The circular chromatic number of series-parallel graphs

Pavol Hell
School of Computing Sciences
Simon Fraser University
Burnaby, BC, Canada
Email: pavol@cs.sfu.ca
and
Xuding Zhu
Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung, Taiwan 80424
Email: zhu@math.nsysu.edu.tw

Abstract

In this paper, we consider the circular chromatic number $\chi_c(G)$ of series-parallel graphs G. It is well known that series-parallel graphs have chromatic number at most 3. Hence their circular chromatic numbers are at most 3. If a series-parallel graph G contains a triangle, then both the chromatic number and the circular chromatic number of G are indeed equal to 3. We shall show that if a series-parallel graph G has girth at least $2\lfloor(3k-1)/2\rfloor$, then $\chi_c(G) \leq 4k/(2k-1)$. The special case $k = 2$ of this result implies that a triangle free series-parallel graph G has circular chromatic number at most $8/3$. Therefore the circular chromatic number of a series-parallel graph (and of a K_4-minor free graph) is either 3 or at most $8/3$. This is in sharp contrast to recent results of Moser [7] and Zhu [13], which imply that the circular chromatic number of K_5-minor free graphs are precisely all rational numbers in the interval $[2, 4]$. We shall also construct examples to demonstrate the sharpness of the bound given in this paper.

1 Introduction

The circular chromatic number of a graph (also known as the "star chromatic number") is a natural generalization of the notion of chromatic number of a graph. Given two integers k, d, such that $k \geq d$, a (k, d)-coloring of a graph G is a coloring c of the vertices of G with colors $0, 1, 2, \cdots, k-1$ such that for any two adjacent vertices x and y of G we have $d \leq |c(x) - c(y)| \leq k - d$. The *circular chromatic number* $\chi_c(G)$ of G is defined as the infimum of the ratio k/d for which there exists a (k, d)-coloring of G. Note that any non-trivial graph has circular chromatic number at least 2. It was shown by Vince [10] (cf. also [2] for a combinatorial proof) that if G is finite then the infimum in this definition is always attained, and hence can be replaced by the minimum.

Note that a $(k, 1)$-coloring of a graph G is just an ordinary k-coloring of G. Therefore we have $\chi_c(G) \leq \chi(G)$ for any graph G. On the other hand, it was shown in [10] that $\chi_c(G) > \chi(G) - 1$. Hence $\chi(G) = \lceil \chi_c(G) \rceil$. In this sense, for a graph G, the circular chromatic number $\chi_c(G)$ is a refinement of the chromatic number $\chi(G)$, and $\chi(G)$ is an approximation of $\chi_c(G)$. The parameter $\chi_c(G)$ has been studied extensively since it was introduced by Vince in 1988 (see [15] for a survey).

Since the infimum can be replaced by the minimum, the circular chromatic number of a finite graph is always rational. On the other hand, it was shown in [10] that for any rational number $r \geq 2$, there exists a finite graph G of circular chromatic number r.

Given a property P of graphs, it is usually an interesting and difficult problem to determine whether or not there exists a graph G which has property P, and whose circular chromatic number is equal to a given rational number r. One such problem was discussed in [12]. For an integer g, consider the property of having girth at least g. It was shown in [12] that for any integer g and for any rational number $r \geq 2$, there exists a graph G which has girth at least g and circular chromatic number r. This result is a generalization of the result of Erdős concerning the existence of graphs with arbitrarily large girth and chromatic number [3].

Another such problem was discussed in [7, 13]; the property considered there is planarity. The authors ask for which rational numbers r there exists a planar graph G with circular chromatic number r. It follows from the Four Color Theorem, that the number r is at most 4. It was shown in [7] that for any rational r between 2 and 3, there is a planar graph G with circular chromatic number r, and it was shown in [13] that for any rational number r between 3 and 4, there is a planar graph G with circular chromatic number r. We conclude that a rational r is the circular chromatic number of a non-trivial planar graph if and only if $2 \le r \le 4$.

