The circular chromatic number of series-parallel graphs with large girth

Chihyun Chien and Xuding Zhu
Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung, Taiwan 80424
Email: chiency, zhu@math.nsysu.edu.tw

Abstract

It was proved by Hell and Zhu that if G is a series-parallel graph of girth at least $2\lfloor (3k-1)/2 \rfloor$, then $\chi_c(G) \leq 4k/(2k-1)$. In this paper, we prove that the girth requirement is sharp, i.e., for any $k \geq 2$, there is a series-parallel graph G of girth $2\lfloor (3k-1)/2 \rfloor - 1$ such that $\chi_c(G) > 4k/(2k-1)$.

1 Introduction

The circular chromatic number (also called the star chromatic number) of a graph is a natural generalization of the notion of chromatic number of a graph. It was introduced by A. Vince [13] in 1988, and has attracted considerable attention since then [1, 2, 6, 7, 8, 11, 14, 18, 19, 20, 21]. Given two integers k, d, such that $k \geq d$, a (k, d)-coloring of a graph G is a coloring c of the vertices of G with colors $0, 1, 2, \cdots, k-1$ such that for any two adjacent vertices x and y of G we have $d \leq |c(x)-c(y)| \leq k-d$. The *circular chromatic number* $\chi_c(G)$ of G is defined as the infimum of the ratio k/d for which there exists a (k, d)-coloring of G. Note that any non-trivial graph has circular chromatic number at least 2. It was shown by Vince [13] (cf. also [2, 14] for combinatorial proofs) that if G is finite then the infimum in this definition is always attained, and hence can be replaced by the minimum.

Note that a $(k, 1)$-coloring of a graph G is just an ordinary k-coloring of G. Therefore we have $\chi_c(G) \leq \chi(G)$ for any graph G. On the other hand, it was shown in [13] that $\chi_c(G) > \chi(G) - 1$. Hence $\chi(G) = \lceil \chi_c(G) \rceil$. In this sense, for a graph G, the circular chromatic number $\chi_c(G)$ is a refinement of the chromatic number $\chi(G)$, and $\chi(G)$ is an approximation of $\chi_c(G)$.

Since the infimum can be replaced by the minimum, the circular chromatic number of a finite graph is always rational. On the other hand, it was shown in [13] that for any rational number $r \geq 2$, there exists a finite graph G of circular chromatic number r. Even more is true: it was shown by Zhu [15] that for any integer g and any rational number $r \geq 2$, there exists a graph G which has girth at least g and circular chromatic number r (also see [16] for a discussion of construction of such graphs). This result is a generalization of the result of Erdős concerning the existence of graphs with arbitrarily large girth and chromatic number [5].

A graph H is called a *minor* of a graph G if H is isomorphic to a graph obtained from a subgraph of G by contracting edges. We say a graph G is *H-minor free* if H is not a minor of G. It is well-known that if restricted to graphs which are H-minor free for a fixed graph H, then the chromatic number will be "small" when the girth is "large".

If we consider the circular chromatic number instead of the chromatic number, an H-minor free graph of girth "large enough" will have circular chromatic number "close to" 2: This can be proved by using the following result of Thomassen [12]: *For any finite graph H, there is an integer $f(H)$ such that every graph of minimum degree 3 and girth at least $f(H)$ contains H as a minor.* Indeed, it can be shown, by induction on the number of vertices, that any H-minor free graph G of girth at least $2kf(H)$ admits a homomorphism to C_{2k+1}, and hence has circular chromatic number at most $2+1/k$. Suppose G is H-minor free and has girth at least $2kf(H)$. Let G' be obtained from G by repeatedly deleting all degree 1 vertices (so that G' has no vertices of degree 1), and let G'' be obtained from G' by suppressing all vertices of degree 2. Then G'' is H-minor free and has minimum degree ≥ 3. Hence by the above result of Thomassen, G'' has a cycle of length $< f(H)$ (unless G is a tree and hence G' the empty graph, which is a trivial case). As G' has girth $\geq 2kf(H)$, we conclude that G' has an induced path P of length $2k$. Now by the induction hypothesis, $G' - P$ admits a homomorphism to C_{2k+1}, and such a homomorphism can be easily extended to the induced path P, and further extended to all the vertices of G. (In [9], a stronger result was proved: If G has large girth, then any H-minor free graph G' which admits an homomorphism to G has circular chromatic number close to 2.)

As graphs embeddable in a fixed surface forbids certain finite graph as a minor, it follows that for any surface S, for any $\epsilon > 0$, there is an integer $g(S, \epsilon)$ such that any graph embeddable in S and with girth at least $g(S, \epsilon)$ has circular chromatic number at most $2 + \epsilon$.

