Chang, D. D.-F. Liu and X. Zhu, *Distance graphs and T-coloring*, manuscript (1997).

[3] J. J. Chen, G. J. Chang and K. C. Huang, *Integral distance graphs*, J. Graph Theory, (1997).

[4] W. Deuber and X. Zhu, *The chromatic number of distance graphs*, Disc. Math. 165/166 (1997) 195-204.

[5] R. B. Eggleton, P. Erdös and D. K. Skilton, *Coloring the real line*, J. Comb. Theory, Series B 39 (1985) 86-100.

[6] R. B. Eggleton, P. Erdös and D. K. Skilton, *Research problem 77*, Disc. Math. 58 (1986) 323.

[7] R. B. Eggleton, P. Erdös and D. K. Skilton, *Update information on research problem 77*, Disc. Math. 69 (1988) 105.

[8] R. B. Eggleton, P. Erdös and D. K. Skilton, *Coloring prime distance graphs*, Graphs and Comb. 32 (1990) 17-32.

[9] J. R. Griggs and D. D.-F. Liu, *The channel assignment problem for mutually adjacent sites*, J. Comb. Theory, Series A 68 (1994) 169-183.

[10] L. Huang and G. Chang, *The circular chromatic number of distance graphs with missing multiple in the distance set*, manuscript, 1998.

[11] A. Kemnitz and H. Kolberg, *Coloring of integer distance graphs*, Tech. Report 96/07, Institute für Mathematik, Technische Universität Braunschweig (1996).

[12] D. D.-F. Liu, *T-coloring and chromatic number of distance graphs*, manuscript (1997).

[13] D. D.-F. Liu and X. Zhu, *Distance graphs with missing multiples in the distance sets*, manuscript, 1997.

[14] M. Voigt, *Coloring of distance graphs*, Ars Comb. (to appear).

[15] M. Voigt, ...,

[16] M. Voigt and H. Walther, *Chromatic number of prime distance graphs*, Disc. Appl. Math. 51 (1994) 197-209.

[17] X. Zhu, *Distance graphs on the real line*, manuscript (1996).

[18] X. Zhu, *Pattern-periodic coloring of distance graphs*, J. Combin. Th. (B), 73(1998), 195-206.

[19] X. Zhu, *The circular chromatic number of a class of distance graphs*, manuscript, 1998.

Omitted details for the proof of Theorem 10:

Assume to the contrary that $c(j+uk)=c(j+vk)$ for some integers j,u,v such that $0\le u<v\le 3$. Assume that $j+uk=kx+y$ and $j+vk=kx'+y'$. Then $(v-u)k=(x'-x)k+(y'-y)$. It follows that k divides $y'-y$, which implies that $y'-y=0$ (because $|y'-y|\le k-1$). As $c(j+uk)=c(j+vk)$, it follows that $x'\equiv x \pmod 4$. This implies that $v-u\equiv 0 \pmod 4$, contrary to the assumption that $0\le u<v\le 3$.

Next, let X be a set of $m-k+1$ consecutive integers. We divide the set $Z^+\cup\{0\}$ into segments I_j, where $I_j=\{4j,4j+1,4j+2,4j+3\}$. Each of the segments I_j contains exactly one integer of per-color 0. Since $\lceil (m-k+1)/4\rceil = n$, we know that X either intersects with $n-k$ of the segments I_j, or $n-k+1$ of the segments I_j. If X intersect with $n-k$ of the segments I_j, then of course X contains at most $n-k$ integers of pre-color 0. Assume that X intersect with $n-k+1$ segments I_j. Since $m+3k+1\equiv 2 \pmod 4$, it follows that $m-k+1\equiv 2 \pmod 4$. Thus $m-k+1=4(n-k)-2$. Let $I_i, I_{i+1},\cdots,I_{i+n-k}$ be the segments that intersect X. Then $|X\cap I_i|=|X\cap I_{i+n-k}|=1$. If the last element of I_i does not have pre-color 0, or the first element of I_{i+n-k} does not have pre-color 0, then X contains at most $n-k$ elements of pre-color 0. Assume that the last element of I_i has pre-color 0, and the first element of I_{i+n-k} also have pre-color 0. Then by the definition of c, $4i=kx+y$ for some $x\equiv 3 \pmod 4$ and $y\le k-1$, and that $4(i+(n-k))=kx'+y'$ for some $x'\equiv 0 \pmod 4$ and $y'\le k-1$. Now $m-k+1=4(i+(n-k))-4i+(y'-y)-2=k(x'-x)+(y'-y)-2$, where $x'-x\equiv 1 \pmod 4$. Hence if $m-k+1=4(c-1)k+d$ for some integers c,d such that $d\le 4k$, then $d<2k$ (because $|y'-y|\le k-2$, and that $m-k+1$ is not a multiple of k), contrary to the assumption that $d>2k$.

