

Fractional Chromatic Number of Distance Graphs Generated by Two-Interval Sets

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Abstract

Let D be a set of positive integers. The distance graph generated by D , denoted by $G(Z, D)$, has the set Z of all integers as the vertex

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set, and two vertices x and y are adjacent whenever $|x - y| \in D$. For integers $1 < a \leq b < m - 1$, denote $D_{a,b,m} = \{1, 2, \dots, a - 1\} \cup \{b + 1, b + 2, \dots, m - 1\}$. For the special case $a = b$, the chromatic number for the family of distance graphs $G(Z, D_{a,a,m})$ was first studied by Eggleton, Erdős and Skilton [5] and was completely solved by Chang, Liu and Zhu [3]. For the general case $a \leq b$, the fractional chromatic number for $G(Z, D_{a,b,m})$ was studied by Lam and Lin [14] and by Wu and Lin [23], in which partial results for special values of a, b, m were obtained. In this article, we completely settle this problem for all possible values of a, b, m .

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1 Introduction

Let D be a set of positive integers. The *distance graph* generated by D , denoted by $G(Z, D)$, has the set Z of all integers as the vertex set, and two vertices x and y are adjacent whenever $|x - y| \in D$. Initiated by Eggleton, Erdős and Skilton [5], the study of distance graphs has attracted considerable attention ([2-8, 11-18, 20-25]).

A *fractional coloring* of a graph G is a mapping f which assigns to each independent set I of G a non-negative weight $f(I)$ such that for each vertex x , $\sum_{x \in I} f(I) \geq 1$. The *fractional chromatic number* $\chi_f(G)$ of G is the least total weight of a fractional coloring for G .

The problem of determining the fractional chromatic number for distance graphs has been studied in different research areas under different names. Firstly, it is equivalent to a sequence density problem in number theory. For a set D of positive integers, a sequence S of non-negative integers is called a *D -sequence* if $a - b \notin D$ for any $a, b \in S$. Let $S(n)$ denote $|\{0, 1, \dots, n-1\} \cap S|$. The upper density and the lower density of S are defined, respectively, by

$$\bar{\delta}(S) = \overline{\lim}_{n \rightarrow \infty} \frac{S(n)}{n}, \quad \underline{\delta}(S) = \underline{\lim}_{n \rightarrow \infty} \frac{S(n)}{n}.$$

We say S has density $\delta(S)$ if $\bar{\delta}(S) = \underline{\delta}(S) = \delta(S)$. The parameter of interest is the maximum density of a D -sequence, defined by

$$\mu(D) = \sup\{\delta(S) : S \text{ is a } D\text{-sequence}\}.$$

The problem of determining or estimating $\mu(D)$ was initially posed by Motzkin in an unpublished problem collection (cf. [1]), and has been studied in [1, 10, 19, 9, 18]. Note that, S is a D -sequence if and only if S (as a set of integers) is an independent set of $G(Z, D)$. It was proved by Chang et al. [3] that for any finite set D ,

$$\mu(D) = \frac{1}{\chi_f(G(Z, D))}.$$

Secondly, the fractional chromatic number of a distance graph is equivalent to an asymptotic problem in T -coloring. The T -coloring problem was motivated by the channel assignment problem introduced by Hale [10], in which an integer broadcast channel is assigned to each of a given set of stations or transmitters so that interference among nearby stations is avoided. Interference is modeled by a set of non-negative integers T containing 0 as the forbidden channel separations. By using a graph G to represent the broadcast network, a valid channel assignment is defined as a T -coloring for G , which is a mapping $f : V(G) \rightarrow Z$ such that $|f(x) - f(y)| \notin T$ whenever $xy \in E$. The *span* of a T -coloring f is the difference between the largest and the smallest numbers in $f(V)$, i.e., $\max\{|f(u) - f(v)| : u, v \in V\}$. Given T and G , the T -span of G , denoted by $\text{sp}_T(G)$, is the minimum span among all T -colorings of G . As for any graph G , $\text{sp}_T(G) \leq \text{sp}_T(K_{\chi(G)})$, it is useful to estimate $\text{sp}_T(K_n)$. Let σ_n denote $\text{sp}_T(K_n)$. Griggs and Liu [9] proved that for any set T the *asymptotic T -coloring ratio*

$$R(T) := \lim_{n \rightarrow \infty} \frac{\sigma_n}{n}$$

exists and is a rational number. It was proved in [3] that for any T , by letting $D = T - \{0\}$, we have $R(T) = \chi_f(G(Z, D))$.

Partially due to its rich connections to other problems, the fractional chromatic number for various classes of distance graphs has been studied in the literature (cf. [2, 3, 17, 18, 23, 14, 24, 25]). If D is a singleton, trivially

$\chi_f(G(Z, D)) = 2$. If $D = \{a, b\}$ and $\gcd(a, b) = 1$, it is known [1] that $\chi_f(G(Z, D)) = \frac{a+b}{\lfloor (a+b)/2 \rfloor}$. For $|D| \geq 3$, the exact values of $\chi_f(G(Z, D))$ are known only for some special sets D . For $D = \{a, b, a+b\}$, upper and lower bounds for $\chi_f(G(Z, D))$ were obtained by Rabinowitz and Proulx [19]. Let $\chi(G)$ and $\omega(G)$ denote, respectively, the chromatic number and the clique number of G . It is easy to see that $\omega(G) \leq \chi_f(G) \leq \chi(G)$ holds for any graph G , and $\chi(G(Z, D)) \leq |D| + 1$ ([4, 20]) if D is finite. In [18], the sets D with $\omega(G(Z, D)) \geq |D|$ were characterized and the value of $\chi_f(G(Z, D))$ for most of this class of graphs, including $D = \{a, b, a+b\}$, was determined.

