Circular consecutive choosability of graphs

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Abstract

This paper considers list circular colouring of graphs in which the colour list assigned to each vertex is an interval of a circle. The *circular consecutive choosability* $ch_{cc}(G)$ of G is defined to be the least t such that for any circle $S(r)$ of length $r \geq \chi_c(G)$, if each vertex x of G is assigned an interval $L(x)$ of $S(r)$ of length t, then there is a circular r-colouring f of G such that $f(x) \in L(x)$. We show that for any finite graph G, $\chi(G) - 1 \leq ch_{cc}(G) < 2\chi_c(G)$. We determine the value of $ch_{cc}(G)$ for complete graphs, trees, even cycles and balanced complete bipartite graphs. Upper and lower bounds for $ch_{cc}(G)$ are given for some other classes of graphs.

1 Introduction

Suppose r is a positive real number. Let $S(r)$ denote the circle obtained from the interval $[0, r]$ by identifying 0 and r into a single point. For any real number x, $[x]_r \in$

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$[0, r)$ denotes the remainder of x upon division of r. For $a, b \in S(r)$, $[a, b]_r = \{t \in S(r) : [t - a]_r \leq [b - a]_r\}$ is the interval of $S(r)$ from a to b along the "increasing" direction. For example, if $r = 3.5$, $[3, 1]_{3.5} = [3, 3.5) \cup [0, 1]$. The length of the interval $[a, b]_r$ is equal to $[b - a]_r$. The *distance* between a and b, denoted by $|a - b|_r$, is the length of the shorter interval among the two intervals $[a, b]_r, [b, a]_r$. In other words, $|a - b|_r = \min\{[a - b]_r, [b - a]_r\} = \min\{|a - b|, r - |a - b|\}$.

For a graph $G = (V, E)$ and a real number $r \geq 1$, a *circular r-colouring* of G is a mapping $f : V(G) \to S(r)$ such that for any edge xy of G, $|f(x) - f(y)|_r \geq 1$. The *circular chromatic number* $\chi_c(G)$ of G is the least r for which G has a circular r-colouring. It is well-known [11, 12] that for any graph G, $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$. So $\chi_c(G)$ is a refinement of $\chi(G)$ and $\chi(G)$ is an approximation of $\chi_c(G)$.

Suppose G is a graph, r is a positive real number. A *$(\star, r)$-circular colour-list assignment* for G is a function L that assigns to each vertex v of G a set $L(v)$ which is the union of disjoint closed intervals of $S(r)$. If for each vertex v, the sum of the lengths of the disjoint intervals in $L(v)$ is equal to t, then L is called a *(t, r)-circular colour-list assignment*. Suppose L is a $(\star, r)$-circular colour-list assignment for a graph G. A *circular L-colouring* of G is a circular r-colouring f of G such that $f(v) \in L(v)$ for each vertex v of G. A graph G is called *circular t-choosable* if for any r and for any (t, r)-circular colour-list assignment L, G has a circular L-colouring. The *circular choosability* $ch_c(G)$ of G (also called the *circular list chromatic number* of G and denoted by $\chi_{c,l}(G)$) is defined in [13] as

$$ch_c(G) = \inf\{t : G \text{ is circular } t\text{-choosable}\}.$$

Let $ch(G)$ be the choosability (also known as the list chromatic number) of G. It is proved in [13] that for any graph G, $ch_c(G) \geq ch(G) - 1$. On the other hand, $ch_c(G) - ch(G)$ can be arbitrarily large. In particular, it is proved in [13] that for any

$\epsilon > 0$, there is a k-degenerate graph G for which $ch_c(G) \geq 2k - \epsilon$.

In this paper, we consider those circular colour-lists L in which each $L(x)$ is an interval of $S(r)$. A $(\star, r)$-*circular consecutive colour-list assignment* of G is a mapping L which assigns to each vertex v of G a closed interval $L(v)$ of $S(r)$. If $L(v)$ has length t for each vertex v, then L is called a (t, r)-*circular consecutive colour-list assignment* of G. We say G is *circular consecutive* (t, r)-*choosable* if for any (t, r)-circular consecutive colour-list assignment L of G, G is circular L-colourable.

Observe that if $r < \chi_c(G)$, then for any circular colour-list assignment L with respect to r, G is not circular L-colourable. Therefore, for the definition to be meaningful, we restrict to real numbers $r \geq \chi_c(G)$.

Definition 1.1 *Suppose $r \geq \chi_c(G)$. The circular consecutive choosability of G with respect to r is defined as*

$$ch^r_{cc}(G) = \inf\{t : G \text{ is circular consecutive } (t, r)\text{-choosable}\}.$$

The circular consecutive choosability of G is defined as

$$ch_{cc}(G) = \sup\{ch^r_{cc}(G) : r \geq \chi_c(G)\}.$$

Equivalently, $ch_{cc}(G)$ is the infimum of those t such that for any $r \geq \chi_c(G)$, G is circular consecutive (t, r)-choosable.

