Circular chromatic index of Cartesian products of graphs

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Abstract

The *circular chromatic index* of a graph G, written $\chi'_c(G)$, is the minimum r permitting a function $f\colon E(G) \to [0,r)$ such that $1 \le |f(e) - f(e')| \le r-1$ whenever e and e' are incident. Let $G = H \square C_{2m+1}$, where $\square$ denotes Cartesian product and H is an $(s-2)$-regular graph of odd order, with $s \equiv 0 \bmod 4$ (thus G is s-regular). We prove that $\chi'_c(G) \ge s + \lfloor \lambda(1-1/s) \rfloor^{-1}$, where λ is the minimum, over all bases of the cycle space of H, of the maximum length of a cycle in the basis. When $H = C_{2k+1}$ and m is large, the lower bound is sharp. In particular, if $m \ge 3k+1$, then $\chi'_c(C_{2k+1} \square C_{2m+1}) = 4 + \lceil 3k/2 \rceil^{-1}$, independent of m.

1 Introduction

The *chromatic index* $\chi'(G)$ of a graph G is the minimum number of colors needed to color the edges so that incident edges receive distinct colors. In the case of a simple graph G (no loops or multiple edges), the famous theorem of Vizing [10] and Gupta [4] yields $\Delta(G) \le \chi'(G) \le \Delta(G) + 1$, where $\Delta(G)$ is the maximum vertex degree in G.

With only two values available, it is common to say that a graph G is *Class 1* if $\chi'(G) = \Delta(G)$ and *Class 2* otherwise. In this paper we consider a refinement of the chromatic index called the "circular chromatic index". It equals $\chi'(G)$ when G is Class 1, and otherwise it lies between $\Delta(G)$ and $\chi'(G)$. To define it, we first describe a vertex coloring parameter.

Given a graph G and a real number r, an *r-coloring* of G is a function $f\colon V(G) \to [0,r)$ such that $1 \le |f(x) - f(y)| \le r-1$ whenever x and y are adjacent. In essence, the set of colors form a circle of circumference r, and the colors assigned to adjacent vertices must differ by at least 1 (in each direction) along the circle.

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The *circular chromatic number* of G, written $\chi_c(G)$, is the infimum of all r such that G admits an r-coloring (the infimum can be replaced with minimum). There are many equivalent formulations of $\chi_c(G)$ (see [12, 13] for surveys and many basic results). The definition here is not the most common but is useful for our results. Due to the elementary result that $\chi(G) - 1 < \chi_c(G) \le \chi(G)$ [9], the parameter χ_c is a refinement of χ, and this has motivated its extensive study over the past decade.

For a graph G, the *line graph* $L(G)$ is the graph with vertex set $E(G)$ whose adjacency relation is the incidence relation for edges in G. The *circular chromatic index* $\chi'_c(G)$ is defined by $\chi'_c(G) = \chi_c(L(G))$. That is, we seek the smallest r permitting an r-coloring of the edges of G. Since $\chi'(G) = \chi(L(G))$, we have $\chi'(G) - 1 < \chi'_c(G) \le \chi'(G)$, and χ'_c is a refinement of χ'. From the definition, $\chi'_c(G)$ is at least the maximum number of pairwise incident edges. Thus $\chi'_c(G) = \chi'(G)$ when G is Class 1. Otherwise, $\Delta(G) < \chi'_c(G) \le \Delta(G) + 1$.

Several papers have been published about χ'_c. It was proved in [2] that all 2-edge-connected graphs with maximum degree at most 3 have circular chromatic index at most 11/3, except for two small graphs with circular chromatic index 4. In [5], it was proved that 2-edge-connected 3-regular graphs of large girth have circular chromatic index close to 3. This result was generalized in [6]: for any positive integer d, graphs with maximum degree d have circular chromatic index arbitrarily close to d if their girth is sufficiently large.

In this paper, we study the behavior of circular chromatic index under a product operation. Given graphs G and H, the *Cartesian product* $G\square H$ is the graph with vertex set $V(G) \times V(H)$ defined by making the pair (u, v) adjacent to the pair (u', v') if (1) $u = u'$ and $vv' \in E(H)$, or (2) $v = v'$ and $uu' \in E(G)$. It has long been known that $\chi(G\square H) = \max\{\chi(G), \chi(H)\}$ [1, 8, 11]. The argument holds as well for χ_c, so the behavior of χ_c is trivial under the Cartesian product.

The behavior of χ'_c is more interesting. If $G\square H$ is Class 1, then $\chi'_c(G\square H) = \Delta(G\square H)$, so we consider only products that are Class 2. The product is Class 1 when G or H is Class 1 [7] or when G and H both have perfect matchings [7]. To avoid Class 1, let G and H be regular graphs with odd order. The product $G\square H$ is then also regular with odd order, and a regular graph is Class 1 if and only if it has an edge-coloring in which every color class is a perfect matching, which does not exist in $G\square H$.