We note that for any fixed graph H (respectively, for any fixed surface S), and for any $\epsilon > 0$ the above discussion can be carried out in such a way that actually produces an integer g such that any H-minor free graph (respectively, any graph embeddable in S) of girth at least g has circular chromatic number at most $2+\epsilon$. However, such a derived integer g is usually far from tight. Recently, Hell et al [3] proved that every planar graph of girth at least $10k-3$ has circular chromatic number at most $2+1/k$ ($k \geq 1$). This result has been improved by Klostermeyer and Zhang [7] who showed that any planar graph of odd girth at least $10k-3$ has circular chromatic number at most $2+1/k$ ($k \geq 1$). However, the result also seems far from sharp, in the sense of girth vs. circular chromatic number of planar graphs.

Given an $\epsilon > 0$, it seems to be difficult to find the smallest integer $g = g(\epsilon)$ such that any H-minor free graph (or any graph embeddable in a surface S) of girth $\geq g$ has circular chromatic number at most $2+\epsilon$. Nevertheless, we shall prove in this paper that the following result proven in [4] is sharp, in the sense of girth vs. circular chromatic number of K_4-minor free graphs:

Theorem 1 ([4]) *If G is a K_4-minor free graph of girth at least $2\lfloor(3k-1)/2\rfloor$, then $\chi_c(G) \leq 4k/(2k-1)$.*

To be precise, we shall prove the following result (which maybe viewed as a complement of Theorem 1):

Theorem 2 *For any $k \geq 2$, there is a K_4-minor free graph G of girth $2\lfloor(3k-1)/2\rfloor - 1$ such that $\chi_c(G) > 4k/(2k-1)$.*

2 Preliminaries

It is well known that a graph G is K_4-minor free if and only if each block of G is a series-parallel graph. Therefore to prove Theorem 2, it suffices to deal with series-parallel graphs.

In the remaining of this paper, we assume that $k \geq 2$ is a fixed integer. Let $p = \lfloor(3k-1)/2\rfloor$. We shall construct a series-parallel graph G of girth $2p-1$ such that G is not $(4k, 2k-1)$-colorable.

First we recall the following definition from [4]. A two-terminal series-parallel graph $(G; x, y)$ is defined recursively as follows:

- Let $V(K_2) = \{0, 1\}$. Then $(K_2; 0, 1)$ is a two-terminal series-parallel graph.

- (The parallel construction.) Let $(G;x,y)$ and $(G';x',y')$ be two disjoint two-terminal series-parallel graphs. Define G'' to be the graph obtained from the union of G and G' by identifying x and x' into a single vertex x'', and identifying y and y' into a single vertex y''. Then $(G'';x'',y'')$ is a two-terminal series-parallel graph.
- (The series construction.) Let again $(G;x,y)$ and $(G';x',y')$ be two disjoint two-terminal series-parallel graphs. Define G'' to be the graph obtained from the union of G and G' by identifying y and x' into a single vertex. Then $(G'';x,y')$ is a two-terminal series-parallel graph.
- There are no other two-terminal series-parallel graphs.

Note that $(G;x,y)$ may be a two-terminal series-parallel graph for some pairs x,y and not for others. A graph G is a series-parallel graph if there exist some two vertices x,y such that $(G;x,y)$ is a two-terminal series-parallel graph. We call the distance in G between the two terminals x,y, the *length* of $(G;x,y)$. If the two terminals are understood, or are of no significance, we shall write G for $(G;x,y)$.

To prove Theorem 2, we shall construct a series-parallel graph of girth $2p-1$ which is not $(4k,2k-1)$-colorable. To maintain that the constructed series-parallel graphs have girth $\geq 2p-1$, we shall only do the parallel constructions for those pairs of graphs G and G' such that the sum of the lengths of the two graphs is at least $2p-1$.

For a two terminal series-parallel graph $(G;x,y)$ we shall denote by $L(G)$ the set of colors j for which there is a $(4k,2k-1)$-coloring c of G such that $c(x)=0$ and $c(y)=j$. Note that if $(G;x,y)$ has a $(4k,2k-1)$-coloring, then there is such a coloring c with $c(x)=0$. Therefore $L(G)=\emptyset$ if and only if G is not $(4k,2k-1)$-colorable. We shall call $L(G)$ *the color set* of G.

Our construction is as follows: Starting from a single edge, we construct a sequence of two-terminal series-parallel graphs G of girth $\geq 2p-1$. At each step, we shall determine $L(G)$ for each constructed two-terminal series-parallel graph G. Once we obtain a two-terminal series-parallel graph G with $L(G)=\emptyset$, then G is the graph we wanted, i.e., G is a series-parallel graph of girth at least $2p-1$ (and hence of girth exactly $2p-1$, by Theorem 1) which is not $(4k,2k-1)$-colorable.

We shall denote by I the set of all colors, i.e., $I=\{0,1,\cdots,4k-1\}$. For two subsets A,B of I, we define $A+B$ as

$$A+B=\{i+j \pmod{4k} : i\in A, j\in B\}.$$

The following three lemmas are straightforward.