It follows from the definition, $ch_{cc}(G) \leq ch_c(G)$ for any graph G. As $ch_c(G) \geq ch(G) - 1$, we know that $ch_c(G)$ can be arbitrarily large for bipartite graphs G. We shall show in Section 4 that $ch_{cc}(G) < 2\chi_c(G) \leq 2\chi(G)$ for any graph G. Thus $ch_c(G)$ cannot be bounded in terms of $ch_{cc}(G)$.

Circular colouring provides a model for many periodic scheduling problems. The vertices of G represent jobs to be scheduled periodically with period r, and adjacent vertices represent jobs that cannot be carried out at the same time. The whole period is the circle $S(r)$. A scheduling is a mapping $f : V(G) \to S(r)$, where $f(x)$ is the starting moment of job x. We assume that each job needs a unit time to complete. So if $x \sim y$, then the distance between $f(x)$ and $f(y)$ on $S(r)$ needs to be at last 1. Such a period scheduling is a circular r-colouring of G. It is natural that for each job x, there are some restrictions on the starting moment of x. This motivates the problem of list circular colouring of G, where we require that $f(x) \in L(x)$ for each x, where $L(x)$ is the set of permissible starting moment of x. It is not unusual that for each vertex x, $L(x)$ is just one interval. In this case, we have circular consecutive list colouring of G.

Another motivation for the study of circular choosability of graphs is the application in inductive proofs of circular colourability of graphs. To prove a graph G is circular r-colourable, one may find an induced subgraph H of G, find a circular r-colouring f of $G - H$ (by induction hypothesis), then extend f to a circular r-colouring of H to obtain a circular r-colouring of G. In the extension, the colours available to vertices of H are restricted. Thus we are facing with a circular list colouring problem. Such techniques have been used in the study of the circular chromatic number of planar graphs of large girth in a sequence of papers [1, 2, 3, 10]. In the inductive proof described above, if a vertex x of H is adjacent to one coloured vertex in G, the set of available colours to x is an interval of $S(r)$. Therefore we are left with a circular consecutive list colouring problem of H.

Circular consecutive choosability is also a generalization of the consecutive choosability of a graph introduced by Waters [9]. In Section 2, we define consecutive choosability of graphs and discuss the relation between circular consecutive choosability and consecutive choosability.

In Section 3, we give two other equivalent definitions of circular consecutive choosability. Section 4 gives upper and lower bounds for the circular consecutive choosability of graphs in terms of their circular chromatic number. In Section 5, the circular consecutive choosability of trees, cycles and complete graphs are determined. In Section 6, we present some relatively sharp upper and lower bounds for $ch^r_{cc}(G)$ for some special classes of graphs.

2 Consecutive choosability

Circular consecutive choosability is a generalization of the concept of consecutive list colouring of graphs introduced by Waters in [9]. For a positive real number t, a *t-interval assignment* of a graph G is a function L which assigns to each vertex x of G a closed interval $L(x)$ of length t. An *L-colouring* of G is a function $f : V(G) \to R$ such that for each vertex x of G, $f(x) \in L(x)$ and for each edge xy of G, $|f(x) - f(y)| \geq 1$. A graph G is said to be t-interval choosable if for any t-interval assignment L, G has an L-colouring. The *consecutive choosability* of G is defined as

$$\tau(G) = \inf\{t : G \text{ is } t\text{-interval choosable}\}.$$

It follows easily from the definition that for any $r \geq \chi_c(G)$, $ch^r_{cc}(G) \geq \tau(G)$. It is proved in [9] that for any graph G, $\tau(G) \geq \chi(G) - 1$. Thus for any graph G, for any $r \geq \chi_c(G)$, $ch_{cc}(G) \geq ch^r_{cc}(G) \geq \chi(G) - 1$. Lemma 2.1 below shows that in some sense, the circular consecutive list colouring is a generalization of consecutive list colouring.

Observe that a mapping $l : V(G) \to S(r)$ corresponds to a (t, r)-circular consecutive colour-list assignment L defined as $L(x) = [l(x), l(x) + t]_r$. A circular r-colouring f is said to be *compatible with* (l, t) if $f(x) \in [l(x), l(x) + t]_r$ for each vertex x. Thus to prove that a graph is circular consecutive (t, r)-choosable, it suffices to show that for any mapping $l : V(G) \to S(r)$, there is a circular r-colouring f compatible with (l, t).