In particular, we consider the product of an odd cycle with a regular graph H of odd order, where the degree of the vertices in H is congruent to 2 modulo 4. We prove that $\chi'_c(H\square C_{2m+1}) \ge s + \lfloor\lambda(1 - 1/s)\rfloor^{-1}$, where λ is the maximum length of the cycles in some basis of the cycle space of H (choosing the basis to make λ smallest gives the best lower bound). We also prove that the bound is sharp when H is an odd cycle and m is large. Indeed, $\chi'_c(H\square C_{2m+1})$ always decreases to a limit as m increases. In particular, if $m \ge 3k + 1$, then $\chi'_c(C_{2k+1}\square C_{2m+1}) = 4 + \lceil 3k/2\rceil^{-1}$, independent of m.

2 Properties of r-Colorings

We view the color set $[0, r)$ for a r-coloring of a graph as the set of real numbers modulo r. Thus we interpret it as a circle C^r of circumference r, by identifying 0 and r. For $a, b \in C^r$, we write $[a, b]_r$ for the set in C^r moving from a to b through increasing values. That is, $[a, b]_r = [a, b]$ when $a \leq b$, while $[a, b]_r = [a, r) \cup [0, b]$ when $a > b$. For convenience, we extend this notation to all real numbers a and b by letting $[a, b]_r = [a_{\text{ mod } r}, b_{\text{ mod } r}]_r$, where $a_{\text{ mod } r}$ and $b_{\text{ mod } r}$ are the remainders of a and b upon division by r. The intervals $[a, b)_r$, $(a, b]_r$ and $(a, b)_r$ are defined similarly. We use $\ell([a, b])$ to denote the length of the interval $[a, b]$, and we define a measure of distance on the circle as $|a - b|_r = \min\{\ell([a, b]_r), \ell([b, a]_r)\}$. An *s-clique* is a set of s pairwise adjacent vertices.

Lemma 2.1 *Let G be a graph and f be an r-coloring of G, where $r = s + \epsilon$ with $s \in \mathbb{N}$ and $\epsilon < 1/2$. If Q is an s-clique in G and $v \in Q$, then each set $[f(v) + i, f(v) + i + \epsilon]_r$ for $0 \leq i \leq s - 1$ contains the color of exactly one vertex in Q. If X and Y are intersecting s-cliques, then for each $x \in X$ there is a unique $y \in Y$ such that $|f(y) - f(x)|_r \leq \epsilon$.*

Proof. Since the colors on vertices of Q must pairwise differ by at least 1, the ith such color after $f(v)$ must be at least i units later along the circle. It cannot be more than $i + \epsilon$ units later, since $s - i$ subsequent colors are encountering in returning to $f(v)$.

Now consider $v \in X \cap Y$. With $x_0 = y_0 = v$, let x_i be the ith vertex of X whose color is encountered moving upward from $f(v)$ around the circle (similarly define y_i). By the preceding paragraph, both $f(x_i)$ and $f(y_i)$ lie in $[f(v) + i, f(v) + i + \epsilon]_r$, for $1 \leq i \leq s - 1$. Hence they differ by at most ϵ. Furthermore, since $\epsilon < 1/2$, the distance between two such intervals is more than ϵ, so y_i is the only vertex of Y whose color is within ϵ of $f(x_i)$. ■

To facilitate proofs, we interpret vertex colorings as edge-weightings of orientations. Let $\vec{G}$ be an orientation of a graph G. For a weight function $w\colon E(G) \to \mathbb{R}$ and a walk W in G, let $w(W)$ denote the sum of the weights along W, where the weight of an edge counts negatively when followed against its direction in $\vec{G}$.

A *tension* on $\vec{G}$ is a weight function w such that $w(C) = 0$ for every cycle C in G. Given a real number r with $r \geq 2$, an *r-tension* is a tension w such that $1 \leq |w(uv)| \leq r - 1$ for every $uv \in E(G)$. An r-coloring f of G generates an r-tension w on an orientation $\vec{G}$ by letting $w(uv) = f(v) - f(u)$ for each $uv \in E(\vec{G})$.

A *modular r-tension* on an orientation $\vec{G}$ is a weight function $w\colon E(G) \to \mathbb{R}$ such that (1) $w(C)$ is a multiple of r whenever C is a cycle in G, and (2) the weight on each edge differs by at least 1 from any multiple of r. Every r-tension is a modular r-tension, so an r-coloring of G generates a modular r-tension on $\vec{G}$ as above.

Conversely, a modular r-tension w on $\vec{G}$ generates an r-coloring f of G as follows. We may assume that G is connected (else do this in each component). Fix a vertex x. For each vertex v, choose an x, v-walk W in G, and choose $f(v) \equiv w(W) \bmod r$ with $0 \leq f(v) < r$.