Lemma 1 *If G^* is obtained from G and G' by a series construction, then the length of G^* is the sum of the lengths of G and G', and that $L(G^*) = L(G) + L(G')$.*

Lemma 2 *If G^* is obtained from G and G' by a parallel construction, then the length of G^* is equal to the minimum of the lengths of G and G', and that $L(G^*) = L(G) \cap L(G')$.*

Lemma 3 $L(K_2) = \{2k-1, 2k, 2k+1\}$.

As each two terminal series-parallel graph G is constructed from K_2 by a sequence of series and parallel constructions, it is straightforward, although tedious sometime, to determine $L(G)$ for any two-terminal series-parallel graph G, by using the three lemmas above.

In the remaining, whenever we construct a new series-parallel graph G^* from two old series-parallel graphs G and G' (either by the series construction or the parallel construction), all we care about is the length of G^* and the color set $L(G^*)$. The inner structure of G^* is of no importance to us. By Lemmas 1 and 2, such information can be obtained from the knowledge of the lengths of G and G', and from $L(G)$ and $L(G')$. Therefore, instead of constructing the series-parallel graph G^* from G and G', we can simply "construct" the set $L(G^*)$ from $L(G)$ and $L(G')$, and record the lengths of the corresponding series-parallel graphs.

Definition 1 *For $i \geq 1$, a subset S of I is called* i-constructible, *if there is a series-parallel graph G of girth at least $2p-1$ which has length i and for which $L(G) = S$.*

By applying Lemmas 1 and 2, we have the following corollary:

Corollary 1 *Suppose A is i-constructible and B is j-constructible. Let $C = A + B$. Then C is $(i+j)$-constructible. Moreover, if $i \leq j$ and $i + j \geq 2p-1$, and $D = A \cap B$, then D is i-constructible.*

As noted above, our goal is to show that the emptyset is i-constructible for some i (and hence for all i), which means that the corresponding two-terminal series-parallel graph has girth at least $2p-1$ and which is not $(4k, 2k-1)$-colorable. We shall repeatedly calculate the sums of two color sets, and the intersections of two color sets. Lemma 4 below, quoted from [4], illustrates the result of adding two color sets each of which is an "interval" of colors.

We shall view the numbers in I as cyclically ordered, and all the additions of the elements of I are carried out modulo $4k$, unless otherwise specified. For two elements a, b of I, we shall denote by $[a, b]$ the set $\{a, a+1, \cdots, b\}$. For example, $[2,5] = \{2,3,4,5\}$ and $[5,2] = \{5,6,\cdots,4k-1,0,1,2\}$.

Lemma 4 *Suppose $X = [a,b]$ and $Y = [a', b']$.*

- *If $|X| + |Y| \geq 4k+1$, then $X + Y = I$;*
- *If $|X| + |Y| \leq 4k$, then $X + Y = [a + a', b + b']$.*

Here the sum $|X| + |Y|$ is the ordinary sum, and the sum $a + a', b + b'$ are carried out modulo $4k$.

3 The proof of Theorem 2

We shall construct, for some integers i, sequences of smaller and smaller i-constructible sets, and eventually show that the emptyset is i-constructible for some integer i (and hence for all i). We start with the simplest series-parallel graphs, i.e., paths.

Lemma 5 *For $i \geq 1$, let I_i be the sets defined as follows:*

1. *If $i \leq 2k-1$ is odd, then $I_i = [2k-i, 2k+i]$.*
2. *If $i \leq 2k-1$ i even, then $I_i = [4k-i, i]$.*
3. *For $i \geq 2k$, $I_i = I$.*

Then I_i is i-constructible.

Proof. The case $i = 1$ follows from Lemma 3. For $i \geq 2$, it is easy to verify that $I_i = I_{i-1} + I_1$. Therefore, by Corollary 1, and by induction on i, I_i is i-constructible. ∎

Indeed, I_i is just the color set of the path of length i.

For the remaining part, the construction methods will be slightly different for different values of $k \pmod 4$. We consider four cases.

Case 1 $k \equiv 1 \pmod 4$.

Assume that $k = 4n + 1$.

Lemma 6 *For* $1 \le m \le \log_2 n + 1$*, let*

1. $I^{(m)}_{6n+1} = [2n + 2^{m-1}, (6n + 2)] \cup [(10n + 2), (14n + 4) - 2^{m-1}]$,

2. $I^{(m)}_{6n} = [2n+2^{m-1}, (6n+1)-2^{m-1}] \cup [(10n+3)+2^{m-1}, (14n+4)-2^{m-1}]$,

3. $I^{(m)}_{4n} = [2^m - 1, (4n + 1) - 2^m] \cup [(12n + 3) + 2^m, (16n + 5) - 2^m]$,

4. $I^{(m)}_{4n+1} = [4n + 2^m, (8n + 2) - 2^m] \cup [(8n + 2) + 2^m, (12n + 4) - 2^m]$,

5. $I^{(m)}_{8n+1} = [2^{m+1} - 1, (16n + 5) - 2^{m+1}]$,

6. $I^{(m)}_{8n} = [2^{m+1} - 1, (8n + 2) - 2^{m+1}] \cup [(8n + 2) + 2^{m+1}, (16n + 5) - 2^{m+1}]$.