Lemma 2.1 *Suppose G is a finite graph on n vertices. If $r \geq n^2 + 1$, then $ch^r_{cc}(G) = \tau(G)$.*

Proof. As observed above, $\tau(G) \le ch^r_{cc}(G)$ for any r. We now prove that $\tau(G) \ge ch^r_{cc}(G)$ for $r \ge n^2+1$. Assume that $\tau(G) = t$, and $r \ge n^2+1$. Let $l : V(G) \to [0, r)$ be an arbitrary mapping. We need to show that G has a circular r-colouring compatible with (l, t). We may assume that $0 = l(x_1) \le l(x_2) \le \cdots \le l(x_n)$. As $t = \tau(G) \le \chi(G)(1 - 1/n) \le n - 1$ (see [9]), it follows that there is an index i such that $[l(x_{i+1}) - l(x_i)]_r \ge t + 1$ (the sum in the indices are modulo n, i.e., $x_{n+1} = x_1$). Let $l'(x_j) = [l(x_j) - l(x_{i+1})]_r$. Then $0 = l'(x_{i+1}) \le l'(x_{i+2}) \le \cdots \le l'(x_n) \le l'(x_1) \le l'(x_2) \le \cdots \le l'(x_i) \le r - (t+1)$. Regard l' as a mapping $l' : V(G) \to R$. It is known [9] that if $\tau(G) = t$ then G is t-interval choosable. So there is a colouring f of G such that $f(x) \in [l'(x), l'(x) + t]$ for each vertex x and $|f(x) - f(y)| \ge 1$ for each edge xy. Now for any vertex x, $0 \le l'(x) \le f(x) \le l'(x) + t \le r - 1$. So for any edge xy, $|f(x) - f(y)| \le r - 1$. Let $g(x) = [f(x) + l(x_{i+1})]_r$. Then g is a circular r-colouring compatible with (l, t). □

Lemma 2.1 shows that the consecutive choosability of a graph G corresponds to the circular consecutive choosability of G with respect to sufficiently large r. Indeed, in the definition of consecutive choosability of G, the intervals assigned to vertices of G are intervals of the real line $\mathbb{R}$, which may be regarded as an infinite circle.

3 Equivalent definitions

This section gives two different definitions of $ch^r_{cc}(G)$. Sometimes these alternate definitions are used in our proofs. A mapping $l : V(G) \to R$ can be viewed as a mapping from $V(G)$ to $S(r)$ by identifying a point x of R with $[x]_r$. By such a convention, the circular consecutive choosability of a graph can be defined alternately as follows.

Lemma 3.1 *Suppose G is a graph and $r > t$ are positive real numbers. Then G is circular consecutive t-choosable with respect to r if and only if for any mapping*

$l : V(G) \to R$, *there is a mapping* $f : V(G) \to R$ *such that the following hold:*

- *For each vertex* v, $l(v) \le f(v) \le l(v) + t$.
- *For any edge* xy *of* G, $\min\{[f(x) - f(y)]_r, [f(y) - f(x)]_r\} \ge 1$.

Proof. Assume G is circular consecutive t-choosable with respect to r. Let $l : V(G) \to R$ be an arbitrary mapping. Let $l' : V(G) \to S(r)$ be defined as $l'(x) = [l(x)]_r$. As G is circular consecutive t-choosable with respect to r, there is a circular r-colouring $g : V(G) \to [0, r)$ of G which is consistent with (l', t). Let $f : V(G) \to R$ be defined as $[f(v)]_r = g(v)$ and $l(v) \le f(v) < l(v) + r$. Since $[f(v) - l(v)]_r = [g(v) - l(v)]_r \le t$, we conclude that $f(v) \le l(v) + t$.

Conversely, assume that for any mapping $l : V(G) \to R$, there is a mapping $f : V(G) \to R$ such that $l(v) \le f(v) \le l(v)+t$ for each vertex v, and $|f(x)-f(y)|_r \ge 1$ for each edge xy. A mapping $l : V(G) \to S(r)$ can be viewed as a mapping from $V(G)$ to R. Then the mapping g defined as $g(v) = [f(v)]_r$ is a circular r-colouring of G which is compatible with (l, t). □

The circular chromatic number of graphs can be defined through (p, q)-colourings. Given integers $p \ge 2q$, a (p, q)-*colouring* of a graph G is a mapping $f : V(G) \to \{0, 1, \cdots, p-1\}$ such that for any edge xy of G, $q \le |f(x) - f(y)| \le p - q$. The circular chromatic number of G can be defined as

$$\chi_c(G) = \inf\{p/q : G \text{ has a } (p, q)\text{-colouring}\}.$$

In [13], it is shown that the circular choosability of graphs can also be defined through (p, q)-colourings.

Definition 3.2 *Suppose* G *is a graph and* $p \ge 2q$ *are positive integers. A* (p, q)-*list assignment* L *is a mapping which assigns to each vertex* v *of* G *a subset* $L(v)$ *of*

$\{0, 1, \cdots, p-1\}$. *An* L*-*(p,q)*-colouring of* G *is a* (p,q)*-colouring* f *of* G *such that for any vertex* v, $f(v) \in L(v)$. *Suppose* t *is a positive real number. A* t*-*(p,q)*-list assignment is a* (p,q)*-list assignment* L *such that for every vertex* v, $|L(v)| \geq tq$.