Proof of Theorem 11:

Let $n=\lceil (m+3k+1)/4\rceil$. Suppose $m+3k+1=4ck+d$, where c,d are integers such that $0<d<4k$ (since $b=0$, we know that $d\neq 0$). If $d\le k$, then define a pre-coloring c of $Z^+ \cup \{0\}$ as follows:

For any non-negative integer i, write i in the form $i = kx + y$, where x, y are non-negative integers such that $y \leq k - 1$. Let $c(i) = x + y \pmod 4$. This is the same pre-coloring as defined in the proof of Theorem 10. Hence for any integer j, the vertices $j, j + k, j + 2k, j + 3k$ have distinct pre-colors. We need to show that any $m - k + 1$ consecutive integers contains at most $n - k$ integers of pre-color 0.

Let X be a set of $m-k+1$ consecutive integers. We divide the set $Z^+ \cup \{0\}$ into segments I_j, where $I_j = \{4j, 4j+1, 4j+2, 4j+3\}$. Each of the segments I_j contains exactly one integer of per-color 0. Since $\lceil (m-k+1)/4 \rceil = n$, we know that X either intersects with $n-k$ of the segments I_j, or $n-k+1$ of the segments I_j. If X intersect with $n - k$ of the segments I_j, then of course X contains at most $n - k$ integers of pre-color 0. Assume that X intersect with $n-k+1$ segments I_j. Since $m+3k+1 \equiv 3 \pmod 4$, it follows that $m-k+1 \equiv 3 \pmod 4$. Thus $m-k+1 = 4(n-k)-1$. Let $I_i, I_{i+1}, \cdots, I_{i+n-k}$ be the segments that intersect X. Then $|X \cap I_i| + |X \cap I_{i+n-k}| = 3$.

First we consider the case that $|X \cap I_i| = 2$ and $|X \cap I_{i+n-k}| + 1$.

If the last two elements of I_i does not have pre-color 0, or the first element of I_{i+n-k} does not have pre-color 0, then X contains at most $n-k$ elements of pre-color 0. Assume that one of the last two elements of I_i has pre-color 0, and the first element of I_{i+n-k} also have pre-color 0. Then by the definition of c, $4i = kx+y$ for some $x \equiv 1$ or2 $\pmod 4$ and $y \leq k-1$, and that $4(i+(n-k)) = kx' + y'$ for some $x' \equiv 0 \pmod 4$ and $y' \leq k-1$. Now $m-k+1 = 4(i+(n-k)) - 4i + (y'-y) - 2 = k(x'-x) + (y'-y) - 2$, where $x' - x \equiv 3$ or $2 \pmod 4$. Hence if $m-k+1 = 4(c-1)k+d$ for some integers c, d such that $d \leq 4k$, then $d > k$ (because $|y'-y| \leq k-2$, and that $m-k+1$ is not a multiple of k), contrary to the assumption that $d \leq k$.

Next we consider the case that $|X \cap I_i| = 1$ and $|X \cap I_{i+n-k}| = 2$.

If the last element of I_i does not have pre-color 0, or the first two elements of I_{i+n-k} do not have pre-color 0, then X contains at most $n-k$ elements of pre-color 0.

Assume that the last two element of I_i has pre-color 0, and one of the first two elements of I_{i+n-k} has pre-color 0. Then by the definition of c, $4i = kx + y$ for some $x \equiv 1 \pmod 4$ and $y \leq k-1$, and that $4(i+(n-k)) = kx'+y'$ for some $x' \equiv 0$ or $3 \pmod 4$ and $y' \leq k-1$. Now $m-k+1 = 4(i+(n-k))-4i+(y'-y)-2 = k(x'-x)+(y'-y)-2$, where $x'-x \equiv 3$ or $2 \pmod 4$. Hence if $m-k+1 = 4(c-1)k+d$ for some integers c, d such that $d \leq 4k$, then $d > k$ (because $|y'-y| \leq k-2$, and that $m-k+1$ is not a multiple of k), contrary to the assumption that $d \leq k$.