Since w is a modular r-tension, $f(v)$ does not depend on the choice of W, and the colors on adjacent vertices differ by at least 1. We call the resulting f an r-coloring *generated from* w. We say "an" here because the coloring depends on the choice of x, but only by a cyclic permutation. We have shown that $\chi_c(G)$ equals the least r such that some orientation $\vec{G}$ has a modular r-tension.

Our lower bound on $\chi'_c(H\square C_{2m+1})$ uses an analogue of girth, employing a parameter obtained from the cycle space of the graph. We obtain a strong lower bound when all the cycles in some basis of the cycle space are short.

Within the binary vector space of dimension $|E(G)|$ with canonical basis vectors indexed by the edges, the *cycle space* of an undirected graph G is the subspace spanned by the incidence vectors of the cycles. The analogue for an orientation $\vec{G}$ is the real vector space spanned by the signed incidence vectors of the cycles. For each cycle C in G, followed in a given direction, the *signed incidence vector* relative to $\vec{G}$ has 1 or -1 in each position for an edge of C, using -1 if and only if the edge is followed against its direction in $\vec{G}$.

For any orientation $\vec{G}$, the same sets of cycles form bases of its cycle space as form bases of the cycle space of the underlying graph G. In either context, the number of nonzero positions in the incidence vector for a cycle is the same. Hence we define the relevant parameter in terms of G. For a basis $\mathcal{B}$ of the cycle space of G, let $\lambda(\mathcal{B})$ denote the maximum length of an element of $\mathcal{B}$. Let $\lambda(G)$ denote the minimum of $\lambda(\mathcal{B})$ over all bases of the cycle space. Note that $\lambda(G)$ may be larger than the girth of G, but never smaller. The smaller the value of $\lambda(G)$, the larger the lower bound we will obtain on $\chi_c(G)$.

Before embarking on the technical lemmas, we pause to motivate their hypotheses. Let $F = H\square C_{2m+1}$. When H is $(s-2)$-regular, F is s-regular. Furthermore, the edges incident to any vertex of F become an s-clique in $L(F)$. Conversely, any two adjacent vertices of $L(F)$ correspond to two incident edges in F and hence lie in an s-clique in $L(F)$. Therefore, we can study r-edge-colorings of F by studying r-colorings of $L(F)$, which we do by studying r-colorings of graphs in which every edge lies in a complete subgraph of order s.

Lemma 2.2 *Let G be a graph such that each edge lies in a complete subgraph of order s. Let G have an r-coloring f such that*

$$r < s + \frac{1}{\lfloor \lambda(G)(1-1/s) \rfloor}.$$

If $\vec{G}$ is an orientation of G, then setting $w(xy) = \lfloor \ell[f(x), f(y))_r \rfloor$ for all $xy \in E(\vec{G})$ defines a modular s-tension on $\vec{G}$.

Proof. Let $\epsilon = r - s$, so $\epsilon < \lfloor \lambda(G)(1-1/s) \rfloor^{-1}$. For an edge xy, let Q be an s-clique containing x and y, and let $f(Q) = \{f(v)\colon\ v \in Q\}$. Let $t = |f(Q) \cap [f(x), f(y))_r|$, so $s - t = |f(Q) \cap [f(y), f(x))_r|$. By Lemma 2.1,

$$t \le \ell([f(x), f(y))_r) \le t + \epsilon \quad \text{and} \quad s - t \le \ell([f(x), f(y))_r) \le s - t + \epsilon. \tag{1}$$

By definition, $w(xy) = \lfloor \ell([f(x), f(y))_r) \rfloor$, so

$$f(y) \in [f(x) + w(xy), f(x) + w(xy) + \epsilon]_r. \tag{2}$$

By (1), $w(yx) = s - w(xy)$. Let $\mathcal{B}$ be a basis of the cycle space such that $\lambda(\mathcal{B}) = \lambda(G) = k$. To prove that w is a modular s-tension (when restricted to an orientation $\vec{G}$ of G), it suffices to show that $w(C) \equiv 0 \bmod s$ for each C whose signed incidence vector lies in $\mathcal{B}$. (Since $w(yx) = s - w(xy)$, the choice of $\vec{G}$ does not matter.)

Let $x_0, \ldots, x_{l-1}$ be the vertices of C in order, and let $x_l = x_0$; note that $l \leq k$. Let $e_i = x_i x_{i+1}$. In testing whether $w(C) \equiv 0 \bmod s$, the orientation of the edges along C does not matter; all orientations yield the same congruence class for $w(C)$. Since the same sets of cycles yield bases under each orientation, in studying C we may assume an orientation with each e_i directed from x_i to x_{i+1}. Now $w(C) \equiv \sum_{i=0}^{l-1} w(e_i) \bmod s$.