A graph H is called a *minor* of a graph G if H is isomorphic to a graph which is obtained from a subgraph of G by contracting edges. We say a graph G is *H-minor free* if H is not a minor of G. As a generalization of the Four Color Problem, Hadwiger conjectured that any graph G with chromatic number at least n contains K_n as a minor. In other words, Hadwiger's conjecture asserts that a K_n-minor free graph has chromatic number at most $n-1$ (and thus circular chromatic number also at most $n-1$). In [14] it was proved that for any rational $2 \le r \le n-2$, there does exist a K_n-minor free graph which has circular chromatic number r. For r between $n-2$ and $n-1$, it remains an open question whether or not there exists a K_n-minor free graph with circular chromatic number r. (For $r > n-1$ the problem involves Hadwiger's conjecture as noted above.)

For the special case $n = 5$ the question has been answered in the affirmative. This is a consequence of the result mentioned above: For any rational number r between 2 and 4 there is a planar (hence K_5-minor free) graph with circular chromatic number r.

In this paper, we consider the case $n = 4$. In other words, we ask which rational r between 2 and 3 are the circular chromatic numbers of K_4-minor free graphs. Surprisingly, we find that there are gaps among these rational numbers. We shall prove that the circular chromatic number of a K_4-minor free graph G is either equal to 3 or is at most $8/3$. This is in sharp contrast to the situation for K_5-minor free graphs (as explained above).

It is well known [5] that a graph G is K_4-minor free if and only if each block of G is a series-parallel graph. It is easy to see that a graph G is (k,d)-colorable if and only if each block of G is (k,d)-colorable. Therefore to prove our results, it suffices to deal with series-parallel graphs.

In proving our result we shall observe that the case $\chi_c(G) = 3$ corresponds to graphs G which contain a triangle, and the case $\chi_c(G) \le 8/3$ to graphs G of girth at least 4. We are thus lead to investigate the effect of the girth of a series-parallel graph G on its circular chromatic number $\chi_c(G)$. It was observed in [8] that for any graph of bounded treewidth, the circular chromatic number is arbitrarily close to 2 if the girth is large enough. Since the treewidth of series-parallel graphs is at most 2, we know that large girth implies that χ_c is close to 2. The main result of this paper is the following theorem which makes this dependence precise, and turns out to be the best possible bound:

Theorem 1.1 *Let k be an integer, $k \ge 2$. If G is a series-parallel graph with girth at least $2\lfloor(3k-1)/2\rfloor$, then $\chi_c(G) \le 4k/(2k-1)$.*

When $k = 2$, we have the following corollary:

Corollary 1.1 *If G is a triangle free series-parallel graph, then $\chi_c(G) \leq 8/3$.*

Since a series-parallel graph containing a triangle has circular chromatic number 3, we also obtain the following corollary:

Corollary 1.2 *If G is a series-parallel graph then either $\chi_c(G) = 3$ or $\chi_c(G) \leq 8/3$.*

Corollary 1.3 *If G is a K_4-minor free graph then either $\chi_c(G) = 3$ or $\chi_c(G) \leq 8/3$.*

The corollary implies that there are no circular chromatic numbers of K_4-minor free graphs in the interval $(8/3, 3)$. Since both 3 and $8/3$ are such circular chromatic numbers (take the triangle and the graph in figure 2) we have found a gap among these rational numbers. We do not know whether each rational number between 2 and $8/3$ is the circular chromatic number of a K_4-minor free graph, but rather suspect that there may be other gaps.

2 Proof of Theorem 1.1

A two-terminal series-parallel graph $(G; x, y)$ is defined recursively as follows:

- Let $V(K_2) = \{0, 1\}$. Then $(K_2; 0, 1)$ is a two-terminal series-parallel graph.
- (The parallel construction.) Let $(G; x, y)$ and $(G'; x', y')$ be two disjoint two-terminal series-parallel graphs. Define G'' to be the graph obtained from the union of G and G' by identifying x and x' into a single vertex x'', and identifying y and y' into a single vertex y''. Then $(G''; x'', y'')$ is a two-terminal series-parallel graph.
- (The series construction.) Let again $(G; x, y)$ and $(G'; x', y')$ be two disjoint two-terminal series-parallel graphs. Define G'' to be the graph obtained from the union of G and G' by identifying y and x' into a single vertex. Then $(G''; x, y')$ is a two-terminal series-parallel graph.
- There are no other two-terminal series-parallel graphs.

