Circular colouring and orientation of graphs

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Abstract

This paper proves that if a graph G has an orientation D such that for each cycle C satisfying $d\ell(C) mod k \in \{1, 2, \cdots, 2d-1\}$ we have $\ell(C)/|C^+| \leq k/d$ and $\ell(C)/|C^-| \leq k/d$, then G has a (k,d)-colouring and hence $\chi_c(G) \leq k/d$. This is a generalization of a result of Tuza [3] concerning the vertex colouring of a graph, and is also a strengthening of a result of Goddyn, Tarsi and Zhang [1] concerning the relation between orientation and circular chromatic number of a graph.

1 Introduction

Let $G = (V, E)$ be a graph, and let D be an orientation of G. For a cycle C of D with a chosen direction of traversal (each cycle has two different directions for traversal), let C^+ be the set of positive edges of C (i.e., whose direction coincide with the direction of the traversal) and let C^- be the set of negative edges of C. Denote by $\ell(C)$ the length of C, i.e., $\ell(C) = |C^+| + |C^-|$. The parameter $\tau(C) = \max\{\ell(C)/|C^-|,\ \ell(C)/|C^+|\}$ measures the "imbalance" of the cycle C. Then $\tau(C) \geq 2$. If $\tau(C) = 2$, then it is perfectly balanced. The bigger is $\tau(C)$ the more imbalanced is the orientation of C. Let $\xi(D) = \max\{\tau(C) : C \text{ is a cycle of } D\}$. Then $\xi(D)$ is a measure of the imbalance of the orientation D of G. A well-knwon result of Minty says that the chromatic number of a graph, in some sense, measures the imbalance of an "optimal" orientation of G. The precise statement is as follows:

Theorem 1 *[2] For any finite graph G,*

$$\chi(G) = \min\{\lceil \xi(D) \rceil : D \text{ is an orientation of } G\}.$$

One may take the equation in Theorem 1 as a definition of the chromatic number of a graph, and treat $\chi(G)$ as a measure of the imbalance of an optimal orientation of G. But in this sense, it is very unnatural that one should take the ceiling of $\xi(D)$ instead of $\xi(D)$ itself in the definition. The only advantage of taking the ceiling function is to get an integer. But why should we forbidden ourselves using non-integer numbers ? Let us take away the ceiling function. What we obtain is the circular chromatic number $\chi_c(G)$ of G:

$$\chi_c(G) = \min\{\xi(D) : D \text{ is an orientation of } G\}.$$

The parameter $\chi_c(G)$ is a refinement of $\chi(G)$ first introduced by Vince in 1988 under the name "star chromatic number" and was denoted by $\chi^*(G)$. There are a few equivalent definitions of $\chi_c(G)$. The equation above (obtained by Goddyn, Tarsi and Zhang [1]) can be treated as one definition. The following definition is the most frequently used one and is the one we shall use in this paper. Suppose $G = (V, E)$ is a graph and $k \geq 2d \geq 1$ are integers. A (k, d)-colouring of G is a mapping $f : V \to \{0, 1, \cdots, k-1\}$ such that for every edge xy of G,

$$d \leq |f(x) - f(y)| \leq k - d.$$

The *circular chromatic number* $\chi_c(G)$ of G is defined as

$$\chi_c(G) = \min\{k/d : \text{ there exists a } (k, d)\text{-colouring of } G\}.$$

The concept of circular chromatic number is a natural generalization of the chromatic number from many different points of view. It has attracted considerable attention in the past decade. Readers are referred to [4] for a survey of research in this area and for other equivalent definitions of $\chi_c(G)$. In this paper, we further explore the relation between circular chromatic number and orientation of a graph. We prove the result stated in the abstract. It generalizes a result of Tuza [3] and a result of Goddyn, Tarsi and Zhang [1], while both results of [3] and of [1] are generalizations of Minty's result mentioned above. The nontrivial direction of Minty's result asserts that if G has an orientation D with $\xi(D) \leq k$ where k is an integer then G is k-colourable. Tuza's result says that to obtain the same conclusion, instead of requiring $\xi(D) \leq k$ which is equivalent to require that $\tau(C) \leq k$ for every cycle C, it suffices to require that $\tau(C) \leq k$ for those cycles C with length $\ell(C) \pmod k = 1$. Goddyn, Tarsi and Zhang's result says that if $\xi(D) \leq k/d$ for some fraction k/d then G is (k,d)-colourable. So Tuza's result aims at obtaining the same conclusion under a weaker condition, while Goddyn, Tarsi and Zhang's result aims at obtaining a more precise conclusion under the same condition. The main result of this paper combines these two features, i.e., a more precise conclusion under a weaker condition.