Since each edge lies in a complete subgraph of order s, (2) applies to each edge, so $f(x_{i+1}) \in [f(x_i) + w(e_i), f(x_i) + w(e_i) + \epsilon]_r$ for $0 \leq i \leq l-1$. Combining the allowed variations in the intervals for all edges of C yields

$$f(x_0) \in [f(x_0) + w(C), f(x_0) + w(C) + l\epsilon]_r. \tag{3}$$

By symmetry, we may choose $f(x_0) = 0$, which reduces (3) to $0 \in [w(C), w(C) + l\epsilon]_r$.

Since $1 \leq w(e_i) \leq s-1$, we have $l \leq w(C) \leq (s-1)l$. Since $w(C)$ is an integer, by choosing q to be $\lceil w(C)/s \rceil$ or $\lfloor w(C)/s \rfloor$ we can write $w(C) = qs + j$ for integers q and j such that $\lceil l/s \rceil \leq q \leq \lfloor l(1-1/s) \rfloor$ and $|j| \leq s-1$. Now

$$[w(C), w(C) + l\epsilon]_r = [j + qr - q\epsilon, j + qr + (l-q)\epsilon]_r = [j - q\epsilon, j + (l-q)\epsilon]_r.$$

Since $q \leq \lfloor l(1-1/s) \rfloor$, we have $q\epsilon \leq \lfloor l(1-1/s) \rfloor \epsilon \leq \lfloor k(1-1/s) \rfloor \epsilon < 1$. Similarly, $q \geq \lceil l/s \rceil$ yields $(l-q)\epsilon \leq \lfloor l(1-1/s) \rfloor \epsilon < 1$. Since $0 \in [w(C), w(C) + l\epsilon]_r \subseteq (j-1, j+1)_r$, we thus have $j = 0$. That is, $w(C) \equiv 0 \bmod s$. Thus w is a modular s-tension on $\vec{G}$. ■

The conclusion of Lemma 2.2 states that G is s-colorable. This is impossible if G is s-regular with odd order, so the lemma implies that $\chi_c(G) \geq s + \lfloor \lambda(G)(1-1/s) \rfloor^{-1}$. With $G = L(H \square C_{2m+1})$, we obtain a lower bound for $\chi'_c(H \square C_{2m+1})$, but it is not the lower bound we seek. The cycle space for G contains copies of the cycle space for H, but it is larger, and it may be that $\lambda(G) > \lambda(H)$, so the bound may be weaker than desired. To improve the bound, we will study subgraphs of G where we can control the value of λ. Before introducing these subgraphs, we prove a technical lemma about the color classes of the colorings generated from the modular s-tension produced by Lemma 2.2.

Lemma 2.3 *Let G be a graph such that each edge lies in a complete subgraph of order s. Suppose that G has an r-coloring f such that*

$$r < s + \frac{1}{\lfloor \lambda(G)(1-1/s) \rfloor}.$$

For a fixed vertex $v^* \in V(G)$ and any $x \in V(G)$, let $g(x) = \lfloor \ell([f(v^*), f(x))_r) \rfloor$. This function g is a proper (integer) s-coloring of G that satisfies the following property: $g(x) = g(x')$ if and only if G has a vertex list $(x, \ldots, x')$ in which any consecutive entries v and v' satisfy $d_G(v, v') = 2$ and $|f(v) - f(v')|_r < 1/2$.

Proof. Call a list $(x, \ldots, x')$ with the specified properties an x, x'-*skiplist*.

Let $\vec{G}$ be an orientation of G. By Lemma 2.2, setting $w(xy) = \lfloor \ell([f(x), f(y))_r) \rfloor$ for all $xy \in E(\vec{G})$ defines a modular s-tension w on $\vec{G}$, and g is an s-coloring of G generated from w. Since the values of w are integers in $\{0, \ldots, s-1\}$, in fact g is a proper (integer) s-coloring of G.

Vertices y and y' with $|f(y) - f(y')|_r < 1/2$ must be nonadjacent. If they have a common neighbor z, then

$$f(z) \in [f(y) + w(yz), f(y) + w(yz) + \epsilon]_r \cap [f(y') + w(y'z), f(y') + w(y'z) + \epsilon]_r.$$

If $|w(y'z) - w(yz)| \geq 1$, then the intervals on the right are disjoint, since $\epsilon < 1/2$ and $|f(y) - f(y')|_r < 1/2$. Therefore $w(yz) = w(y'z)$, which yields $g(y) = g(y')$. Therefore, all vertices in an x, x'-skiplist have the same color under g; in particular, $g(x) = g(x')$.