Note that $(G; x, y)$ may be a two-terminal series-parallel graph for some pairs x, y and not for others. A graph G is a series-parallel graph if there exist some two vertices x, y such that $(G; x, y)$ is a two-terminal series-parallel graph. For all the series-parallel graphs in the remaining part, there are always two terminals which are clearly indicated in the context. We call the distance in G between the two terminals x, y, the *length* of $(G; x, y)$.

We shall prove Theorem 1.1 by induction on the number of steps in the construction of $(G; x, y)$. Let k be an integer, $k \geq 2$, and let $I = \{0, 1, \cdots, 4k-1\}$. We shall view the numbers in I as cyclically ordered, as illustrated in Fig. 1, and all the additions of the elements of I are carried out modulo $4k$, unless otherwise specified. For two elements a, b of I, we shall denote by $[a, b]$ the set $\{a, a+1, \cdots, b\}$. For example, $[2, 5] = \{2, 3, 4, 5\}$ and $[5, 2] = \{5, 6, \cdots, 4k-1, 0, 1, 2\}$.

For $i \geq 1$, we define sets I_i as follows: If $i \leq 2k-1$ is odd, then $I_i = [2k-i, 2k+i]$. If $i \leq 2k-1$ is even, then $I_i = [4k-i, i]$. For $i \geq 2k$, let $I_i = I$.

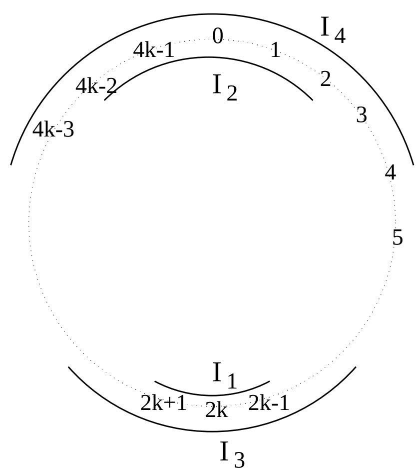

Figure 1: Illustration of the sets I_j

For two subsets X, Y of I, define $X + Y = \{i + j : i \in X, j \in Y\}$.

Lemma 2.1 *Suppose $X = [a, b]$ and $Y = [a', b']$.*

- *If $|X| + |Y| \geq 4k+1$, then $X + Y = I$;*
- *If $|X| + |Y| \leq 4k$, then $X + Y = [a + a', b + b']$.*

In the calculation of $|X| + |Y|$, the ordinary addition is carried out. (All other additions are modulo $4k$). The proof of Lemma 2.1 is straightforward and is omitted.

Corollary 2.1 *For any positive integers i and j we have $I_i + I_j = I_{i+j}$.*

We now introduce sets J_i which are the ones actually used in our proof. They can be introduced directly (cf. below), but their properties are better seen when they are viewed as subsets of the simpler sets I_i. In these definitions (and subsequent proofs) there is a number which plays a central role; we shall denote it by p and define it as

$$p = \lfloor (3k-1)/2 \rfloor.$$

- if $1 \le i \le k-1$ or $i \ge 2k-1$, then $J_i = I_i$,
- if $p \le i \le 2k-2$, then J_i is recursively defined from J_{i+1} by letting $J_i = I_i \cap J_{i+1}$,
- if $k \le i \le p-1$, then $J_i = I_i \cap J_{2p-i}$.

With some calculation, explicit formulae for the sets J_i can be found as follows (ignoring the cases $1 \le i \le k-1$ and $i \ge 2k-1$ where $J_i = I_i$, cf. above):

If k is even, then in the range $k \le i \le p$, we have

- $J_i = [2k-i, 3k-i-1] \cup [i+k+1, 2k+i]$, when i is odd, and
- $J_i = [i+1-k, i] \cup [4k-i, 5k-i-1]$, when i is even.

If k is odd, then in the range $k \le i \le p$, we have

- $J_i = [2k-i, 3k-i] \cup [i+k, 2k+i]$, when i is odd, and
- $J_i = [i-k, i] \cup [4k-i, 5k-i]$, when i is even.