The sets $I^{(j)}_i$ *defined above are i-constructible.*

Proof. We prove Lemma 6 by induction on m. Note that since $k = 4n + 1$, we have $p = 6n + 1$. We can do parallel construction to two two-terminal series-parallel graphs G and G' if and only if the sum of the lengths of G and G' is at least $2p - 1 = 12n + 1$.

If $m = 1$, let $A = I_{4n} \cap I_{8n+1}$, then straightforward calculation shows that

$$A = [1, 4n] \cup [12n + 4, 16n + 3].$$

Since I_{4n} and I_{8n+1} are $4n$-constructible and $(8n + 1)$-constructible, respectively (by Lemma 5), and since $4n + 8n + 1 = 12n + 1 = 2p - 1$, it follows from Corollary 1 that A is $4n$-constructible.

Let $B = I_{4n+1} \cap I_{8n}$, then

$$B = [4n + 1, 8n] \cup [8n + 4, 12n + 3],$$

and similarly as above, B is $(4n+1)$-constructible.

Let $I_{6n+1}^{(1)} = I_{6n+1} \cap I_{6n+2}$, then straightforward calculation shows that

$$I_{6n+1}^{(1)} = [2n+1, 6n+2] \cup [10n+2, 14n+3],$$

and similarly as above, $I_{6n+1}^{(1)}$ is $(6n+1)$-constructible.

Let $I_{6n}^{(1)} = I_{6n} \cap I_{6n+1}$, then

$$I_{6n}^{(1)} = [2n+1, 6n] \cup [10n+4, 14n+3],$$

and similarly, $I_{6n}^{(1)}$ is $6n$-constructible.

Let $C = I_{6n}^{(1)} + I_{6n}^{(1)}$, then

$$C = [4n+2, 12n+2] \cup [12n+5, 4n-1],$$

and C is $12n$-constructible.

Let $I_{4n}^{(1)} = A \cap C$, then

$$I_{4n}^{(1)} = [1, 4n-1] \cup [12n+5, 16n+3],$$

and $I_{4n}^{(1)}$ is $4n$-constructible (because A is $4n$-constructible, C is $12n$-constructible, and $4n + 12n \geq 2p - 1$).

Let $I_{4n+1}^{(1)} = B \cap C$, then

$$I_{4n+1}^{(1)} = [4n+2, 8n] \cup [8n+4, 12n+2],$$

and similarly as above, $I_{4n+1}^{(1)}$ is $(4n+1)$-constructible.

Let $I_{8n+1}^{(1)} = (I_{4n}^{(1)} + I_{4n+1}^{(1)})$, then

$$I_{8n+1}^{(1)} = [3, 16n+1],$$

and $I_{8n+1}^{(1)}$ is $(8n+1)$-constructible, because $I_{4n}^{(1)} + I_{4n+1}^{(1)}$ is $(8n+1)$-constructible (by Corollary 1) and I_{8n+1} is also $(8n+1)$-constructible.

Let $I_{8n}^{(1)} = (I_{4n}^{(1)} + I_{4n}^{(1)}) \cap I_{8n+1}^{(1)}$, then

$$I_{8n}^{(1)} = [3, 8n-2] \cup [8n+6, 16n+1],$$

and similarly as above, $I_{8n}^{(1)}$ is $8n$-constructible.

This completes the proof of the $m = 1$ case of Lemma 6.

Suppose Lemma 6 is true for $m = i$, i.e., the sets $I_{6n+1}^{(i)}$, $I_{6n}^{(i)}$, $I_{4n}^{(i)}$, $I_{4n+1}^{(i)}$, $I_{8n+1}^{(i)}$, $I_{8n}^{(i)}$ defined as in Lemma 6 are $(6n+1)$-, $6n$-, $4n$-, $(4n+1)$-, $(8n+1)$- $8n$-constructible, respectively. Assume $i + 1 \leq \log_2 n + 1$. We shall use the induction hypothesis to prove that $I_{6n+1}^{(i+1)}$, $I_{6n}^{(i+1)}$, $I_{4n}^{(i+1)}$, $I_{4n+1}^{(i+1)}$, $I_{8n+1}^{(i+1)}$, $I_{8n}^{(i+1)}$ defined as in Lemma 6 are $(6n+1)$-, $6n$-, $12n$-, $4n$-, $(4n+1)$-, $(8n+1)$-, $8n$-constructible, respectively.