Conversely, suppose that $g(x) = g(x')$. Let $v_0, \ldots, v_t$ be the vertices along an x, x'-path in G, with $x = v_0$ and $x' = v_t$. For $0 \leq i \leq t-1$, let X_i be an s-clique of G containing v_i and v_{i+1}. Select auxiliary vertices $x_0, \ldots, x_t$ as follows. Having selected $x_0, \ldots, x_{i-1}$ (starting with $x_0 = v_0 = x$), observe that $v_i \in X_{i-1} \cap X_i$. By Lemma 2.1, there is a unique vertex $x_i \in X_i$ with $|f(x_i) - f(x_{i-1})|_r \leq \epsilon < 1/2$. Applying the preceding paragraph with $y = x_i$ and $y' = x_{i-1}$ yields $g(x_i) = g(x_{i-1})$. Finally, $x_t = x'$, since $x_t, x' \in X_t$ and $g(x_t) = g(x) = g(x')$. Now $(x_0, \ldots, x_t)$ is an x, x'-skiplist. ∎

The crucial consequence of Lemma 2.3 is that the partition of G into color classes under g does not depend on the choice of v^*.

3 A Lower Bound on $\chi'_c(H \square C_{2m+1})$

We specialize again to the study of $\chi'_c(H \square C_{2m+1})$. When H is $(s-2)$-regular with odd order, the product $H \square C_{2m+1}$ is s-regular with odd order and hence is Class 2. Thus $\chi'_c(H \square C_{2m+1}) > s$. We improve this lower bound when s is divisible by 4.

Let $V(C_{2m+1}) = \{v_0, \ldots, v_{2m}\}$, indexed in order; treat subscripts modulo $2m+1$. The ith *layer* H_i of $H \square C_{2m+1}$ is the subgraph induced by $V(H) \times \{v_i\}$. Each layer H_i is isomorphic to H. For $e \in E(H)$ and $x \in V(H)$, let e^i and x^i denote the copies of e and x in H_i. We call $\bigcup_{i=0}^{2m} E(H_i)$ the *horizontal edges* of $H \square C_{2m+1}$.

For $x \in V(H)$, let l_x^i denote the edge $x^i x^{i+1}$ in $H \square C_{2m+1}$. Let $L_i = \{l_x^i \colon x \in V(H)\}$; we call L_i the ith *link* of $H \square C_{2m+1}$ and call $\bigcup_{i=0}^{2m} L_i$ the *vertical edges* of $H \square C_{2m+1}$.

In a graph G whose vertices all have degree s or 1, any two incident edges are incident at a vertex of degree s. Therefore, in $L(G)$ every edge lies in a complete subgraph of order s. We will be applying the results of Section 2 to subgraphs of $H\Box C_{2m+1}$ having the form $L_{i-1}\cup H_i\cup L_i$, where every vertex has degree s or 1. We also need the following observation.

Lemma 3.1 *For any graph G, the equality $\lambda(L(G)) = \lambda(G)$ holds.*

Proof. Since cycles in G turn into cycles in $L(G)$ and must be spanned by any basis for $L(G)$, we have $\lambda(L(G)) \geq \lambda(G)$. Also, a basis for the cycle space of G (indexed by edges) can be augmented to a basis for the cycle space of $L(G)$ (indexed by vertices) by adding the incidence vectors of triangles in $L(G)$ consisting of three edges in G having a common endpoint. The added vectors have weight 3, so $\lambda(L(G)) \leq \lambda(G)$. ■

Theorem 3.2 *If H is an $(s-2)$-regular graph of odd order, where $4 \mid s$, then*

$$\chi_c'(H\Box C_{2m+1}) \geq s + \frac{1}{\lfloor \lambda(H)(1-1/s)\rfloor}.$$

Proof. If not, then $H\Box C_{2m+1}$ has an $(s+\epsilon)$-edge-coloring f, where $\epsilon < \lfloor \lambda(H)(1-1/s)\rfloor^{-1}$.

Let G_i be the subgraph of $L(H\Box C_{2m+1})$ induced by $L_{i-1}\cup E(H_i)\cup L_i$ (as defined above). Each edge of G_i lies in a complete subgraph of order s. Let $\mathcal{T}$ be the set of triangles in G_i. If $\mathcal{B}$ is a basis of the cycle space of $L(H_i)$, then $\mathcal{B}\cup\mathcal{T}$ contains a basis of the cycle space of G_i. Thus $\lambda(G_i) = \lambda(L(H_i)) = \lambda(L(H)) = \lambda(H)$, using $H_i \cong H$ and Lemma 3.1.

For each G_i, Lemma 2.3 states that the function g_i defined by fixing $v^* \in V(G_i)$ and setting $g_i(x) = \lfloor \ell([f(v^*), f(x))_r)\rfloor$ for all $x \in V(G_i)$ is a proper (integer) s-coloring of G_i. Since this g_i depends only on the global r-coloring f and the choice of v^*, the restrictions to L_i of the partitions of $V(G_i)$ and $V(G_{i+1})$ into color classes under g_i and g_{i+1} are the same when v^* is chosen to be an element of L_i.

Furthermore, Lemma 2.3 implies that the partition of $V(G_i)$ into color classes does not depend on the choice of v^*; it is determined only by values of f and distances between vertices in G_i. We conclude that no matter how v_i^* and v_{i+1}^* are chosen in specifying g_i and g_{i+1}, the resulting partitions of L_i into color classes are the same.