In the range $p+1 \le i \le 2k-2$, we have (regardless of the parity of k)

- $J_i = [2k-i, i+1] \cup [4k-i-1, 2k+i]$, when i is odd, and
- $J_i = [2k-i-1, i] \cup [4k-i, 2k+i+1]$, when i is even.

Theorem 2.1 *Let $k \ge 2$ and $t \ge 1$ be integers. Suppose $(G; x, y)$ is a series-parallel graph of length t. If G has girth at least $2\lfloor (3k-1)/2 \rfloor$, then for any colors $0 \le a, b \le 4k-1$ such that $|a-b| \in J_t$, there is a $(4k, 2k-1)$-coloring c of G such that $c(x) = a$ and $c(y) = b$.*

Note that in terms of the number p introduced above, the bound on the girth is equal to $2p$. Theorem 1.1 will follow immediately from Theorem 2.1. The proof of Theorem 2.1 is based on the following two lemmas.

Lemma 2.2 *If $i \leq i'$ and $i + i' \geq 2p$, then $J_i \subseteq J_{i'}$.*

Proof. If $i \leq k-2$, then $i' \geq 2k$ and $J_i \subseteq I = J_{i'}$. If $i = k-1$ then either k is odd and $i' \geq 2k$, or k is even and $i' \geq 2k-1$. In the former case, $J_{k-1} \subseteq I = J_{i'}$. In the latter case $J_{k-1} \subseteq [1, 4k-1] \subseteq J_{i'}$. If $i \geq p$, then it follows from the definition that $J_i \subseteq J_{i+1}$, and hence $J_i \subseteq J_{i'}$. If $k \leq i \leq p-1$, then $2p - i \geq p$ and it follows from the definition that $J_i \subseteq J_{2p-i} \subseteq J_{i'}$.

∎

Lemma 2.3 *If $i + i' = i''$, then $J_{i''} \subseteq J_i + J_{i'}$.*

Proof. If $k = 2$ then we have $J_1 = I_1 = [3,5]$, $J_2 = \{1,2,6,7\}$, $J_3 = I_3 = [1,7]$, and all other $J_i = I_i = [0,7]$. According to Corollary 2.1 we only need to verify that $J_{2+i'} \subseteq J_2 + J_{i'}$, which is easily done for each of $i' = 1,2,3$.

In the following we assume that $k \geq 3$. The proofs are straightforward, but somewhat tedious. We only sketch a few of the cases to indicate the flavour of the calculations. The reader can easily complete the verifications in the other cases along the same lines. We may assume that $i \leq i'$.

If $i, i' \leq k-1$ then $J_i = I_i$, $J_{i'} = I_{i'}$ and $J_{i''} \subseteq I_{i''}$, and hence the conclusion follows from Corollary 2.1.

If $k \leq i, i' \leq p$ then $J_{i''} = I$, and it is easy to check that also $J_i + J_{i'} = I$. For instance when k, i, i' are all even, we have $J_i = [i+1-k, i] \cup [4k-i, 5k-i-1]$, $J_{i'} = [i'+1-k, i'] \cup [4k-i', 5k-i'-1]$, and so by Lemma 2.1

$$J_i + J_{i'} = [i''+2-2k, i''] \cup [4k-i'', 2k-i''-2] \cup [3k+i-i'+1, 5k+i-i'-1] \cup [3k-i+i'+1, 5k-i+i'-1].$$

Since at least one of the last two intervals covers $[3k+1, k-1]$, we only have to verify that the first two intervals cover $[k, 3k]$, which is easy to do since $2k \leq i'' \leq 2p < 3k$.

If $i, i' \geq p+1$ then we also have $J_{i''} = I$. It follows from Lemma 2.2 that $J_{p+1} \subseteq J_i$ and $J_{p+1} \subseteq J_{i'}$. Thus it suffices to verify that $J_{p+1} + J_{p+1} = I$, which is easy to do. For instance, when k is even, say $k = 2m$, then $p = 3m-1$ and J_{p+1} is $[m, 3m+1] \cup [5m-1, 7m]$ if $p+1$ is odd, or $[m-1, 3m] \cup [5m, 7m+1]$ if $p+1$ is even. In any case, $[m, 3m] \cup [5m, 7m] \subseteq J_{p+1}$. It follows from Lemma 2.1 that $J_{p+1} + J_{p+1} \supseteq [2m, 6m] \cup [6m, 2m] = I$.