By the induction hypothesis,

$$I_{4n}^{(i)} = [2^i - 1, (4n+1) - 2^i] \cup [(12n+3) + 2^i, (16n+5) - 2^i]$$

$$I_{8n+1}^{(i)} = [2^{i+1} - 1, (16n+5) - 2^{i+1}]$$

are $4n$- and $(8n+1)$-constructible, respectively. Let $X = I_{4n}^{(i)} \cap I_{8n+1}^{(i)}$ Since $i + 1 \leq \log_2 n + 1$, we have $2^i \leq n$. This implies that

$$((4n+1) - 2^i) - (2^{i+1} - 1) = (4n+2) - 3 \times 2^i \geq 0,$$

$$((16n+5) - 2^{i+1}) - ((12n+3) + 2^i) = (4n+2) - 3 \times 2^i \geq 0.$$

So

$$2^i - 1 \leq 2^{i+1} - 1 \leq 4n+1) - 2^i,$$
$$(12n+3) + 2^i \leq (16n+5) - 2^{i+1} \leq (16n+5) - 2^i.$$

Therefore

$$X = I_{4n}^{(i)} \cap I_{8n+1}^{(i)} = [2^{i+1} - 1, (4n+1) - 2^i] \cup [(12n+3) + 2^i, (16n+5) - 2^{i+1}].$$

Similarly,

$$Y = I_{4n+1}^{(i)} \cap I_{8n}^{(i)} = [4n + 2^i, (8n+2) - 2^{i+1}] \cup [(8n+2) + 2^{i+1}, (12n+4) - 2^i].$$

By Corollary 1, X and Y are $4n$-, $(4n+1)$-constructible, respectively.

Let $I_{6n+1}^{(i+1)} = (Y + I_{2n}) \cap I_{6n+1}^{(i)}$. Firstly, by applying lemma 4, we have

$$Y + I_{2n} = [2n + 2^i, (10n+2) - 2^{i+1}] \cup [(6n+2) + 2^{i+1}, (14n+4) - 2^i].$$

Since $((10n+2) - 2^{i+1}) - ((6n+2) + 2^{i+1}) = 4n - 2^{i+2} \geq 0$ (as $i+1 \leq \log_2 n + 1$), the two intervals in the formula above intersect, and hence their union is a single interval, i.e.,

$$Y + I_{2n} = [2n + 2^i, (14n+4) - 2^i].$$

Similarly as the argument in the previous paragraph, we conclude that

$$(Y + I_{2n}) \cap I_{6n+1}^{(i)} = [2n + 2^i, (6n+2)] \cup [(10n+2), (14n+4) - 2^i].$$

By Corollary 1, $I_{6n+1}^{(i+1)} = (Y + I_{2n}) \cap I_{6n+1}^{(i)}$ is $(6n+1)$-constructible.

Let $I_{6n}^{(i+1)} = (X + I_{2n}) \cap I_{6n+1}^{(i+1)}$. Then by the same argument as above

$$I_{6n}^{(i+1)} = [2n + 2^i, (6n+1) - 2^i] \cup [(10n+3) + 2^i, (14n+4) - 2^i]$$

and $I_{6n}^{(i+1)}$ is $6n$-constructible, because $X + I_{2n}$ is $6n$-constructible, and $I_{6n+1}^{(i+1)}$ is $(6n+1)$-constructible.

Let $Z = I_{6n}^{(i+1)} + I_{6n}^{(i+1)}$. Then

$$Z = [4n + 2^{i+1}, (12n+4) - 2^{i+1}] \cup [(12n+3) + 2^{i+1}, (4n+1) - 2^{i+1}]$$

and Z is $12n$-constructible by Corollary 1.

Let $I_{4n}^{(i+1)} = X \cap Z$. Then

$$I_{4n}^{(i+1)} = [2^{i+1} - 1, (4n+1) - 2^{i+1}] \cup [(12n+3) + 2^{i+1}, (16n+5) - 2^{i+1})]$$

and $I_{4n}^{(i+1)}$ is $4n$-constructible, because X is $4n$-constructible, and Z is $12n$-constructible.

Let $I_{(4n+1)}^{(i+1)} = Y \cap Z$. Then

$$I_{(4n+1)}^{(i+1)} = [4n + 2^{i+1}, (8n+2) - 2^{i+1}] \cup [(8n+2) + 2^{i+1}, (12n+4) - 2^{i+1}]$$

and $I_{(4n+1)}^{(i+1)}$ is $(4n+1)$-constructible, because Y is $(4n+1)$-constructible, and Z is $12n$-constructible.

Let $I_{8n+1}^{(i+1)} = I_{4n+1}^{(i+1)} + I_{4n}^{(i+1)}$. Then

$$I_{8n+1}^{(i+1)} = [2^{i+2} - 1, (16n+5) - 2^{i+2}]$$

and $I_{8n+1}^{(i+1)}$ is $(8n+1)$-constructible, because $I_{4n+1}^{(i+1)} + I_{4n}^{(i+1)}$ and $I_{8n+1}^{(i)}$ are both $(8n+1)$-constructible.