For any two integers $a \leq b$, let $[a, b]$ denote the interval of consecutive integers $\{a, a+1, \dots, b\}$. It is known that if $D = [a, b]$, then $\chi_f(G(Z, D)) = (a+b)/a$ [9, 2]. For the sets D of the form $D = [1, m] - \{k, 2k, \dots, sk\}$ for integers m, k and s , the values of $\chi_f(G(Z, D))$ were determined in [17].

For $1 < a \leq b < m - 1$, let $D_{a,b,m}$ denote the two-interval set

$$D_{a,b,m} = [1, a-1] \cup [b+1, m-1].$$

Note, if $a = b$, then $D_{a,a,m} = [1, m-1] - \{a\}$. The chromatic number for $G(Z, D_{a,a,m})$ was first studied by Eggleton, Erdős and Skilton [5] and the problem was completely solved in [3]. For the general case $a \leq b$, both the fractional chromatic number and the chromatic number for $G(Z, D_{a,b,m})$ were studied by Wu and Lin [23], and by Lam and Lin [14]. Some partial results were obtained. In this article, we completely determine the fractional chromatic number of $G(Z, D_{a,b,m})$ for all $1 < a \leq b < m - 1$.

2 The main result and some preliminaries

For some special cases, the values of $\chi_f(G(Z, D_{a,b,m}))$ for the two-interval set $D_{a,b,m} = [1, a-1] \cup [b+1, m-1]$ were solved in [23] and [14]. If $b < 2a$, then $\chi_f(G(Z, D_{a,b,m}))$ is determined in [23]. Let $\Delta = m - b$. If $\Delta \leq a$ or $\Delta \geq 2a$, then $\chi_f(G(Z, D_{a,b,m}))$ is determined in [14]. Some other special cases (which cannot be easily described) are discussed in [14].

The main result of this article is the following which completely determines the value of $\chi_f(G(Z, D_{a,b,m}))$ for all $1 < a \leq b < m - 1$.

Theorem 1 For integers $1 < a \leq b < m - 1$. Suppose $G = G(Z, D_{a,b,m})$ where $D_{a,b,m} = [1, a - 1] \cup [b + 1, m - 1]$. Let $\Delta = m - b$, $s = \lfloor b/a \rfloor$, and $q = \lfloor m/\Delta \rfloor$.

- If $\Delta \geq 2a$, then $\chi_f(G) = (sa + m)/(s + 1)$.
- If $\Delta \leq a$, then $\chi_f(G) = \max\{a, m/(s + 1)\}$.
- If $a < \Delta < 2a$, then

$$\chi_f(G) = \begin{cases} \frac{sa+m}{s+1}, & \text{if } 2qa \leq m < a + q\Delta \text{ or} \\ & \text{if } m \geq (2q + 1)a; \\ \frac{m}{q}, & \text{if } m < \min\{q\Delta + a, 2qa\}; \\ \frac{(2q-1)m+a}{2q^2}, & \text{if } q\Delta + a \leq m < (2q + 1)a. \end{cases}$$

The cases for $\Delta \geq 2a$ and $\Delta \leq a$ were solved in [14]. However, for completeness, we include these cases in the statement and give a short proof for them.

Recall the result in [3] mentioned in Section 1, the fractional chromatic number of G is equal to the reciprocal of $\mu(D_{a,b,m})$, which is the maximum density of a $D_{a,b,m}$ -sequence. Let $I = \{x_1, x_2, \dots\}$ be a $D_{a,b,m}$ -sequence where $x_i < x_{i+1}$. Let $\delta_i = x_{i+1} - x_i$. The sequence $\Omega = (\delta_1, \delta_2, \dots)$ is called the *gap sequence* of I . In the following, we call a sequence $(\delta_1, \delta_2, \dots)$ a *D-gap sequence* if it is the gap sequence of a D -sequence. Observe that a sequence $(\delta_1, \delta_2, \dots)$ is a D -gap sequence if and only if for any $j \leq j'$, $\sum_{i=j}^{j'} \delta_i \notin D$. In particular, the following observation is frequently used, usually implicitly, in our proofs.

- A sequence $(\delta_1, \delta_2, \dots)$ is a $D_{a,b,m}$ -gap sequence if and only if
 - (1) $\delta_i \geq a$ for each i ; and
 - (2) for any $j \leq j'$, either $\sum_{i=j}^{j'} \delta_i \leq b$ or $\sum_{i=j}^{j'} \delta_i \geq m$.