We divide the remaining verifications into three cases:

Case 1 $k \le i \le p$ and $i' \ge p+1$.

In this case we have again $J_{i''} = I$ and $J_{p+1} \subseteq J_{i'}$. It is again straightforward to verify that $J_i + J_{i'} \supseteq I$. For instance, if $k = 2m$ is even then, as above, we have $[m, 3m] \cup [5m, 7m] \subseteq J_{p+1}$. If also i is even, then $J_i = [i+1-2m, i] \cup [8m-i, 10m-1-i]$ and

$$J_i+J_{i'} \supseteq J_i+J_{p+1} \supseteq ([i+1-2m, i]\cup[8m-i, 10m-1-i])+([m, 3m]\cup[5m, 7m]).$$

The sum of the last two intervals with $[8m-i, 10m-1-i]$ is (by Lemma 2.1)

$$[m-i, 5m-1-i] \cup [5m-i, m-1-i] = I.$$

Case 2 $1 \le i \le k-1$ and $i' \ge p+1$. In this case $J_{i''}$ may or may not be equal to I. We illustrate some of the arguments on the case of even k, i, i' (and thus also i''). We have

$$J_i = [4k-i, i], J_{i'} = [2k-i'-1, i'] \cup [4k-i', 2k+i'+1],$$

and by Lemma 2.1, $J_i+J_{i'} = [6k-i''-1, i''] \cup [4k-i'', 2k+i''+1]$. If $i'' \ge 2k-1$ then $i'' \ge 2k$ (since it is even) and $J_i + J_{i'} \supseteq [0, 2k] \cup [2k, 1] = I = J_{i''}$. If $i'' =\le 2k-2$, then by definition

$$J_{i''} = [2k-i''-1, i''] \cup [4k-i'', 2k+i''+1] = J_i + J_{i'}.$$

Other arguments needed in this proof are illustrated in the case when k, i are even and i' (and hence also i'') is odd. If $i'' \ge 2k$, then $i'' \ge 2k+1$ (as i'' is odd) and $i + (i'-1) \ge 2k$. Since $J_{i'-1} \subseteq J_{i'}$, and we have proved that $J_i + J_{i'-1} = I$ if $i'-1 \ge p+1$ (cf. the previous paragraph), so $J_i + J_{i'} = I$ if $i' \ge p+2$. In case $i' = p+1$, then it is also straightforward to verify that $J_i + J_{i'} = I$. Thus we may assume that $i + i' \le 2k-1$. Then

$$J_i = [4k-i, i],$$

$$J_{i'} = [2k-i', i'+1] \cup [4k-i'-1, 2k+i'],$$

and

$$J_i + J_{i'} = [2k-i'', i''+1] \cup [4k-i''-1, 2k+i''] = J_{i''}.$$

Case 3 $1 \le i \le k-1$ and $k \le i' \le p$.

Suppose for instance that all k, i, i' (and hence also i'') are even. Then

$$J_i = [4k-i, i], J_{i'} = [i'+1-k, i'] \cup [4k-i', 5k-i'-1],$$

and $J_i + J_{i'} = [3k - i + i' + 1, i''] \cup [4k - i'', 5k + i - i' - 1]$. We now consider the possible values of the (positive) quantity $i' - i$. If $i' - i \leq k - 2$, then $5k + i - i' - 1 = k - 1 - (i' - i)$, while $3k - i + i' + 1$ is reduced modulo $4k$. It is then easy to see that $J_i + J_{i'} = [4k - i'', i''] = I_{i''} \supseteq J_{i''}$. Otherwise $i' - i \geq k$ (being even), and $J_i + J_{i'} = [(i' - i) + 1 - k, i''] \cup [4k - i'', 5k - 1 - (i' - i)]$. Now $J_{i''}$ is either $[i'' - k + 1, i''] \cup [4k - i'', 5k - i'' - 1]$ (if $i'' \leq p$) or $[2k - i'' - 1, i''] \cup [4k - i'', 2k + i'' + 1]$ (if $i'' \geq p + 1$). Note that $i'' \leq 2k - 2$ because $i' \leq p$ and $i' - i \geq k$. In either case, $J_{i''} \subseteq J_i + J_{i'}$.