Let $I_{8n}^{(i+1)} = (I_{4n}^{(i+1)} + I_{4n}^{(i+1)}) \cap I_{8n+1}^{(i+1)}$. Then

$$I_{8n}^{(i+1)} = [2^{i+2} - 1, (8n+2) - 2^{i+2}] \cup [(8n+2) + 2^{i+2}, (16n+5) - 2^{i+2}]$$

and $I_{8n}^{(i+1)}$ is $8n$-constructible, because $I_{4n}^{(i+1)} + I_{4n}^{(i+1)}$ and $I_{8n}^{(i)}$ are both $8n$-constructible, and $I_{8n+1}^{(i+1)}$ is $(8n+1)$-constructible. This completes the proof of Lemma 6. ∎

Case 2 $k \equiv 2 \pmod 4$.

Assume that $k = 4n + 2$.

Lemma 7 *For $1 \le m \le \log_2 n + 1$, let*

1. $I_{6n+2}^{(m)} = [(2n+1), (6n+3) - 2^{m-1}] \cup [(10n+5) + 2^{m-1}, (14n+7)]$,

2. $I_{6n+1}^{(m)} = [(2n+2) + 2^{m-1}, (6n+3) - 2^{m-1}] \cup [(10n+5) + 2^{m-1}, (14n+6) - 2^{m-1}]$,

3. $I_{4n}^{(m)} = [2^{m-1}, (4n+1) - 2^m] \cup [(12n+7) + 2^m, (16n+8) - 2^{m-1}]$,

4. $I_{4n+1}^{(m)} = [(4n+2) + 2^m, (8n+4) - 2^m] \cup [(8n+4) + 2^m, (12n+6) - 2^m]$,

5. $I_{8n+3}^{(m)} = [2^m, (16n+8) - 2^m]$,

6. $I_{8n+2}^{(m)} = [2^m, (8n+4) - 2^{m+1}] \cup [(8n+4) + 2^{m+1}, (16n+8) - 2^m]$.

The sets $I_i^{(j)}$ defined above are i-constructible.

Proof. The proof of Lemma 7 is similar to the proof of Lemma 6. We shall only give the rules of how to construct the sets $I_i^{(1)}$, and how to construct the sets $I_i^{(j+1)}$ from the sets $I_i^{(j)}$. Since $k = 4n+2$, we have $p = 6n+2$, hence $2p-1 = 12n+3$. Therefore, we can do parallel construction to two-terminal series-parallel graphs G and G' if and only if the sum of the lengths of the two graphs is at least $12n+3$.

The sets $I_i^{(1)}$ are constructed as follows:

1. $I_{6n+2}^{(1)} = I_{6n+2} \cap I_{6n+3}$,

2. $I_{6n+1}^{(1)} = I_{6n+1} \cap I_{6n+2}$,

3. $I_{4n}^{(1)} = (I_{6n+1}^{(1)} + I_{6n+1}^{(1)}) \cap I_{4n} \cap I_{8n+3}$,

4. $I_{4n+1}^{(1)} = (I_{6n+1}^{(1)} + I_{6n+1}^{(1)}) \cap I_{4n+1} \cap I_{8n+2}$,

5. $I^{(1)}_{8n+3} = I^{(1)}_{4n+1} + I_{4n+2}$,

6. $I^{(1)}_{8n+2} = (I^{(1)}_{4n+1} + I^{(1)}_{4n+1}) \cap I^{(1)}_{8n+3}$.

The sets $I_i^{(j+1)}$ are constructed from the sets $I_i^{(j)}$ as follows:

1. $I^{(j+1)}_{6n+2} = ((I^{(j)}_{4n+1} \cap I^{(j)}_{8n+2}) + I_{2n+1}) \cap I^{(j)}_{6n+2}$,

2. $I^{(j+1)}_{6n+1} = ((I^{(j)}_{4n} \cap I^{(j)}_{8n+3}) + I_{2n+1}) \cap I^{(j+1)}_{6n+2}$,

3. $I^{(j+1)}_{4n} = (I^{(j+1)}_{6n+1} + I^{(j+1)}_{6n+1}) \cap I^{(j)}_{4n} \cap I^{(j)}_{8n+3}$,

4. $I^{(j+1)}_{4n+1} = (I^{(j+1)}_{6n+1} + I^{(j+1)}_{6n+1}) \cap I^{(j)}_{4n+1} \cap I^{(j)}_{8n+2}$,

5. $I^{(j+1)}_{8n+3} = I^{(j+1)}_{4n+1} + I_{4n+2}$,

6. $I^{(j+1)}_{8n+2} = (I^{(j+1)}_{4n+1} + I^{(j+1)}_{4n+1}) \cap I^{(j+1)}_{8n+3}$.

∎

Case 3 $k \equiv 3 \pmod 4$.

Assume that $k = 4n + 3$.