Each vertex x^i of the product has two incident vertical edges, namely l_x^i and l_x^{i-1}. We say that a color j is a *vertical color at x^i* if some vertical edge incident to x^i has color j under g_i. For each $x^i \in V(H_i)$, the s incident edges of G_i have distinct colors. Therefore a color j is a vertical color at x^i if and only if no edge of H_i incident to x^i has color j under g_i. Since H has odd order, and the number of vertices of H_i incident to edges of H_i with color j is even, we conclude that j is a vertical color at an odd number of vertices of H_i. In other words, in the partition of $L_{i-1}\cup L_i$ formed by the color classes under g_i, each class has odd size.

Let C_i^+ [respectively, C_i^-] be the set of colors used by g_i on an odd number of edges of L_i [respectively, L_{i-1}]. Since each class under g_i has odd size in $L_i\cup L_{i-1}$, we conclude that $j \in C_i^-$ if and only if $j \notin C_i^+$.

Since $|L_i|$ and $|L_{i-1}|$ are odd, it follows that $\left|C_i^+\right|$ and $\left|C_i^-\right|$ are also odd. Since $\left|C_i^+\right| + \left|C_i^-\right| = s$ and s is divisible by 4, it follows that $\left|C_i^+\right| \neq \left|C_i^-\right|$. Since g_i and g_{i+1} induce the same partitions of L_i, it follows that $\left|C_{i+1}^-\right| = \left|C_i^+\right|$, and hence also $\left|C_{i+1}^+\right| = \left|C_i^-\right|$. Now the values of $\left|C_i^+\right|$ must alternate between two distinct values as i runs through all $2m+1$ subscripts, which is impossible since $2m+1$ is odd. ■

4 An Upper Bound on $\chi'_c(H \square C_{2m+1})$

In this section, we obtain an upper bound on $\chi'_c(H \square C_{2m+1})$ for some H. As a consequence, we show that $\chi'_c(H \square C_{2m+1}) - \Delta(H \square C_{2m+1})$ can be bounded above by a number that is arbitrarily close to $\chi'_c(H) - \Delta(H)$ by making m sufficiently large.

We show first that increasing m cannot increase the circular chromatic index. We simply use the coloring of one layer on three consecutive layers in the larger graph and re-use the colorings on its neighboring links.

Lemma 4.1 *If $m' \geq m$, then $\chi'_c(H \square C_{2m'+1}) \leq \chi'_c(H \square C_{2m+1})$.*

Proof. It suffices to prove that $\chi'_c(H \square C_{h+2}) \leq \chi'_c(H \square C_h)$ for all h. Let f be an r-edge-coloring of $H \square C_h$). Form an r-edge-coloring of $H \square C_h$ as follows. Color the layers $H_0, \ldots, H_{h-1}$ and links $L_0, \ldots, L_{h-1}$ as under f. Color the layers H_h and H_{h+1} the same as H_{h-1}. Color the links L_h and L_{h+1} the same as L_{h-2} and L_{h-1}, respectively. Now the colors on any two incident edges of $H \square C_{h+2}$ under f' are also colors on two incident edges of $L(H \square C_h)$ under f. Thus f' is also an r-edge-coloring.

The colors on any two adjacent vertices of $L(H \square C_{2m+3})$ under f' are also colors on two adjacent vertices of $L(H \square C_{2m+1})$ under f. Thus f' is also an r-coloring. ■

Since $\chi'_c(H \square C_{2m+1}) \geq \Delta(H \square C_{2m+1}) = \Delta(H) + 2$ for all m, Lemma 4.1 implies that $\chi'_c(H \square C_{2m+1})$ has a limit as $m \to \infty$. In Section 5 we show that this limit is attained when H is an odd cycle, and we compute its value.

To prove the upper bound, we need a standard result about circular coloring.

Lemma 4.2 (See [13]) *If a graph G has a r-coloring f with $r = p/q$ where $p, q \in \mathbb{N}$, then it has an r-coloring f' such that the colors under f' are multiples of $1/q$, and such that if $xy \in E(G)$, then $|f'(x) - f'(y)|_r$ differs by less than $1/q$ from $|f(x) - f(y)|_r$.*

Proof. Let $f'(x) = \lfloor qf(x) \rfloor / q$ (such multiplication arguments were used as early as [3]). Note that $f'(x)$ is the largest multiple of $1/q$ that does not exceed $f(x)$. Under this transformation, $|f'(x) - f'(y)|_r$ equals $|f(x) - f(y)|$ if the latter is a multiple of $1/q$. Otherwise, the difference shifts to the next larger or next smaller multiple of $1/q$.

In particular, if the colors assigned to two vertices differ by at least a/q before the transformation, for some positive integer a (such as $a = q$), then they also differ by at least a/q after the transformation. Thus f' is an r-coloring. ■

Given an r-edge-coloring of a graph H, a *color gap* for a vertex x of H is a maximal open interval on the circle C^r that contains no color used on an edge incident to x.