By definition,

$$\mu(D_{a,b,m}) = \max \lim_{n \rightarrow \infty} \frac{|I \cap [0, n - 1]|}{n},$$

where the maximum is taken over all $D_{a,b,m}$ -sequences I . Hence

$$\chi_f(G) = \frac{1}{\mu(D_{a,b,m})} = \min \lim_{n \rightarrow \infty} \frac{n}{|I \cap [0, n-1]|} = \min \lim_{k \rightarrow \infty} \sum_{i=1}^k \frac{\delta_i}{k}.$$

Again, the minimum is taken over all $D_{a,b,m}$ -sequences I with gap sequence $(\delta_1, \delta_2, \dots)$.

For an interval of integers $[a, b]$, we call its cardinality $|[a, b]|$ the *length* of $[a, b]$. Given a $D_{a,b,m}$ -gap sequence $Y = (\delta_1, \delta_2, \delta_3, \dots)$, the *average gap length* of Y is $\lim_{k \rightarrow \infty} \sum_{i=1}^k \frac{\delta_i}{k}$ (if exists). Thus to determine the fractional chromatic number of $G(Z, D_{a,b,m})$, it amounts to determine the minimum average gap length of a $D_{a,b,m}$ -gap sequence. Usually, the gap sequences we concern are periodic. For a periodic gap sequence, it suffices to present one period of the sequence. We shall denote by $\langle y_1, y_2, \dots, y_k \rangle$ the infinite periodic sequence with period k . That is, $\langle y_1, y_2, \dots, y_k \rangle = (y_1, y_2, \dots, y_j, \dots)$ where for $j > k$, $y_j = y_{j-k}$. For convenience, we denote by $p \otimes t$, for any integers p and t , the p repetitions of t . For example, $\langle 3 \otimes 5, 2 \otimes 7 \rangle$ is the periodic sequence $\langle 5, 5, 5, 7, 7 \rangle = (5, 5, 5, 7, 7, 5, 5, 5, 7, 7, \dots)$.

We now give a short proof for the cases $\Delta \leq a$ and $\Delta \geq 2a$. As each gap of a $D_{a,b,m}$ -gap sequence is at least a , we have $\chi_f(G) \geq a$. If $m \leq (s+1)a$, then $\langle a \rangle$ is a $D_{a,b,m}$ -gap sequence with average gap length a . Hence $\chi_f(G) = a$. Assume $m > (s+1)a$ and $\Delta \leq a$. Then the sequence $\langle s \otimes a, m - sa \rangle$ is a $D_{a,b,m}$ -gap sequence of average gap length $m/(s+1)$. So $\chi_f(G) \leq m/(s+1)$. On the other hand, for any $D_{a,b,m}$ -gap sequence $(\delta_1, \delta_2, \dots)$, since $\sum_{i=1}^{s+1} \delta_i \geq (s+1)a \geq b+1$, we must have $\sum_{i=1}^{s+1} \delta_i \geq m$. Hence the average gap length is at least $m/(s+1)$. So $\chi_f(G) = m/(s+1)$.

Assume $\Delta \geq 2a$. It is easy to verify that the sequence $\langle s \otimes a, m \rangle$ is a $D_{a,b,m}$ -gap sequence with average gap length $(m+sa)/(s+1)$. Hence $\chi_f(G) \leq (m+sa)/(s+1)$. On the other hand, if $\chi_f(G) = 1/\mu(D_{a,b,m}) < (m+sa)/(s+1)$, then there is a $D_{a,b,m}$ -sequence I with $|[0, sa+m-1] \cap I| \geq s+2$. Without loss of generality, we may assume $0 \in I$. Let $I' = \{i : i \in I, i \leq b\} \cup \{i - m + a : i \in I, i \geq m - a\}$. It is easy to verify that $|I| = |I'|$, $I' \subseteq [0, (s+1)a - 1]$ and for any $x, y \in I'$, $|x - y| \geq a$. This is in contrary to the assumption that $|I| \geq s+2$. Therefore we have $\chi_f(G) = (m+sa)/(s+1)$.

3 Proof of the upper bound

In the rest of the paper, we assume that $a < \Delta < 2a$, and let

$$\tau(D_{a,b,m}) = \begin{cases} \frac{sa+m}{s+1}, & \text{if } 2qa \leq m < a + q\Delta \text{ or} \\ & \text{if } m \geq (2q+1)a; \\ \frac{m}{q}, & \text{if } m < \min\{q\Delta + a, 2qa\}; \\ \frac{(2q-1)m+a}{2q^2}, & \text{if } q\Delta + a \leq m < (2q+1)a. \end{cases}$$

In this section, we prove that $\chi_f(G) \leq \tau(D_{a,b,m})$. This amounts to present a $D_{a,b,m}$ -gap sequence whose average gap length is at most $\tau(D_{a,b,m})$.

Lemma 2 *Suppose $G = G(Z, D_{a,b,m})$. Then $\chi_f(G) \leq \tau(D_{a,b,m})$.*

Proof. First note that the following are two $D_{a,b,m}$ -gap sequences:

$$\langle s \otimes a, m \rangle \quad \text{and} \quad \langle (q-1) \otimes \Delta, m - ((q-1)\Delta) \rangle,$$

where the average gap lengths, respectively, are $(sa+m)/(s+1)$ and m/q . This proves the result for all the cases, except the very last one.