It is shown in [13] that we can define the circular choosability of G as

$$ch_c(G) = \inf\{t : \text{for any } p \geq 2q, \text{ for any } t\text{-}(p,q)\text{-list assignment } L,\ G \text{ is } L\text{-}(p,q)\text{-colourable.}\}$$

Similarly, the circular consecutive choosability of graphs can also be defined through (p,q)-colourings. Given a positive integer p, and $a, b \in \{0, 1, \cdots, p-1\}$. The *circular integral interval* $[a,b]_p$ is defined as

$$[a,b]_p = \{a, a+1, a+2, \cdots, b\},$$

where the additions are modulo p. The *length* $|[a,b]_p|$ of the interval $[a,b]_p$ is the cardinality of the set $[a,b]_p$. For example $[2,5]_8 = \{2,3,4,5\}$ and $[5,2]_8 = \{5,6,7,0,1,2\}$, and these two intervals have lengths 4 and 6, respectively. We are interested in (p,q)-list assignments L such that for each vertex x, $L(x)$ is a circular integral interval. Once the length of the interval $L(x)$ is known, then $L(x)$ is determined by its left end point. Thus we have the following definition.

Definition 3.3 *Suppose* G *is a graph and* p, q *are positive integers such that* $p/q \geq \chi_c(G)$, *and* s *is a positive integer. Let* $l : V(G) \to \{0, 1, \cdots, p-1\}$ *be a mapping. A* (p,q)*-colouring* f *of* G *is compatible with* (l,s) *if for any vertex* x, $f(x) \in [l(x), l(x)+s-1]_p$, *i.e.,* $[f(x) - l(x)]_p \leq s-1$.

Observe that the circular consecutive integral interval starting from $l(x)$ and of cardinality s is the interval $[l(x), l(x)+s-1]_p$. So in the definition above, we require that $[f(x) - l(x)]_p \leq s-1$ (instead of $[f(x) - l(x)]_p \leq s$).

Definition 3.4 *Suppose G is a graph, p, q are positive integers such that $p/q \geq \chi_c(G)$, and s is a positive integer. We say a graph G is* circular consecutive (p, q)-s-choosable *if for any mapping $l : V(G) \to \{0, 1, \cdots, p-1\}$, G has a (p, q)-colouring f which is compatible with (l, s). We define the consecutive (p, q)-choosability of G as*

$$ch_{p,q}(G) = \min\{s : G \text{ is circular consecutive } (p, q)\text{-}s\text{-choosable}\}.$$

The following lemma shows that the definition of $ch_{p,q}(G)$ is determined by $ch^r_{cc}(G)$ for $r = p/q$.

Lemma 3.5 *For any graph G and for any $r = p/q \geq \chi_c(G)$,*

$$ch_{p,q}(G) = \lfloor ch^r_{cc}(G)q \rfloor + 1.$$

Proof. First we show that $ch_{p,q}(G) \leq \lfloor q\ ch^r_{cc}(G) \rfloor + 1$. Assume $ch^r_{cc}(G) = t$. Let $s = \lfloor qt \rfloor + 1$. Let $l : V(G) \to \{0, 1, \cdots, p-1\}$ be an arbitrary mapping. We need to show that G has a (p, q)-colouring compatible with (l, s).

Let $l' : V(G) \to [0, r)$ be defined as $l'(x) = l(x)/q$. As G is circular consecutive t-choosable with respect to r, there is a circular r-colouring f of G which is compatible with (l', t). It is easy to verify that $\phi(x) = \lfloor f(x)q \rfloor$ is an (p, q)-colouring of G compatible with (l, s). Thus $ch_{p,q}(G) \leq \lfloor q\ ch^r_{cc}(G) \rfloor + 1$.

Now we show that $ch_{p,q}(G) \geq \lfloor q\ ch^r_{cc}(G) \rfloor + 1$. This is equivalent to show that $ch^r_{cc}(G) < ch_{p,q}(G)/q$.

Assume G has n vertices, and let $\epsilon < 1/n$. Assume $ch_{p,q}(G) = s$ and let $t = (s-\epsilon)/q$. Let l be an arbitrary mapping from $V(G)$ to $[0, r)$. We shall show that G has a circular r-colouring compatible with (l, t).

For $b \in [0, 1)$, let $l'_b(x) = \lceil l(x)q + b \rceil$. As $ch_{p,q}(G) = s$, G has a (p, q)-colouring f'_b compatible with (l'_b, s), i.e., f'_b is a (p, q)-colouring of G with $f'_b(x) \in [l'_b(x), l'_b(x)+s-1]_p$.