Theorem 4.3 *Let H be a graph having a p/q-edge-coloring f such that every vertex x of H has a color gap of length at least 3. If p is odd and $2m+1 \ge p$, then $\chi'_c(H \Box C_{2m+1}) \le p/q$.*

Proof. By Lemma 4.1, it suffices to prove this when $2m+1 = p$. By Lemma4.2 (applied to $L(H)$), we may assume that each $f(e)$ is a multiple of $1/q$, still with each vertex having a color gap of length at least 3 (using $a = 3q$ in that argument). For each $x \in V(H)$, let $(a_x, b_x)_{p/q}$ be a color gap under f with length at least 3.

We produce a p/q-edge-coloring ϕ for $H \Box C_{2m+1}$. We use the same coloring f in each layer, except that the colors in each layer increase by one unit from the colors on the corresponding edges in the previous layer. Since $2m = p - 1 = q(p/q) - 1$, the colors on layer H_0 are also one unit (modulo p/q) above the corresponding colors on H_{2m}. This is achieved by letting $\phi(e^i) = f(e) + i \bmod p/q$ for each $e \in E(H)$ and $0 \le i \le 2m$.

It now suffices to use the color gaps to fit in colors for the vertical edges. Specifically, we set $\phi(l_x^i) = a_x + 2 + i \bmod p/q$ for each $x \in V(H)$ and $0 \le i \le 2m$. Since no horizontal edge at x^i receives a color in $(a_x + i, a_x + i + 3)$, the colors $a_x + i + 1$ and $a_x + i + 2$ are available for l_x^{i-1} and l_x^i, respectively, when viewed from x^i. Furthermore, ϕ achieves this assignment simultaneously for the vertical edges at all x^j. Hence for all incident edges, the assigned colors differ by at least 1. ■

For any graph G, let $\partial(G) = \chi'_c(G) - \Delta(G)$. Thus G is Class 1 if and only if $\partial(G) = 0$, and otherwise $0 < \partial(G) \le 1$.

Corollary 4.4 *For any graph H, $\lim_{m\to\infty} \partial(H \Box C_{2m+1}) \le \partial(H)$.*

Proof. The limit exists, using $\Delta(H \Box C_{2m+1}) \ge \Delta(H) + 2$ and Lemma 4.1. It suffices to show, given $\epsilon > 0$, that $\partial(H \Box C_{2m+1}) \le \partial(H) + \epsilon$ when m is sufficiently large.

Choose $p, q \in \mathbb{N}$ with p odd such that $\chi'_c(H) \le p/q \le \chi'_c(H) + \epsilon$. Let f be a p/q-edge-coloring of H. Also f can be viewed as a $(p/q+2)$-edge-coloring of H. For $x \in V(H)$, let b_x and a_x be the minimum and maximum colors in $[0, p/q)$ used on edges incident to x, respectively. Since $\ell((a_x, b_x)_{p/q}) \ge 1$, also $\ell((a_x, b_x)_{p/q+2}) \ge 3$. Relative to f as a $(p/q+2)$-edge-coloring, each vertex of H thus has a color gap of length at least 3. By Theorem 4.3, $\chi'_c(H \Box C_{2m+1}) \le p/q + 2 \le \Delta(H \Box C_{2m+1}) + \epsilon$ when $2m+1 \ge p$. ■

Recall that $H \Box H'$ is Class 1 when H or H' is Class 1. That is, $\partial(H) = 0$ or $\partial(H') = 0$ implies $\partial(H \Box H') = 0$. It is natural to ask if $\partial(H \Box H') \le \min\{\partial(H), \partial(H')\}$ always holds.

It does not, by the following example. Let $H = C_{2k+1}$ and $H' = C_{2m+1}$. Since $\chi'_c(C_{2m+1}) = 2 + 1/m$, we can make $\partial(H')$ arbitarily small. However, $\lambda(H) = 2k + 1$, so Theorem 3.2 yields $\partial(H \Box H') \geq \lfloor (6k+3)/4 \rfloor^{-1} = \lceil 3k/2 \rceil^{-1}$, independent of m.

On the other hand, $\lceil 3k/2 \rceil^{-1} < k^{-1} = \partial(C_{2k+1})$. Based on this and Theorem 4.3 and other examples, we propose the following conjecture.

Conjecture 4.5 *For any graphs H and H', $\partial(H \Box H') \leq \max\{\partial(H), \partial(H')\}$.*

5 Tightness of the lower bound

As noted above, Theorem 3.2 implies that $\chi'_c(C_{2k+1} \Box C_{2m+1}) \geq 4 + \lceil 3k/2 \rceil^{-1}$ for all m. In this section, we prove that the bound is sharp when $m \geq 3k+1$. This proves Conjecture 4.5 for products of two odd cycles when one is at least three times as long as the other.