For the last case, $q\Delta + a \leq m < (2q+1)a$, the gap sequence is more complicated. We shall define some special sequences, then combine them to form the required periodic sequence.

For $i = 1, 2, \dots, q-1$, let Y_i and Y'_i and Z be finite sequences of integers defined as follows:

$$\begin{aligned} Y_i &= (i \otimes \Delta, a, (q-1-i) \otimes \Delta, m - (a + (q-1)\Delta)) \\ Y'_i &= ((i-1) \otimes \Delta, \Delta + a, (q-1-i) \otimes \Delta, m - (a + (q-1)\Delta)) \\ Z &= (a) \end{aligned}$$

Let

$$Y'_q = ((q-1) \otimes \Delta, m - (q-1)\Delta).$$

For finite sequences $A = (a_1, a_2, \dots, a_s)$ and $B = (b_1, b_2, \dots, b_t)$, the *concatenation* of A and B , denoted by AB , is the sequence

$$AB = (a_1, a_2, \dots, a_s, b_1, b_2, \dots, b_t).$$

The concatenation of sequences is associative. Thus for finite sequences A_1, A_2, \dots, A_t , the sequence $A_1 A_2 \dots A_t$ is well-defined. Define the periodic gap sequence as

$$\langle Y'_q Y_{q-1} Y'_{q-1} Y_{q-2} Y'_{q-2} \dots Y_1 Y'_1 Z \rangle.$$

Now we show that this sequence is indeed a $D_{a,b,m}$ -gap sequence. Since

$$m - (a + (q-1)\Delta) = m - q\Delta - a + \Delta \geq \Delta > a,$$

each entry of the sequence is at least a . It remains to show that the sum of any number of consecutive entries of the sequence is either at most b or at least m . Observe that the sum of the entries in each Y_i or Y'_i is equal to m . Consider the sum of any t consecutive entries in the sequence. Straightforward calculation shows that if $t \geq q+1$, then the sum is at least m ; if $t \leq q-1$, then the sum is at most b ; if $t = q$, then the sum is either equal to m or at most b . (Here we use the condition that $(q-1)\Delta + a \leq (q-1)\Delta + m - q\Delta = b$.) Thus the sequence defined above is a $D_{a,b,m}$ -gap sequence.

Straightforward calculation shows that this gap sequence has average gap length $\frac{(2q-1)m+a}{2q^2}$. \blacksquare

4 Proof of the lower bound

To complete the proof of Theorem 1, it remains to show that $\chi_f(G) \geq \tau(D_{a,b,m})$. To this end, we need some more definitions.

In the following, we assume that $I = \{x_1, x_2, \dots\}$ is a $D_{a,b,m}$ -sequence, i.e., an independent set in $G = G(Z, D_{a,b,m})$. We shall prove that the gap sequence of I has average gap length at least $\tau(D_{a,b,m})$.

Let

$$L = \{i : x_{i+1} - x_i \geq \Delta\}.$$

For each $x_i \in I$, we associate it with a set X_i of integers as follows.

$$X_i = \begin{cases} [x_i, x_i + \Delta - 1], & \text{if } i \in L; \\ [x_i, x_i + a - 1] \cup [x_i + m, x_i + m + a - 1], & \text{if } i \notin L. \end{cases}$$

Lemma 3 *If $i \neq j$, then $X_i \cap X_j = \emptyset$.*

Proof. Assume $i < j$. If $i \in L$, then $X_i = [x_i, x_i + \Delta - 1]$ and by definition, $x_j \geq x_i + \Delta$. As $t \in X_j$ implies that $t \geq x_j$, we have $X_i \cap X_j = \emptyset$. Assume $i \notin L$. Then $X_i = [x_i, x_i + a - 1] \cup [x_i + m, x_i + m + a - 1]$. As $x_j \geq x_i + a$, we know that $X_j \cap [x_i, x_i + a - 1] = \emptyset$. Assume $X_j \cap [x_i + m, x_i + m + a - 1] \neq \emptyset$. Then by the definition of X_j , we have either $x_j \in [x_i + m - \Delta + 1, x_i + m - 1]$ or $x_j \in [x_i + m, x_i + m + a - 1]$. The former case implies $b + 1 \leq x_j - x_i \leq m - 1$; and the latter case implies $b + 1 \leq x_j - x_{i+1} \leq m - 1$ (since $i \notin L$, we have $a \leq x_{i+1} - x_i < \Delta$). For both cases, it contradicts the assumption that I is a $D_{a,b,m}$ -sequence. ■

We call intervals of the form $[x_i + m, x_i + m + a - 1]$ for $i \notin L$ *Type-B I-intervals*. Intervals of the form $[x_i, x_i + \Delta - 1]$ for $i \in L$, and intervals of the form $[x_i, x_i + a - 1]$ for $i \notin L$ are called *Type-A I-intervals*. Both Type-A and Type-B *I-intervals* are referred as *I-intervals*. The length of an *I-interval* is either Δ or a , and they are called, respectively, *long* or *short I-intervals*.