If $d \geq 3k$, then define a pre-coloring c of $Z^+ \cup \{0\}$ as follows:

For any non-negative integer i, write i in the form $i = kx + y$, where x, y are non-negative integers such that $y \leq k-1$.

1. If $x \equiv 0 \pmod 4$, then let $c(i) = y \pmod 4$.

2. If $x \equiv 1 \pmod 4$, then let $c(i) = 3 + y \pmod 4$.

3. If $x \equiv 2 \pmod 4$, then let $c(i) = 2 + y \pmod 4$.

4. If $x \equiv 3 \pmod 4$, then let $c(i) = 1 + y \pmod 4$.

Again this is one of the pre-coloring defined in the proof of Theorem 10, where it is proved that that for any integer j, the vertices $j, j+k, j+2k, j+3k$ have distinct pre-colors. We need to show that any set of $m-k+1$ consecutive integers contains at most $n-k$ integers of pre-color 0.

Let X be a set of $m-k+1$ consecutive integers. We divide the set $Z^+ \cup \{0\}$ into segments I_j, where $I_j = \{4j, 4j+1, 4j+2, 4j+3\}$. Each of the segments I_j contains exactly one integer of per-color 0. Since $\lceil (m-k+1)/4 \rceil = n$, we know that X either intersects with $n-k$ of the segments I_j, or $n-k+1$ of the segments I_j. If

X intersect with $n-k$ of the segments I_j, then of course X contains at most $n-k$ integers of pre-color 0.

Assume that X intersect with $n-k+1$ segments I_j. Since $m+3k+1 \equiv 3 \pmod 4$, it follows that $m-k+1 \equiv 3 \pmod 4$. Thus $m-k+1 = 4(n-k)-1$. Let $I_i, I_{i+1}, \cdots, I_{i+n-k}$ be the segments that intersect X. Then $|X \cap I_i| + |X \cap I_{i+n-k}| = 3$.

First we consider the case that $|X \cap I_i| = 2$ and $|X \cap I_{i+n-k}| + 1$.

If the last two elements of I_i does not have pre-color 0, or the first element of I_{i+n-k} does not have pre-color 0, then X contains at most $n-k$ elements of pre-color 0. Assume that one of the last two elements of I_i has pre-color 0, and the first element of I_{i+n-k} also have pre-color 0. Then by the definition of c, $4i = kx+y$ for some $x \equiv 2$ or3 $\pmod 4$ and $y \leq k-1$, and that $4(i+(n-k)) = kx'+y'$ for some $x' \equiv 0 \pmod 4$ and $y' \leq k-1$. Now $m-k+1 = 4(i+(n-k))-4i+(y'-y)-2 = k(x'-x)+(y'-y)-2$, where $x'-x \equiv 1$ or $2 \pmod 4$. Hence if $m-k+1 = 4(c-1)k+d$ for some integers c, d such that $d \leq 4k$, then $d < 3k$ (because $|y'-y| \leq k-2$, and that $m-k+1$ is not a multiple of k), contrary to the assumption that $d \geq 3k$.

Next we consider the case that $|X \cap I_i| = 1$ and $|X \cap I_{i+n-k}| = 2$.

If the last element of I_i does not have pre-color 0, or the first two elements of I_{i+n-k} do not have pre-color 0, then X contains at most $n-k$ elements of pre-color 0. Assume that the last two element of I_i has pre-color 0, and one of the first two elements of I_{i+n-k} has pre-color 0. Then by the definition of c, $4i = kx+y$ for some $x \equiv 3 \pmod 4$ and $y \leq k-1$, and that $4(i+(n-k)) = kx'+y'$ for some $x' \equiv 0$ or $1 \pmod 4$ and $y' \leq k-1$. Now $m-k+1 = 4(i+(n-k))-4i+(y'-y)-2 = k(x'-x)+(y'-y)-2$, where $x'-x \equiv 3$ or $2 \pmod 4$. Hence if $m-k+1 = 4(c-1)k+d$ for some integers c, d such that $d \leq 4k$, then $d < 3k$ (because $|y'-y| \leq k-2$, and that $m-k+1$ is not a multiple of k), contrary to the assumption that $d \geq 3k$.