Let $f_b(x) = (f'_b(x) - b)/q$. Now we estimate $[f_b(x) - l(x)]_r$. By definition,

$$[f_b(x) - l(x)]_r = [f'_b(x)/q - b/q - l'_b(x)/q + l'_b(x)/q - l(x)]_r.$$

Thus

$$q[f_b(x) - l(x)]_r \leq [f'_b(x) - b - l'_b(x)]_p + l'_b(x) - l(x)q \leq (s-1) - b + l'_b(x) - l(x)q.$$

Thus

$$\begin{aligned}
& [f_b(x) - l(x)]_r \leq t \\
\Leftrightarrow \quad & q[f_b(x) - l(x)]_r \leq qt = s - \epsilon \\
\Leftarrow \quad & (s-1) - b + l'_b(x) - l(x)q \leq s - \epsilon \\
\Leftrightarrow \quad & \lceil l(x)q + b \rceil - (l(x)q + b) \leq 1 - \epsilon.
\end{aligned}$$

The inequality $\lceil l(x)q + b \rceil - (l(x)q + b) \leq 1 - \epsilon$ holds provided that the fractional part of $l(x)q + b$ is greater than or equal to ϵ. Thus there is a subset A_x of $[0, 1)$ which is an open interval of $S(1)$ of length ϵ such that if $b \notin A_x$, then $[f_b(x) - l(x)]_r \leq t$. As $\epsilon < 1/n$, there is a $b \in [0, 1)$ such that $[f_b(x) - l(x)]_r \leq t$ for all vertices x of G, i.e., f_b is a circular r-colouring of G compatible with (l, t). Thus G is circular consecutive t-choosable, and hence $ch^r_{cc}(G) < ch_{p,q}(G)/q$. □

Corollary 3.6 *Suppose G is a finite graph and $r = p/q \geq \chi_c(G)$. Then*

$$ch^r_{cc}(G) = \lim_{s \to \infty} ch_{ps,qs}(G)/qs.$$

Corollary 3.6 can be regarded as another definition of $ch^r_{cc}(G)$ for rational $r \geq \chi_c(G)$. As $ch_{cc}(G) = \sup\{ch^r_{cc} : r \geq \chi_c(G)\}$, it follows that $ch_{cc}(G) \geq \lim_{q \to \infty} \sup\{ch_{p,q}(G)/q : p \geq \chi_c(G)q\}$. In the following we shall show that equality holds.

Lemma 3.7 *Suppose $r' > r$. Then $ch^{r'}_{cc}(G) \leq \frac{r'}{r} ch^r_{cc}(G) - \frac{r'}{r} + 1$.*

Proof. Suppose $ch^r_{cc}(G) = t$ and let $t' = tr'/r - r'/r + 1$. Let l' be an arbitrary mapping from $V(G)$ to $[0, r')$. Let $l(x) = l'(x)r/r'$ be a mapping from $V(G)$ to $[0, r)$. As $ch^r_{cc}(G) = t$, there is a circular r-colouring f of G compatible with (l, t). Let f' be defined as $f'(x) = [l'(x) + a(x)]_{r'}$, where

$$a(x) = \min\{[f(x) - l(x)]_r r'/r, t'\}.$$

We shall show that f' is a circular r'-colouring of G compatible with (l', t'). By definition, $[f'(x) - l'(x)]_{r'} \leq t'$. So we only need to show that f' is indeed a circular r'-colouring of G. By definition, either $f'(x) = f(x)r'/r$, or

$$[f(x) - l(x)]_r r'/r > t' = tr'/r - r'/r + 1 = [f'(x) - l'(x)]_{r'}.$$

In the latter case, we have

$$[f(x)r'/r - l(x)r'/r]_{r'} > t'.$$

Since $[f(x) - l(x)]_r \leq t$, we have

$$[f(x)r'/r - l(x)r'/r]_{r'} \leq tr'/r = t' + r'/r - 1 = [f'(x) - l'(x)]_{r'} + r'/r - 1.$$

So in any case,

$$f'(x) \in [f(x)r'/r - r'/r + 1, f(x)r'/r]_{r'}.$$

Therefore for any edge xy of G, we have

$$[f'(x) - f'(y)]_{r'} \geq [f(x) - f(y)]_r r'/r - r'/r + 1 \geq 1.$$

Hence f' is indeed a circular r'-colouring of G. □

We shall see in next section that the bound provided in Lemma 3.7 is tight for complete bipartite graphs. Now we use this lemma to prove that $ch_{cc}(G) = \sup\{ch_{p,q}(G)/q : p \geq \chi_c(G)q\}$.