Lemma 4 *If $T = [x_i, x_i + a - 1]$ is a short Type-A I-interval, then the first I-interval $T' = [u, v]$ with $u \geq x_i + a$ is Type-A.*

Proof. Assume to the contrary that $T' = [u, v] = [x_j + m, x_j + m + a - 1]$ for some j . As $x_j + m \geq x_i + a$, which implies $x_i - x_j \leq m - a$, we have $x_i - x_j \leq b$. So $x_j + m \geq x_i + \Delta$. In addition, since T is a short Type-A *I-interval*, $x_{i+1} < x_i + \Delta$. Hence, $x_{i+1} < x_j + m$, contradicting the choice of T' . ■

Lemma 5 *There are at most s short consecutive I-intervals that are of the same type.*

Proof. First we show that there are at most s short consecutive Type-A *I-intervals*. Assume $T_1 = [u_1, v_1], T_2 = [u_2, v_2], \dots, T_j = [u_j, v_j]$ are consecutive *I-intervals* and T_1, T_2, \dots, T_{j-1} are short and Type-A. By Lemma 4, T_j is also Type-A. So $u_1, u_2, \dots, u_j \in I$. We prove by induction on i that $u_i \leq u_1 + b$ for $i = 1, 2, \dots, j$. It is trivial for $i = 1$. Assume $i < j$ and $u_i \leq u_1 + b$. By definition of *I-intervals*, $u_{i+1} - u_i < \Delta$. Hence $u_{i+1} < u_i + \Delta \leq u_1 + m$. As $u_1, u_{i+1} \in I$, it follows that $u_{i+1} \leq u_1 + b$.

Because $s = \lfloor b/a \rfloor$ and $|T_i| \geq a$, we conclude that there are at most s consecutive short Type-A I -intervals. By definition, consecutive Type-B I -intervals correspond to consecutive short Type-A I -intervals. So the result follows. \blacksquare

Suppose T is an I -interval. Define the *weight* of T by

$$w(T) = \begin{cases} 1, & \text{if } T \text{ is long;} \\ 1/2, & \text{if } T \text{ is short.} \end{cases}$$

For any interval of integers $[u, v]$, let

$$w([u, v]) = \sum_{T \text{ is an } I\text{-interval and } T \subseteq [u, v]} w(T).$$

By definition, every integer in I creates either a long interval of weight 1 or two short intervals of weight 1/2 each. By Lemma 3, all these intervals are disjoint, and by definition the two short intervals induced by an integer in I are of distance $m - a$ apart. Hence, by Lemma 5, for any n ,

$$w([0, n-1]) - s/2 \leq |I \cap [0, n-1]| \leq w([0, n-1]) + s/2.$$

Thus to prove that $\lim_{n \rightarrow \infty} \frac{n}{|I \cap [0, n-1]|} \geq \tau(D_{a,b,m})$, it suffices to show that $\lim_{n \rightarrow \infty} \frac{n}{w([0, n-1])} \geq \tau(D_{a,b,m})$.

An interval $W = [x, y]$ of integers is called *neat* if every I -interval is either contained in W or disjoint from W . Suppose W is a neat interval. We define the *X-ratio* of W to be

$$r(W) = \frac{|W|}{w(W)}.$$

To prove that $\lim_{n \rightarrow \infty} \frac{n}{|I \cap [0, n-1]|} \geq \tau(D_{a,b,m})$, it suffices to find integers $a_1 < a_2 < \dots$ such that for any i , $R_i = [a_i, a_{i+1} - 1]$ is a neat interval and $r(R_i) \geq \tau(D_{a,b,m})$.

We say an integer p has property (*) if

(*) *for the first Type-B I -interval $[u, u + a - 1]$ with $u \geq p$, we have $u \geq p + \Delta$.*

Lemma 6 *Each $x_i \in I$ has property (*). Moreover, if $i \in L$, then $x_i + m$ also has property (*) and $[x_i, x_i + m - 1]$ is neat.*

Proof. If $i \notin L$, by Lemma 4, x_i has property (*). Assume $i \in L$. By definition, x_i has property (*). Suppose $x_i + m$ does not have property (*). Then, there exists some u with $x_i + m \leq u < x_i + m + \Delta$ such that $[u, u + a - 1]$ is a Type-B I -interval. By definition, $u - m \in I$ and $[u - m, u - m + a - 1]$ is Type-A. This is impossible as $x_i \leq u - m < x_i + \Delta \leq x_{i+1}$ but $i \in L$. Hence, $x_i + m$ has property (*).

Now, assume to the contrary that $[x_i, x_i + m - 1]$ is not neat. Let $T = [u, v]$ be an I -interval that $T \cap [x_i, x_i + m - 1] \neq \emptyset$ and $T \not\subseteq [x_i, x_i + m - 1]$. By definition and as $i \in L$, T must be Type-A. Hence, $u \in I$. Let $u = x_t$ for some t . Then $x_i + m - \Delta + 1 \leq x_t \leq x_i + m - 1$. This implies $b + 1 \leq x_t - x_i \leq m - 1$, a contradiction. ■

To complete the proof of Theorem 1, it suffices to find an infinite sequence of integers $a_1 < a_2 < \dots$ such that the following hold for all i :

- (1) a_i has property (*),
- (2) $R_i = [a_i, a_{i+1} - 1]$ is neat, and
- (3) $r(R_i) \geq \tau(D_{a,b,m})$.