This completes the proof of Lemma 2.3. ∎

Proof of Theorem 2.1. Let $(G; x, y)$ be a two terminal series-parallel graph of length t and girth at least $2p$. We shall prove Theorem 2.1 by induction on the number of steps in the construction of $(G; x, y)$.

If $(G; x, y) = (K_2; 0, 1)$, then it follows from the definition that for any colors a and b, if $|a - b| \in J_1$, then $c(x) = a$ and $c(y) = b$ is a $(4k, 2k - 1)$-coloring of G.

Assume now that Theorem 2.1 is true for two-terminal series parallel graphs $(G; x, y)$ and $(G'; x', y')$, and that $(G''; x'', y'')$ is obtained from $(G; x, y)$ and $(G'; x', y')$ by a parallel operation. Assume that the length of $(G; x, y)$ is t and the length of $(G'; x', y')$ is t', and without loss of generality, assume that $t \leq t'$. It is obvious that the length of $(G''; x'', y'')$ is the minimum of the lengths of $(G; x, y)$ and $(G'; x', y')$, i.e., t. Since G'' has girth at least $2p$, it follows that $t + t' \geq 2p$, and by Lemma 2.2, $J_t \subseteq J_{t'}$. By the induction hypotheses, for any colors a, b such that $|a - b| \in J_t \subseteq J_{t'}$, there exists a $(4k, 2k - 1)$-coloring c of G such that $c(x) = a$ and $c(y) = b$; and there exists a $(4k, 2k - 1)$-coloring c' of G' such that $c(x') = a$ and $c(y') = b$. Let c'' be the union of c and c', i.e., $c''(v) = c(v)$ if $v \in V(G)$ and $c''(v) = c'(v)$ if $v \in V(G')$. Then c'' is a $(4k, 2k - 1)$-coloring of G'', such that $c''(x'') = a$ and $c''(y'') = b$.

Finally, assume now that Theorem 2.1 is true for series parallel graphs $(G; x, y)$ and $(G'; x', y')$, and that $(G''; x, y')$ is obtained from $(G; x, y)$ and $(G'; x', y')$ by a series operation. Assume that the lengths of $(G; x, y)$ and $(G'; x, y)$ are t and t' respectively. Then clearly the length of $(G''; x'', y'')$ is $t'' = t + t'$. Suppose a, b are colors such that $|a - b| \in J_{t''}$. We shall find a $(4k, 2k - 1)$-coloring of G'' such that $c''(x) = a$ and $c''(y') = b$.

By Lemma 2.3, there exist $j \in J_t$ and $j' \in J_{t'}$ such that $j + j' = |a - b|$. Without loss of generality, we may assume that $j + j' = b - a$. By the induction hypothesis, there is a $(4k, 2k - 1)$-coloring c of G such that $c(x) = 0$ and $c(y) = j$, and there exists a $(4k, 2k - 1)$-coloring c' of G' such that $c'(x') = 0$ and $c'(y') = j'$. Suppose $a + j = s$. Let c'' be the coloring of G'' defined as

$c''(v) = c(v) + a$ if $v \in V(G)$, and $c''(v) = c'(v) + s$ if $v \in V(G)$. Then it is easy to see that c'' is a $(4k, 2k-1)$-coloring of G'' such that $c''(x) = a$ and $c''(y') = b$. This completes the proof of Theorem 2.1. ∎

3 Tightness of the bound

First of all, we shall give an example of a series parallel graph G which is triangle free, and whose circular chromatic number is equal to 8/3. This graph G is depicted in Fig. 2 below. The numbers besides the vertices give a $(8,3)$-coloring of the graph. Therefore $\chi_c(G) \leq 8/3$ (this also follows from Corollary 1). To prove that $\chi_c(G)$ cannot be strictly less than 8/3, it suffices to note that $\chi_c(G) = k/d$ for some $k \leq |V(G)| = 12$, and the largest fraction k/d which is less than 8/3 and satisfying $k \leq 12$ is 5/2. We leave it to the readers to verify that G cannot be $(5,2)$-colored.