Lemma 5.1 *If there exist integers α, β, q with $0 < q \leq m/2$ such that $|\alpha| + |\beta| = 2k + 1$ and $\alpha q + \beta(q+1) \equiv 0 \bmod 4q+1$, then $\chi'_c(C_{2k+1} \Box C_{2m+1}) \leq 4 + 1/q$.*

Proof. By Theorem 4.3 with $p = 4q+1$, it suffices to produce a $(4+1/q)$-edge-coloring f of C_{2k+1} such that every vertex x of C_{2k+1} has a color gap of length at least 3. Since C_{2k+1} is 2-regular, and we use a color circle of length $4 + 1/q$, the condition on f becomes "If e and e' are incident edges in C_{2k+1}, then $1 \leq |f(e') - f(e)|_{(4+1/q)} \leq 1 + 1/q$." Multiplying by q, we further transform this to seeking integers $z_1, \ldots, z_{2k+1}$ modulo $4q+1$ such that neighboring integers differ by q or $q+1$.

In the hypothesis, we may assume by symmetry that $\alpha \geq 0$. We construct the first α and last $|\beta|$ integers as separate arithmetic progressions, with common difference q for the first α and $q+1$ for the last $|\beta|$. For $1 \leq i \leq \alpha$, let $z_i = iq$ (this portion is empty if $\alpha = 0$). For $1 \leq i \leq |\beta|$, let $z_{\alpha+i} = \alpha q + \epsilon i(q+1)$, where $\epsilon = 1$ if $\beta > 0$ and $\epsilon = -1$ if $\beta < 0$.

The construction enforces the needed differences until just before the end; we need only compare z_{2k+1} and z_1. Since $z_{2k+1} = \alpha q + \beta(q+1) \equiv 0 \bmod 4q+1$, indeed z_{2k+1} and z_1 differ by q. ■

Theorem 5.2 *If $m \geq 3k+1$, then $\chi'_c(C_{2k+1} \Box C_{2m+1}) = 4 + \lceil 3k/2 \rceil^{-1}$.*

Proof. We have noted that Theorem 3.2 gives the lower bound. It suffices to find integers α, β, q satisfying the hypotheses of Lemma 5.1 with $q = \lceil 3k/2 \rceil = \lfloor (6k+3)/4 \rfloor$.

Let $r = \lfloor (k-1)/2 \rfloor$, so $k = 2r + s$ with $1 \leq s \leq 2$. Now $q = 3r + s + 1$. Let $\alpha = s - 1$ and $\beta = -(4r + s + 2)$. We have $|\alpha| + |\beta| = (4r + 2s + 1) = 2k + 1$ and

$$\alpha q + \beta(q+1) = (s-1)q - (4r+s+2)(q+1) = -(4q+1)(r+1),$$

where the last computation uses $q = 3r + s + 1$. Thus $\alpha q + \beta(q+1) \equiv 0 \bmod (4q+1)$, and Lemma 5.1 applies. ■

References

[1] O. Aberth, On the sum of graphs. *Rev. Fr. Rech. Opér.* 33 (1964), 353–358.

[2] P. Afshani, M. R. Ghandehari, M. Ghandehari, H. Hatami, R. Tusserkani, and X. Zhu. Circular chromatic index of graphs of maximum degree at most 3 *J. Graph Th.* 49 (2005), 325–335.

[3] J. A. Bondy and P. Hell. A note on the star chromatic number. *J. Graph Th.* 14 (1990), 479–482.

[4] R. P. Gupta, The chromatic index and the degree of a graph, (Abstract 66T-429), *Not. Amer. Math. Soc.* **13** (1966), 719.

[5] T. Kaiser, D. Král, and R. Škrekovski, A revival of the girth conjecture. *J. Combin. Th. (B)* 92 (2004), 41–53.

[6] T. Kaiser, D. Král, R. Škrekovski, and X. Zhu, The circular chromatic index of graphs of high girth. *J. Combin. Th. (B)* 97 (2007), 1–13.

[7] A. Kotzig, 1-factorizations of Cartesian products of regular graphs. *J. Graph Th.* 3 (1979), 23–34.

[8] G. Sabidussi, Graphs with given group and given graph-theoretical properties. *Canad. J. Math.* 9 (1957), 515–525.

[9] A. Vince, Star chromatic number. *J. Graph Th.* 12 (1988), 551–559.

[10] V. G. Vizing, On an estimate of the chromatic class of a p-graph (in Russian). *Diskret. Analiz.* 3 (1964), 24–30.

[11] V. G. Vizing, The cartesian product of graphs (in Russian). *Vyčisl. Sistemy* 9 (1963), 30–43.

[12] X. Zhu, Circular chromatic number: a survey. *Discrete Math.* 229 (2001), 371–410.

[13] X. Zhu, Recent developments in circular colouring of graphs, in *Topics in Discrete Mathematics*, *Algorithms Combin.* 26 (Springer, Berlin, 2006), 497–550.