We shall construct such a sequence of integers $a_1 < a_2 < \dots$ inductively. Initially, set $a_1 = x_1$. By Lemma 6, a_1 has property (*). Assume we have determined a_1, a_2, \dots, a_i , where (1 - 3) in the above are satisfied. We shall determine a_{i+1} so that (1 - 3) still hold.

Let $[u, v]$ be the first I -interval with $u \geq a_i$. If $[u, v]$ is Type-B, then as a_i has property (*), $u \geq a_i + \Delta$. Let $a_{i+1} = x_t$, where x_t is the smallest element of I for which $x_t > a_i$. Then all the I -intervals contained in $R_i = [a_i, a_{i+1} - 1]$ are Type-B, and R_i is neat. Assume R_i contains j Type-B I -intervals. By Lemma 5, $j \leq s$. Since $w(R_i) = j/2$ and $|R_i| \geq \Delta + ja$, it follows that

$$r(R_i) \geq \frac{2(\Delta + ja)}{j} \geq 2a + \frac{2\Delta}{s} \geq \tau(D_{a,b,m}).$$

(Observe that $\frac{sa+m}{s+1} < a + \frac{b}{s+1} + \frac{\Delta}{s+1} < 2a + \frac{\Delta}{s+1}$. If $m < 2qa$, then $\frac{m}{q} < 2a$. If $m < (2q+1)a$, then $\frac{(2q-1)m+a}{2q^2} < 2a$.) Moreover, by Lemma 6, $a_{i+1} = x_t$ has property (*). Thus (1 - 3) in the above are satisfied.

In the following, assume $[u, v]$ is Type-A. Then $u \in I$. Let x_h be the first element of I such that $x_h \geq u$ and $h \in L$. Let $a_{i+1} = x_h + m$. By Lemma 6, $R_i = [a_i, a_{i+1} - 1]$ is neat and a_{i+1} has property (*).

It remains to show (3). Assume the interval $[u, x_h - 1]$ contains j I -intervals for some $j \geq 0$. By Lemma 4, all the I -intervals contained in $[u, x_h - 1]$ are Type-A and short.

Since an I -interval of weight 1 has length Δ and an I -interval of weight $1/2$ has length $a > \Delta/2$, so for any interval T of length m , we have

$$w(T) \leq \begin{cases} q, & \text{if } m < q\Delta + a; \\ q + \frac{1}{2}, & \text{if } m \geq q\Delta + a. \end{cases}$$

Because $R_i = [a_i, x_h - 1] \cup [x_h, x_h + m - 1]$, it follows that

$$w(R_i) \leq \begin{cases} q + \frac{j}{2}, & \text{if } m < q\Delta + a; \\ q + \frac{j+1}{2}, & \text{if } m \geq q\Delta + a. \end{cases}$$

Now we consider three cases.

Case 1 $m < q\Delta + a$. As $|R_i| \geq ja + m$, by the above discussion, $r(R_i) \geq \frac{ja+m}{q+j/2}$. Observe that $\frac{ja+m}{q+j/2}$ is a function of j which is increasing if $m \leq 2qa$ and decreasing if $m \geq 2qa$. Hence, as $j \leq s$, we have

- if $m \geq 2qa$, then $r(R_i) \geq \frac{sa+m}{q+\frac{s}{2}} \geq \frac{sa+m}{s+1}$;
- if $m < 2qa$, then $r(R_i) \geq \frac{0a+m}{q+0} \geq \frac{m}{q}$.

Hence, (3) holds.

Case 2 $m \geq (2q+1)a$. Similar to Case 1, we have $r(R_i) \geq \frac{ja+m}{q+(j+1)/2}$. Because $m \geq (2q+1)a$, which implies that $\frac{ja+m}{q+(j+1)/2}$ is a decreasing function of j , we conclude that $r(R_i) \geq \frac{sa+m}{q+(s+1)/2}$. As $\frac{b}{a} = \frac{m}{a} - \frac{\Delta}{a} \geq 2q+1-2$, we

have $s = \lfloor b/a \rfloor \geq 2q - 1$, i.e., $q \leq (s + 1)/2$. Hence $r(R_i) \geq (sa + m)/(s + 1)$, so (3) holds.

Case 3 $a + q\Delta \leq m < (2q + 1)a$. Then $r(R_i) \geq \frac{ja+m}{q+(j+1)/2}$. Because $m < (2q + 1)a$, $\frac{ja+m}{q+(j+1)/2}$ is an increasing function of j . If $j \geq 1$, then $r(R_i) \geq \frac{a+m}{q+1} > \frac{(2q-1)m+a}{2q^2}$. If $j = 0$ and $w(R_i) \leq q$, then $r(R_i) \geq \frac{m}{q} > \frac{(2q-1)m+a}{2q^2}$, and we are done.