Theorem 3.8 *For any finite graph G, $ch_{cc}(G) = \lim_{q\to\infty} \sup\{ch_{p,q}(G)/q : p \geq \chi_c(G)q\}$.*

Proof. As noted above, $ch_{cc}(G) \geq \lim_{q\to\infty} \sup\{ch_{p,q}(G)/q : p \geq \chi_c(G)q\}$. To prove that the equality holds, it suffices to show that for any real number $r \geq \chi_c(G)$ and for any $\epsilon > 0$, there exist $p/q \geq \chi_c(G)$ such that

$$ch^r_{cc}(G) \leq ch_{p,q}(G)/q + \epsilon.$$

If $r = \chi_c(G)$, then $r = p/q$ is a rational number and hence $ch^r_{cc}(G) \leq ch_{p,q}(G)/q$. Assume $r > \chi_c(G)$ and that $ch^r_{cc}(G) = t$. Let $r' = p/q$ be a rational such that $\chi_c(G) \leq r' \leq r$ and $r/r' < (r+\varepsilon)/r < (t'+\epsilon)/t'$, where $t' = ch^{r'}_{cc}(G)$. By Lemma 3.7 and Corollary 3.6,

$$t \leq rt'/r' - r/r' + 1 < rt'/r' < t' + \epsilon = ch^{r'}_{cc}(G) + \epsilon = \lim_{s\to\infty} ch_{ps,qs}(G)/(qs) + \epsilon.$$

Thus for some positive integer s, $ch^r_{cc}(G) \leq ch_{ps,qs}(G)/(qs) + \epsilon$. □

4 Some general bounds on $ch_{cc}(G)$

First we consider $ch^r_{cc}(G)$ for the case that $r = \chi_c(G)$. The following result is parallel to a result in [9].

Lemma 4.1 *If $r = \chi_c(G)$ and G has n vertices, then $ch^r_{cc}(G) \leq r(1 - 1/n)$.*

Proof. Let $t = r(1-1/n)$, and let l be an arbitrary mapping from $V(G)$ to $[0, r)$. Let $f : V(G) \to S(r)$ be a circular r-colouring of G. For $b \in [0, r)$, let $f_b(x) = [f(x) + b]_r$ for $x \in V(G)$. Then each f_b is a circular r-colouring of G. For each vertex x, let $A_x = (l(x) + t - f(x), l(x) + t + r/n - f(x))_r$. It is straightforward to verify that for any $b \notin A_x$, $[f_b(x) - l(x)]_r \leq t$. As G has n vertices, and A_x has length r/n, so $S(r) \setminus \cup_{x\in V(G)} A_x \neq \emptyset$. I.e., there is a $b \in [0, r)$ such that for any $x \in V(G)$,

$[f_b(x) - l(x)]_r \le t$. Hence f_b is a circular r-colouring of G compatible with (l,t). So $ch^r_{cc}(G) \le t = r(1 - 1/n)$. □

Lemma 4.2 *For any $r \ge \chi_c(G)$, $ch^{2r}_{cc}(G) \le ch^r_{cc}(G)$.*

Proof. Suppose $ch^r_{cc}(G) = t$ and $l : V(G) \to [0, 2r)$ is an arbitrary mapping. Let $l' : V(G) \to [0, r)$ be defined as $l'(x) = [l(x)]_r$. In other words, $l'(x) = l(x)$ if $l(x) \in [0, r)$ and $l'(x) = l(x) - r$ otherwise. As $ch^r_{cc}(G) = t$, G has a circular r-colouring f' which is compatible with (l', t). For any $x \in V(G)$, let $f(x) \in [l(x), l(x) + r)_{2r}$ be the unique number such that $[f(x)]_r = f'(x)$. Then for any vertex x of G, $[f(x) - l(x)]_{2r} = [f'(x) - l'(x)]_r \le t$. Thus f is a circular $2r$-colouring which is compatible with (l,t). □

Corollary 4.3 *For any graph G, $ch_{cc}(G) = \sup\{ch^r_{cc}(G) : \chi_c(G) \le r < 2\chi_c(G)\}$.*

Theorem 4.4 *Suppose G is a graph on n vertices and r is a real number greater than or equal to $\chi_c(G)$. Then*

$$\chi(G) - 1 \le ch^r_{cc}(G) \le r - \frac{r}{|V(G)|} - \frac{r}{\chi_c(G)} + 1.$$

Proof. The lower bound follows from an earlier observation. For the upper bound, let $r_0 = \chi_c(G)$, it follows from Lemma 3.7 that

$$ch^r_{cc}(G) \le \frac{r}{r_0} ch^{r_0}_{cc}(G) - \frac{r}{r_0} + 1.$$

By Lemma 4.1, $ch^{r_0}_{cc}(G) \le r_0(1 - 1/n)$. So $ch^r_{cc}(G) \le r - \frac{r}{|V(G)|} - \frac{r}{\chi_c(G)} + 1$. □

Since $ch_{cc}(G) = \sup_{\chi_c(G) \le r < 2\chi_c(G)} ch^r_{cc}(G)$, we have the following corollary.