2 The main result

Theorem 2 *Suppose G has an orientation D such that for each cycle C satisfying $d\ell(C) mod k \in \{1, 2, \cdots, 2d-1\}$ we have $\tau(C) \leq k/d$, then G has a (k,d)-colouring.*

Proof. Tuza's result in [3] is the special case as $d = 1$ of Theorem 2. The proof of Theorem 2 is parallel to Tuza's proof of that special case. Let r be a fixed vertex of D. Each spanning tree T of D is considered as rooted at r. Given such a spanning tree T, we define the weight $w_T(x)$ of a vertex x of D (with respect to T) recursively as follows:

- $w_T(r) = 0$;

- If xy is an edge of T oriented from x to y and $w_T(x)$ has already been defined, then $w_T(y) = w_T(x) - k + d$;

- If xy is an edge of T oriented from x to y and $w_T(y)$ has already been defined, then $w_T(x) = w_T(y) + d$.

Since T is a spanning tree, for each vertex x of D, $w_T(x)$ is uniquely defined. Then we define the weight $w(T)$ of T as

$$w(T) = \Sigma_{x \in V(D)} w_T(x).$$

Choose a rooted spanning tree T of D which has the maximum weight. Let $f : V \to \{0, 1, \cdots, k-1\}$ be defined as $f(x) = w_T(x) \pmod k$. We shall show that f is a (k, d)-colouring of G.

Assume to the contrary that G has an edge xy such that either $|f(x) - f(y)| < d$ or $|f(x) - f(y)| > k - d$. Without loss of generality, we assume that the edge xy is oriented from x to y in D.

First we consider the case that x is not on the y-r-path of T and y is not on the x-r-path of T. If $w_T(x) - w_T(y) < d$, then we delete the edge of T connecting x to its father, and add the edge xy (so that y becomes the father of x). Then we obtain a spanning tree T' for which $w_{T'}(v) \geq w_T(v)$ for each v, and $w_{T'}(v) > w_T(v)$ for every decedents of x, including x itself. So $w(T') > w(T)$, contrary to our choice of T. Thus $w_T(x) - w_T(y) \geq d$. If $w_T(x) - w_T(y) > k - d$. Now we delete the edge of T connecting y to its father and add the edge xy (so that x becomes the father of y). Then we obtain a spanning tree T' for which $w_{T'}(v) \geq w_T(v)$ for each v, and $w_{T'}(v) > w_T(v)$ for every decedents of y, including y itself. Again contrary to our choice of T. So $d \leq w_T(x) - w_T(y) \leq k - d$, which implies that $d \leq f(x) - f(y) \leq k - d$ (as $f(x) - f(y) = w_T(x) - w_T(y) \pmod k$), contrary to our assumption.

Next we consider the case that y is on the x-r-path of T. Then $w_T(x) - w_T(y) \geq d$ (for otherwise, using the same method as in the previous paragraph, we can obtain another rooted tree T' with $w(T') > w(T)$). Assume $w_T(x) - w_T(y) = ak + j$ for some integers a, j such that $a \geq 0$ and $0 \leq j \leq k - 1$. Then $f(x) - f(y) = j$. Hence $j \in \{0, 1, \cdots, d-1, k-d+1, k-d+2, \cdots, k-1\}$.