Assume $j = 0$ and $w(R_i) = q + 1/2$. Then $u = x_h$ and $r(R_i) \geq m/(q + 1/2)$. As $\frac{m}{q+1/2} < \frac{(2q-1)m+a}{2q^2} = \tau(D_{a,b,m})$, this “ a_{i+1} ” does not satisfy our requirement. We need to find a different a_{i+1} so that (1 - 3) are satisfied. In the following, we re-name the interval $[u, u + m - 1]$ just obtained by R_i^1 . (The correct R_i is not found yet.)

Since $w(R_i^1) = q + 1/2$, R_i^1 contains a short I -interval. Let $p_1 \leq q$ be the total weight of I -intervals preceding the last short I -interval in R_i^1 . As $w(R_i^1) = q + 1/2$ and the first I -interval of R_i^1 is long, we know that $p_1 \geq 1$ is an integer.

Before reaching the correct interval R_i , we may need a (finite) sequence of intervals R_i^j , where R_i^1 is just the first one of them. In the following, we describe the inductive step of finding R_i^j .

Suppose z is an integer, $1 \leq z \leq 2q - 1$, and for $j = 1, 2, \dots, z$, we have obtained $R_i^j = [x_{i_j}, x_{i_j} + m - 1]$ with the following properties:

- $x_{i_j} \in I$ and $i_j \in L$, and for $j \geq 2$, $x_{i_{j-1}} + m \leq x_{i_j} < x_{i_{j-1}} + m + a$.
- $w(R_i^j) = q + 1/2$.

Observe that if $w(R_i^j) = q + 1/2$, the I -intervals in R_i^j must be “tightly packed”. Namely, if a neat sub-interval H of R_i^j has length $\geq \alpha\Delta + \beta a$, where α, β are non-negative integers, then $w(H) \geq \alpha + \beta/2$. For otherwise, $w(R_i^j)$ will be less than $q + 1/2$.

Let p_j be the total weight of I -intervals preceding the last short I -interval in R_i^j . Since $w(R_i^j) = q + 1/2$, R_i^j does contain a short I -interval. Since the first interval of R_i^j is a long interval, we have $p_j \geq 1$.

Let $[x_{s'}, x_{s'} + \Delta - 1]$ be the first long I -interval with $x_{s'} \geq x_{i_z} + m$. If $x_{s'} \geq x_{i_z} + m + a$, let $a_{i+1} = x_{s'}$. Then $R_i = [a_i, a_{i+1} - 1]$ is neat, $|R_i| \geq zm + ja$ for some $j \geq 1$, and $w(R_i) \leq z(q + 1/2) + j/2$. Hence

$$r(R_i) \geq \frac{zm + ja}{z(q + \frac{1}{2})} + \frac{j}{2} \geq \frac{zm + a}{z(q + \frac{1}{2})} + \frac{1}{2} \geq \frac{(2q - 1)m + a}{2q^2}.$$

Note, all the I -intervals contained in $[x_{i_z} + m, x_{s'} - 1]$, if any, are short. The last inequality in the above holds since $z \leq 2q - 1$ and $\frac{zm+a}{z(q+\frac{1}{2})}$ is a decreasing function of z . As $a_{i+1} \in I$ has property (*), we are done for this case.

Assume $x_{s'} \leq x_{i_z} + m + a - 1$. Let $R_i^{z+1} = [x_{s'}, x_{s'} + m - 1]$. If $w(R_i^{z+1}) \leq q$, then let $a_{i+1} = x_{s'} + m$. By Lemma 6, $R_i = [a_i, a_{i+1} - 1]$ is neat and a_{i+1} has property (*). To verify (3), we note that $w(R_i) \leq (z + 1)q + z/2$ and

$$r(R_i) \geq \frac{(z + 1)m}{(z + 1)q + \frac{z}{2}} \geq \frac{2qm}{2q^2 + q - \frac{1}{2}} \geq \frac{(2q - 1)m + a}{2q^2}.$$

The second inequality in the above holds because $\frac{(z+1)m}{(z+1)q+z/2}$ is a decreasing function of z . The third inequality holds because $m \geq a(q + 1)$. Thus we assume that $w(R_i^{z+1}) = q + 1/2$.

Claim: $p_{z+1} \leq p_z$. Moreover, if $p_{z+1} = p_z$, then the last short I -interval contained in R_i^z is Type-A, and the last short I -interval in R_i^{z+1} is Type-B.

Proof of Claim. Let $T = [u, u + a - 1]$ and $T' = [u', u' + a - 1]$ be the last short I -interval in R_i^z and R_i^{z+1} , respectively. If T' is Type-B, then $T'' = [u' - m, u' - m + a - 1]$ is a short Type-A I -interval contained in R_i^z . Note, as $|[u' - m, x_{i_{s'}} - 1]| = |[u', x_{i_{s'}} + m - 1]|$ and $x_{i_{s'}} \geq x_{i_z} + m$, we have $|[x_{i_z}, u' - m - 1]| \geq |[x_{i_{s'}}, u' - 1]|$. Hence, $[x_{i_z}, u' - m - 1]$ is capable of containing I -intervals of total weight at least p_{z+1} . As the I -intervals in R_i^z are ‘‘tightly packed,’’ the I -intervals contained in $[x_{i_z}, u' - m - 1]$ has total weight at least p_{z+1} . Therefore $p_z \geq p_{z+1}$, and if the equality holds then the last short I -interval in R_i^z is of Type A.