Corollary 4.5 *Suppose G is a graph on n vertices. Then*

$$\chi(G) - 1 \le ch_{cc}(G) \le 2\chi_c(G)(1 - 1/n) - 1.$$

We shall see later that the upper bound for $ch_{cc}(G)$ in terms of $\chi_c(G)$ in Corollary 4.5 is best possible.

5 Trees, cycles and complete graphs

This section determines the circular consecutive choosability of some special graphs. First we determine the consecutive choosability of trees.

Theorem 5.1 *Let T be a tree on n vertices. Then $ch_{cc}(T) = 2(1 - \frac{1}{n})$.*

Proof. As a tree with at least an edge has circular chromatic number 2 (cf. [8]) , we only need to consider $ch_{p,q}(T)$ for $p/q \geq 2$. It is proved in [6] that for any $p \geq 2q$, for any list assignment $L : V(T) \to \mathcal{P}(\{0, 1, \cdots, p-1\})$, if for any subtree T' of T, $\sum_{v \in V(T')} |L(v)| \geq 2q(|V(T')| - 1) + 1$, then there is a (p, q)-colouring f of T such that $f(v) \in L(v)$ for all v. On the other hand, if $f : V(T) \to Z^{\geq 0}$ is a mapping such that $\sum_{v \in V(T)} f(v) < 2q(|V(T)| - 1) + 1$, then there is a list assignment $L : V(T) \to \mathcal{P}(\{0, 1, \cdots, p-1\})$ such that each $L(v)$ is an interval of length $f(v)$ and G is not L-(p, q)-colourable. This implies that $ch_{p,q}(T) = \lceil 2q(1 - \frac{1}{n}) + \frac{1}{n} \rceil$. By Corollary 3.8, $ch_{cc}(T) = \lim_{q \to \infty} \sup\{ch_{p,q}(T)/q : p \geq 2q\} = 2(1 - \frac{1}{n})$. □

Next we consider the complete graphs K_n.

Theorem 5.2 *For any integer $n \geq 1$, $ch_{cc}(K_n) = n - 1$.*

Proof. As $\chi_c(K_n) = n$ (cf. [8]), we only need to consider$ch^r_{cc}(K_n)$ for $r \geq n$. By Corollary 4.5, $ch_{cc}(K_n) \geq n-1$. It remains to show that for any $r \geq n$, $ch^r_{cc}(K_n) \leq n-1$. We prove it by induction on n. For $n = 2$, the conclusion follows from Theorem 5.1. Assume $n \geq 3$ and let $l : V(K_n) \to S(r)$ be an arbitrary mapping. Without loss of generality, assume that $l(v_0) = 0$ (the vertex set of K_n is assumed to be $v_0, v_1, v_2, \cdots, v_{n-1}$). By induction hypothesis, there is a circular r-colouring f' of $K_n - v_0$ which is compatible with $(l, n-2)$. We may assume that $f'(v_1) < f'(v_2) < \cdots < f'(v_{n-1})$. Now we define a circular r-colouring f of K_n as follows:

$$\begin{aligned} f(v_0) &= \min\{1, f'(v_1)\} \\ f(v_i) &= \max\{f'(v_i), f(v_{i-1}) + 1\}, \text{ for } i = 1, 2, \cdots, n-1. \end{aligned}$$

It can be easily proved by induction that for each $i \geq 1$, either $f(v_i) = f'(v_i)$ or $f(v_i) = f(v_0) + i \leq f'(v_i) + 1$. In particular, either $f(v_{n-1}) = f'(v_{n-1})$ or $f(v_{n-1}) = f(v_0) + n - 1$. In any case, $f(v_{n-1}) - f(v_0) \leq r - 1$. Thus f is a circular r-colouring of K_n. As $0 \leq f(v_0) = [f(v_0) - l(v_0)]_r \leq 1 \leq n - 1$ and for $i \geq 1$, $[f(v_i) - l(v_i)]_r \leq [f'(v_i) - l(v_i)]_r + 1 \leq n - 1$, we conclude that f is compatible with $(l, n-1)$. □