Lemma 8 *For* $1 \le m \le \log_2(n+1) + 1$, *let*

1. $I^{(m)}_{6n+4} = [(2n+1), (6n+5) - 2^{m-1}] \cup [(10n+7) + 2^{m-1}, (14n+11)]$,

2. $I^{(m)}_{6n+3} = [(2n+2) + 2^{m-1}, (6n+5) - 2^{m-1}] \cup [(10n+7) + 2^{m-1}, (14n+10) - 2^{m-1}]$,

3. $I^{(m)}_{4n+2} = [2^m - 1, (4n+3) - 2^m] \cup [(12n+9) + 2^m, (16n+13) - 2^m]$, *4.* $I^{(m)}_{4n+3} = [(4n+2)+2^m, (8n+6)-2^m] \cup [(8n+6)+2^m, (12n+10)-2^m]$,

5. $I^{(m)}_{8n+5} = [2^{m+1}-1, (16n+13)-2^{m+1}]$,

6. $I^{(m)}_{8n+4} = [2^{m+1}-1, (8n+6)-2^{m+1}] \cup [(8n+6)+2^{m+1}, (16n+13)-2^{m+1}]$.

The sets $I^{(j)}_i$ defined above are i-constructible.

Proof. Similarly, we shall only give the rules of how to construct the sets $I^{(1)}_i$, and how to construct the sets $I^{(j+1)}_i$ from the sets $I^{(j)}_i$. Since $k = 4n+3$, we have $p = 6n+4$, hence $2p-1 = 12n+7$. Therefore, we can do parallel construction to two-terminal series-parallel graphs G and G' if and only if the sum of the lengths of the two graphs is at least $12n+7$.

The sets $I^{(1)}_i$ are constructed as follows:

1. $I^{(1)}_{6n+4} = I_{6n+4} \cap I_{6n+5}$,

2. $I^{(1)}_{6n+3} = I_{6n+3} \cap I_{6n+4}$,

3. $I^{(1)}_{4n+2} = (I^{(1)}_{6n+3} + I^{(1)}_{6n+3}) \cap I_{4n+2} \cap I_{8n+5}$,

4. $I^{(1)}_{4n+3} = (I^{(1)}_{6n+3} + I^{(1)}_{6n+3}) \cap I_{4n+3} \cap I_{8n+4}$,

5. $I^{(1)}_{8n+5} = I^{(1)}_{4n+2} + I^{(1)}_{4n+3}$,

6. $I^{(1)}_{8n+4} = (I^{(1)}_{4n+2} + I^{(1)}_{4n+2}) \cap I^{(1)}_{8n+5}$.

The sets $I^{(j+1)}_i$ are constructed from the sets $I^{(j)}_i$ as follows:

1. $I^{(j+1)}_{6n+4} = ((I^{(j)}_{4n+3} \cap I^{(j)}_{8n+4}) + I_{2n+1}) \cap I^{(j)}_{6n+4}$,

2. $I^{(j+1)}_{6n+3} = ((I^{(j)}_{4n+2} \cap I^{(j)}_{8n+5}) + I_{2n+1}) \cap I^{(j+1)}_{6n+4}$,

3. $I_{4n+2}^{(j+1)} = (I_{6n+3}^{(j+1)} + I_{6n+3}^{(j+1)}) \cap I_{4n+2}^{(j)} \cap I_{8n+5}^{(j)}$,

4. $I_{4n+3}^{(j+1)} = (I_{6n+3}^{(j+1)} + I_{6n+3}^{(j+1)}) \cap I_{4n+3}^{(j)} \cap I_{8n+4}^{(j)}$,

5. $I_{8n+5}^{(j+1)} = I_{4n+2}^{(j+1)} + I_{4n+3}^{(j+1)}$,

6. $I_{8n+4}^{(j+1)} = (I_{4n+2}^{(j+1)} + I_{4n+2}^{(j+1)}) \cap I_{8n+5}^{(j+1)}$.

$\blacksquare$

Case 4 $k \equiv 0 \pmod 4$.

Assume that $k = 4n$.

Lemma 9 *For* $1 \le m \le \log_2 n + 1$, *let*

1. $I_{6n-1}^{(m)} = [2n + 2^{m-1}, 6n] \cup [10n, 14n - 2^{m-1}]$,

2. $I_{6n-2}^{(m)} = [2n + 2^{m-1}, (6n-1) - 2^{m-1}] \cup [(10n+1) + 2^{m-1}, 14n - 2^{m-1}]$,

3. $I_{4n-2}^{(m)} = [2^{m-1}, (4n-1) - 2^m] \cup [(12n+1) + 2^m, 16n - 2^{m-1}]$,

4. $I_{4n-1}^{(m)} = [4n + 2^m, 8n - 2^m] \cup [8n + 2^m, 12n - 2^m]$,

5. $I_{8n-1}^{(m)} = [2^m, 16n - 2^m]$,

6. $I_{8n-2}^{(m)} = [2^m, 8n - 2^{m+1}] \cup [8n + 2^{m+1}, 16n - 2^m]$.