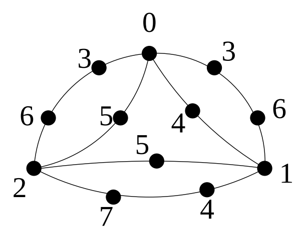

Figure 2: A series-parallel graph of girth 5 and circular chromatic number 8/3

This example can be generalized as in Fig. 3 below. The number k (or $k+1$) on a line means that that line represents a path of k (or $k+1$) edges. It is not difficult to verify that the graph G has circular chromatic number $4k/(2k-1)$.

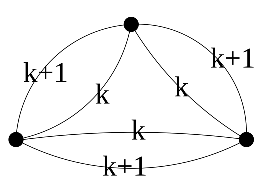

Figure 3: A series-parallel graph of girth $2k+1$ with $\chi_c > 4k/(k-1)$

For an integer $k \geq 2$, let $g(k)$ be the smallest integer such that any series-parallel graph of girth at least $g(k)$ are $(4k, 2k-1)$-colorable. In other words, $g(k)$ is such an integer that every series-parallel graph G of girth at least $g(k)$ has circular chromatic number at most $4k/(2k-1)$, and that there exists a series-parallel graph G of girth $g(k)-1$ which has circular chromatic number greater than $4k/(2k-1)$.

As a corollary of Theorem 1.1 and the example in Fig. 3, we have the following lower and upper bounds for $g(k)$:

Corollary 3.1 *For any integer $k \geq 2$, we have $2k+2 \leq g(k) \leq 2\lfloor(3k-1)/2\rfloor$.*

Proof. Theorem 1.1 implies that $g(k) \leq 2\lfloor(3k-1)/2\rfloor$. Examples in Fig. 3 shows that there are series-parallel graphs of girth $2k+1$ whose circular chromatic number is greater than $4k/(2k-1)$. Hence $g(k) \geq 2k+2$. ∎

It is very likely that the upper bound for $g(k)$ in Corollary 3.1 is the exact value of $g(k)$. For $k=2$, it follows from Corollary 3.1 that $g(2)=4$.

We shall present examples show that the upper bound in Corollary 3.1 is the exact value for $k=3$ and $k=4$ as well.

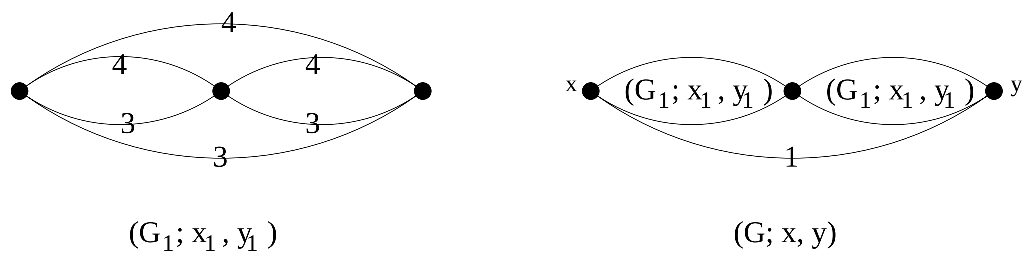

Figure 4: A series-parallel graph og girth 7 and circular chromatic number 12/5

As in Fig. 3, the number t on a line means that that line represents a path of t edges. Obviously the graph $(G;x,y)$ in Fig. 4 is a series-parallel graph of girth 7. We shall show that $\chi_c(G) > 12/5$. Given a series-parallel graph $(H;a,b)$, we define the *admissible set* for $(H;a,b)$, denoted by $A((H;a,b))$, as follows:

$$A((H;a,b)) = \{i : \text{ there exists a } (12,5)\text{-coloring } c \text{ of } H \text{ such that } |c(a)-c(b)| = i\}.$$

It follows from the definition that if $(H;a,b)$ is a copy of K_2, then $A((H;a,b)) = \{5,6,7\}$. It also follows straightforward from the definition that