Let p be the number of edges on the x-r-path of T oriented toward the root, and let q be the number of edges on this path oriented away from the root. By the definition of the weight, we know that $w_T(x) - w_T(y) = pd - q(k-d)$. Therefore we have $(p+q)d \pmod k = j$, and $(p+q+1)d \pmod k = j + d \pmod k \in \{1, 2, \cdots, 2d-1\}$. Note that the cycle C consisting the x-r-path and the edge xy is a cycle of length $\ell(C) = p + q + 1$. Hence $d\ell(C) mod k \in \{1, 2, \cdots, 2d-1\}$. By our assumption, $\tau(C) \leq k/d$, which implies $|C^+|/|C^-| \leq (k-d)/d$, here we choose the direction of traversal of C so that those edge on the x-r-path oriented towards r belongs to C^+. Hence $|C^+| = p$ and $|C^-| = q + 1$, which implies that $pd \leq q(k-d) + k - d$. Hence $w_T(x) - w_T(y) = pd - q(k-d) \leq k - d$. As we already know that $w_T(x) - w_T(y) \geq d$, we conclude that $d \leq f(x) - f(y) \leq k - d$, contrary to our assumption.

Finally we consider the case that x is on the y-r-path of T. Assume $w_T(y) - w_T(x) = ak + j$ for some integers a, j such that $0 \leq j \leq k-1$. Then $f(y) - f(x) \pmod{k} = j$. Hence $j \in \{0, 1, \cdots, d-1, k-d+1, k-d+2, \cdots, k-1\}$. Since $w_T(y) - w_T(x) \geq -k+d$ (for the same reason as above), hence either $a \geq 0$ or $a = -1$ and $j \in \{k-d+1, k-d+2, \cdots, k-1\}$. In any case, $w_T(y) - w_T(x) \geq -d+1$.

Similarly as in the previous case, let p be the number of edges on the y-r-path of T oriented toward the root, and let q be the number of edges on this path oriented away from the root. By the definition of the weight, we know that $w_T(y) - w_T(x) = pd - q(k-d)$. Therefore $(p+q)d \pmod{k} = j$. Therefore $(p+q+1)d \pmod{k} = j + d \pmod{k} \in \{1, 2, \cdots, 2d-1\}$. Again, the cycle C consisting the y-r-path and the edge xy is a cycle of length $p+q+1$. By our assumption, $|C^+|/|C^-| \leq (k-d)/d$, here we also choose the direction of traversal of C so that those edge on the x-r-path oriented towards r belongs to C^+. Hence $|C^+| = p+1$ and $|C^-| = q$, which implies that $pd + d \leq q(k-d)$. Hence $w_T(y) - w_T(x) = pd - q(k-d) \leq -d$, contrary to our previous conclusion. ∎

Corollary 3 *If for each cycle C of G, $d\ell(C) \bmod k \notin \{1, 2, \cdots, 2d-1\}$, then G is (k,d)-colourable.*

The special case $d = 1$ of Corollary 3 was contained in [3]:

Corollary 4 *[3] If for each cycle C of G, $\ell(C) \bmod k \neq 1$, then G is k-colourable.*

Since $\chi_c(G) \leq \chi(G)$, Corollary 3 can also be used to derive upper bound for the chromatic number of G. To be precise, the same condition of Corollary 3 implies that G is $\lceil k/d \rceil$-colourable. An interesting observation is that such derived conditions are different from the condition of Corollary 4. None implies the other. For example, suppose G contains a cycle of length 13, then for $d = 2$ and $k = 11$, the condition of Corollary 3 is not violated (by this cycle), hence one may still derive the conclusion that G is $(11, 2)$-colourable (if other cycles of G also do not violate the condition), and hence 6-colourable. But once a graph G has a cycle of length 13, then one cannot use Corollary 4 to show that G is 6-colourable.

It was shown in [3] that once an orientation D of G is given which satisfies the condition that $\tau(C) \leq k$ for every cycle C of length $\ell(C) \pmod{k} = 1$, then one can construct a k-colouring of G in linear time. The same conclusion is true for (k, d)-colouring. Namely, if an orientation D of G is given such that $\tau(C) \leq k/d$ for every cycle C satusfying $d\ell(C) \pmod{k} \in \{1, 2, \cdots, 2d-1\}$, then a (k, d)-colouring of G can be constructed in linear time.

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