The sets $I_i^{(j)}$ *defined above are i-constructible.*

Proof. Similarly, we shall only give the rules of how to construct the sets $I_i^{(1)}$, and how to construct the sets $I_i^{(j+1)}$ from the sets $I_i^{(j)}$. Since $k = 4n$, we have $p = 6n - 1$, hence $2p - 1 = 12n - 3$. Therefore, we can do parallel construction to two-terminal series-parallel graphs G and G' if and only if the sum of the lengths of the two graphs is at least $12n - 3$.

The sets $I_i^{(1)}$ are constructed as follows:

1. $I^{(1)}_{6n-1} = I_{6n-1} \cap I_{6n}$,

2. $I^{(1)}_{6n-2} = I_{6n-2} \cap I_{6n-1}$,

3. $I^{(1)}_{4n-2} = (I^{(1)}_{6n-2} + I^{(1)}_{6n-2}) \cap I_{4n-2} \cap I_{8n-1}$,

4. $I^{(1)}_{4n-1} = (I^{(1)}_{6n-2} + I^{(1)}_{6n-2}) \cap I_{4n-1} \cap I_{8n-2}$,

5. $I^{(1)}_{8n-1} = I^{(1)}_{4n-1} + I_{4n}$,

6. $I^{(1)}_{8n-2} = (I^{(1)}_{4n-1} + I^{(1)}_{4n-1}) \cap I^{(1)}_{8n-1}$.

The sets $I_i^{(j+1)}$ are constructed from the sets $I_i^{(j)}$ as follows:

1. $I^{(j+1)}_{6n-1} = ((I^{(j)}_{4n-1} \cap I^{(j)}_{8n-2}) + I_{2n}) \cap I^{(j)}_{6n-1}$,

2. $I^{(j+1)}_{6n-2} = ((I^{(j)}_{4n-2} \cap I^{(j)}_{8n-1}) + I_{2n}) \cap I^{(j+1)}_{6n-1}$,

3. $I^{(j+1)}_{4n-2} = (I^{(j+1)}_{6n-2} + I^{(j+1)}_{6n-2}) \cap I^{(j)}_{4n-2} \cap I^{(j)}_{8n-1}$,

4. $I^{(j+1)}_{4n-1} = (I^{(j+1)}_{6n-2} + I^{(j+1)}_{6n-2}) \cap I^{(j)}_{4n-1} \cap I^{(j)}_{8n-2}$,

5. $I^{(j+1)}_{8n-1} = I^{(j+1)}_{4n-1} + I_{4n}$,

6. $I^{(j+1)}_{8n-2} = (I^{(j+1)}_{4n-1} + I^{(j+1)}_{4n-1}) \cap I^{(j+1)}_{8n-1}$.

∎

Proof of Theorem 2 In each of the four cases above, as m becomes larger and larger, the i-constructible sets become smaller and smaller. This process can continue until for some i, the emptyset becomes i-constructible. In the statement of the four lemmas above, an upper bound for m is given. This is merely because of the difficulty in expressing the emptyset as an interval of the form $[a,b]$. Indeed, by our convention, for any a, b, the interval $[a,b]$ is always nonempty.

Consider the case $k = 4n+1$. Suppose we have carried out the inductive constructions of the sets to the mth step, where $m = \lfloor \log_2 n + 1 \rfloor$ (which implies that $n+1 \leq 2^m \leq 2n$, and hence $2n+2 \leq 2^{m+1} \leq 4n$). Now we may continue in the same way, and try to carry out one more induction step. However, in doing so we shall indeed conclude that the empty set is i-constructible for some i.

Let us follow the construction described in the proof of Lemma 6 for one more step. Let $X = I_{4n}^{(m)} \cap I_{8n+1}^{(m)}$. Then

$$X = [2^{m+1} - 1, (4n+1) - 2^m] \cup [(12n+3) + 2^m, (16n+5) - 2^{m+1}]$$

is $4n$-constructible. Let $Y = I_{4n+1}^{(m)} \cap I_{8n}^{(m)}$, $I_{6n+1}^{(m+1)} = (Y + I_{2n}) \cap I_{6n}^{(m)}$, $I_{6n}^{(m+1)} = (X + I_{2n}) \cap I_{6n+1}^{(m+1)}$, and $Z = I_{6n}^{(m+1)} + I_{6n}^{(m+1)}$. Then $Y, I_{6n+1}^{(m+1)}, I_{6n}^{(m+1)}$ and Z are $(4n+1)$-, $(6n+1)$-, $6n$- and $12n$-constructible, respectively. The same calculation shows that

$$Z = [4n + 2^{m+1}, (12n+4) - 2^{m+1}] \cup [(12n+3) + 2^{m+1}, (4n+1) - 2^{m+1}].$$

Let $I_{4n}^{(m+1)} = X \cap Z$. Then $I_{4n}^{(m+1)}$ is $4n$-constructible. However, by noting that $2^{m+1} \geq 2n+2$, we can conclude that $X \cap Z = \emptyset$. Therefore the empty set is $4n$-constructible. As observed earlier, this means that the corresponding series-parallel graph is not $(4k, 2k-1)$-colorable.

The other cases are similar, and we omit the details. ∎

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