- if $(H;a,b)$ is obtained from $(H_1;a_1,b_1)$ and $(H_2;a_2,b_2)$ by a parallel operation, then

$$A((H;a,b)) = A((H_1;a_1,b_1)) \cap A((H_2;a_2,b_2)).$$

- if $(H; a, b)$ is obtained from $(H_1; a_1, b_1)$ and $(H_2; a_2, b_2)$ by a series operation, then

$$A((H; a, b)) = \{i+j \pmod{12} : i \in A((H_1; a_1, b_1)), j \in A((H_2; a_2, b_2))\}.$$

With these observations, straightforward calculation shows that

$$A((G_1; x_1, y_1)) = \{4, 8\}$$

and

$$A((G; x, y)) = \emptyset.$$

This means that G is not $(12, 5)$-colorable.

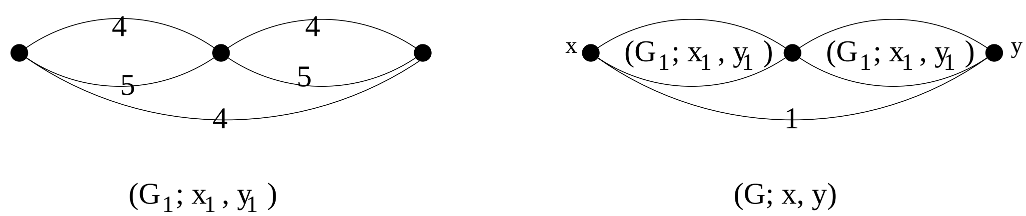

Figure 5: A series-parallel graph og girth 9 and circular chromatic number 16/7

We have actually proved the sharpness of the upper bound for $g(k)$ in Corollary 3.1 for more values of k, based on similar constructions. It would be nice to construct examples for general k.

Addendum: After the first version of this paper was finished, Chien and Zhu [4] have accomplished the general construction, showing that in fact $g(k) = 2\lfloor(3k-1)/2\rfloor$. Thus the bound given in our Theorem 1.1 is the best possible, in the sense that for each integer $k \geq 2$ there exists a series-parallel graph of girth $2\lfloor(3k-1)/2\rfloor - 1$ with $\chi_c(G) > 4k/(2k-1)$.

References

[1] H. L. Abbott and B. Zhou, *The star chromatic number of a graph*, J. Graph Theory **17** (1993), 349-360.

[2] J. A. Bondy and P. Hell, *A note on the star chromatic number*, J. Graph Theory **14** (1990), 479-482.

[3] P. Erős, *Graph theory and probability*, Canad. J. Math. 11(1959), 34-38.

[4] C. Chien and X. Zhu, *The circular chromatic number of series-parallel graphs with large girth*, manuscript, 1999.

[5] M. Grötschel, L. Lovász and A. Schrijver, **Geometric Algorithms and Combinatorial Optimization**, Second edition, Springer-Verlag, pp. 274, 1993.

[6] D. R. Guichard, *Acyclic graph coloring and the complexity of the star chromatic number*, J. Graph Theory 17 (1993), 129-134.

[7] D. Moser, *The star-chromatic number of planar graphs*, J. Graph Theory 24(1997), 33-43.

[8] J. Nešetřil and X. Zhu, *On bounded treewidth daulity of graphs*, J. Graph Theory 23 (1996), 151-162.

[9] E. Steffen and X. Zhu, *On the star chromatic numbers of graphs*, Combinatorica, 16 (1996) 439-448.

[10] A. Vince, *Star chromatic number*, J. Graph Theory 12 (1988), 551-559.

[11] X. Zhu, *Star chromatic numbers and products of graphs*, J. Graph Theory **16** (1992), 557-569.

[12] X. Zhu, *Uniquely H-colorable graphs with large girth*, J. Graph Theory, **23** (1996), 33-41.

[13] X. Zhu, *Planar graphs with circular chromatic numbers between* 3 *and* 4, J. Combinatorial Theory Series B, to appear.

[14] X. Zhu, *Circular chromatic number and graph minor*, preprint, 1997.

[15] X. Zhu, *Circular chromatic number, a survey*, preprint, 1997.