Assume T' is Type-A. Thus $u' = x_{i^*} \in I$ for some i^* . Since T' is short, $x_{i^*+1} \leq x_{i^*} + \Delta - 1$. Note, as $s' \in L$ and $x_{i^*}, x_{i^*+1} \in I$, we have $x_{i^*+1} \leq x_{s'} + b$ and $[x_{i^*} - m + 1, x_{i^*+1} - b - 1] \cap I = \emptyset$.

Consider the interval $[x_{i_z}, x_{i^*+1} - b - 1]$. If this is a sub-interval of R_i^z ,

then since

$$|[x_{i_z}, x_{i^*+1} - b - 1]| \geq |[x_{i_{s'}}, x_{i^*+1} - 1]| + \Delta,$$

and the interval $[x_{i_z}, x_{i^*+1} - b - 1]$ is tightly packed, we conclude that the total weight of the I -intervals that intersect with $[x_{i_z}, x_{i^*+1} - b - 1]$ is at least $p_{z+1} + 1 + 1/2$. Moreover, since $|[x_{i^*} - m + 1, x_{i^*+1} - b - 1]| \geq \Delta + a - 1$ and $[x_{i^*} - m + 1, x_{i^*+1} - b - 1] \cap I = \emptyset$, we conclude that the last I -interval intersecting with $[x_{i_z}, x_{i^*+1} - b - 1]$ is Type-B. The total weight of the I -intervals of R_i^z preceding this Type-B I -interval is at least $p_{z+1} + 1$. Therefore, $p_{z+1} < p_z$.

Assume $[x_{i_z}, x_{i^*+1} - b - 1]$ is not a sub-interval of R_i^z . Then $x_{i^*+1} - b - 1 \geq x_{i_z} + m$. Since $x_{i_z} \geq x_{s'} - m - a + 1$, we have $x_{i^*+1} \geq x_{s'} + b - a + 3$. This implies that $[x_{i^*+1} + 1, x_{s'} + m] \cap I = \emptyset$, and $[x_{i^*+1}, x_{i^*+1} + \Delta - 1]$ is the last I -interval contained in R_i^{z+1} . Hence $p_{z+1} = q - 1$. If $p_z \leq q - 2$, then the conclusion follows.

Assume $p_z = q - 1$. Then the last I -interval in R_i^z is a long interval. Since $[x_{i^*} - m + 1, x_{i_z} + m - 1] \cap I = \emptyset$, the last integer of I in R_i^z is not greater than $x_{i^*} - m$, implying the last short I -interval in R_i^z is contained in $[x_{i_z}, x_{i^*} - m - 1]$. Therefore, the interval $[x_{i_z}, x_{i^*} - m - 1]$ has length at least $(q - 1)\Delta + a$. Moreover, since

$$|[x_{i^*} - m, x_{s'} - 1]| \geq |[x_{i^*} - m, x_{i^*+1} - b - 1]| \geq \Delta + a,$$

we conclude, $[x_{i_z}, x_{s'} - 1]$ has length at least $q\Delta + 2a$. Let $a_{i+1} = x_{s'}$. Then

$$r(R_i) \geq \frac{(z-1)m + q\Delta + 2a}{z(q + \frac{1}{2})} \geq \frac{(2q-2)m + q\Delta + 2a}{(2q-1)(q + \frac{1}{2})}.$$

The second inequality holds because the formula is a decreasing function on z and $z \geq 2q - 1$. To complete the proof of the Claim, it suffices to show

$$\frac{(2q-2)m + q\Delta + 2a}{(2q-1)(q + \frac{1}{2})} \geq \frac{(2q-1)m + a}{2q^2}.$$

Write $m = q\Delta + 2a - \lambda$, where $0 < \lambda \leq a$. The above inequality is equivalent to

$$2q^2\lambda - (2q^2 - 1/2)a - m(1/2 - q) \geq 0.$$

By definition, we have:

$$(1) \lambda \geq 2a - \Delta + 1 \text{ (since } q = \lfloor m/\Delta \rfloor)$$

$$(2) \Delta \leq 2a - 1 \text{ (since } 2a > \Delta)$$

Therefore,

$$\begin{aligned} & 2q^2\lambda - (2q^2 - 1/2)a - m(1/2 - q) \\ = & (2q^2 - q + 1/2)\lambda - a(2q^2 - 2q + 1/2) - \Delta(q/2 - q^2) \\ \geq & a(2q^2 + 1/2) - \Delta(q^2 - q/2 + 1/2) + (2q^2 - q + 1/2) \quad (\text{by (1)}) \\ \geq & a(q - 1/2) + 3q^2 - (3q)/2 + 1 \quad (\text{by (2)}) \\ \geq & 0 \quad (\text{since } q \geq 1) \end{aligned}$$

This completes the proof of the Claim. \square

Since $p_i \geq 1$, so p_{2q} does not exist. Thus the procedure above terminates at the k -th step for some $k \leq 2q$, when the valid a_{i+1} is obtained. This completes the proof of Theorem 1. \blacksquare

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