Lemma 5.3 *For any integer $n \geq 3$, $ch_{cc}(C_n) \geq 2$.*

Proof. The case $n = 3$ follows from Theorem 5.2. First we show that $ch_{cc}(C_n) \geq 2$ for any $n \geq 4$. As $\chi_c(C_n) \leq \chi(C_n) \leq 3$, it suffices to prove that for some $r \geq 3$, $ch^r_{cc}(C_n) \geq 2$. Let $\delta > 0$ be a real number such that $r = n(1-\delta/2) > 3$. We shall prove that $ch^r_{cc}(C_n) > 2 - \delta$. Assume the vertices of C_n are $v_0, v_1, \cdots, v_{n-1}$, in which v_i is adjacent to v_{i+1} for $i = 0, 1, \cdots, n-1$. Assume to the contrary that $ch^r_{cc}(C_n) \leq 2 - \delta$. Let $l : V(C_n) \to R$ be defined as $l(v_i) = j(1 - \delta/2)$. Since $ch^r_{cc}(G) \leq 2 - \delta$, there is a mapping $f : V(C_n) \to R$ such that $l(v_i) \leq f(v_i) \leq l(v_i)+2-\delta$ and $|f(v_i)-f(v_{i+1})|_r \geq 1$ for $i = 0, 1, \cdots, n-1$ (where the summation in the indices are modulo n). First we claim that $f(v_{i+1}) > f(v_i)$ for $i = 0, 1, \cdots, n-1$. For otherwise, we should have $f(v_i) \geq f(v_{i+1})+1$, which implies that $f(v_i) \geq l(v_{i+1})+1 = (i+1)(1-\delta/2)+1 = l(v_i)+2-\delta/2$, in contrary to our assumption. Since $f(v_{i+1}) - f(v_i) \geq 1$ for $i = 0, 1, \cdots, n-1$, we have $f(v_{n-1}) - f(v_0) \geq n - 1$. If $f(v_{n-1}) - f(v_0) < r$, then $|f(v_{n-1}) - f(v_0)|_r = r-(f(v_{n-1})-f(v_0)) \leq r-(n-1) < 1$, which is a contradiction. If $f(v_{n-1})-f(v_0) \geq r$, then $|f(v_{n-1})-f(v_0)|_r = f(v_{n-1})-f(v_0)-r \leq l(v_{n-1})+2-\delta-l(v_0)-r = 1-\delta/2 < 1$, which is again a contradiction. □

Lemma 5.4 *If $n \geq 2$ is even and $r \geq 2$, then $ch_{cc}(C_n) \leq 2$. If n is odd and $r \geq 3$, then $ch^r_{cc}(C_n) \leq 2$.*

Proof. By Corollary 3.6, it suffices to prove that if n is even, then for any $p/q \geq 2$, $ch_{p,q}(C_n) \leq 2q + 1$. If n is odd, then for any $p/q \geq 3$, $ch_{p,q}(C_n) \leq 2q + 1$. Let $\ell : V(C_n) \to \{0, 1, \cdots, p-1\}$ is an arbitrary mapping. Let $L(v) = [\ell(v), \ell(v) + 2q]_p$. We shall show that there is a (p, q)-colouring f of C_n that is compatible with $(\ell, 2q+1)$, i.e., $f(v_i) \in L(v_i)$ for all i.

The case $n = 2$ follows from Theorem 5.1, as the two parallel edges of C_2 can be replaced by a single edge. The case $n = 3$ follows from Theorem 5.2 and Lemma 3.5. Assume $n \geq 4$, and the above statement is true for $n - 2$.

For colours $i, j \in \{0, 1, \cdots, p-1\}$, if $|i - j| \geq q$ then we say colour i is *adjacent* to colour j. We write $i \sim j$ if i and j are adjacent colours, and let $N(i) = \{j : i \sim j\}$, $\bar{N}(i) = \{j : i \not\sim j\}$. In other words, $N(i) = [i + q, i + p - q]_p$ is an interval of length $p - 2q + 1$ and $\bar{N}(i) = [i - q + 1, i + q - 1]_p$ is an interval of length $2q - 1$. As for any $v \in V(C_n)$, $L(v)$ has length $2q + 1$ and for any colour i, $\bar{N}(i)$ has length $2q - 1$, we conclude that $L(v) \cap N(i) \neq \emptyset$. As $L(v)$ and $N(i)$ are intervals, we know that if $N(i)$ contains a colour $j \notin L(v)$, then $N(i)$ contains at least one of the two end colours of the interval $L(v)$.

If $\ell(v_i)$ is adjacent to $\ell(v_{i+1})$ for $i = 0, 1, \cdots, n-1$ (where summation in the indices are modulo n), then $f(v_i) = \ell(v_i)$ is a required (p, q)-colouring of C_n. Assume this is not the case. Without loss of generality, assume that $\ell(v_{n-2})$ is not adjacent to $\ell(v_{n-1})$. Consider the cycle C_{n-2} with vertices $v_0, v_1, \cdots, v_{n-3}$, with the restriction of L to $\{v_0, v_1, \cdots, v_{n-3}\}$ as a colour-list assignment to C_{n-2}. By induction hypothesis, there is a (p, q)-colouring f of C_{n-2} such that for each i, $f(v_i) \in L(v_i)$.

If $f(v_{n-3}) \in L(v_{n-1})$, then choose a colour $j \in N(f(v_{n-3})) \cap L(v_{n-2})$. Extending f by letting $f(v_{n-2}) = j$ and $f(v_{n-1}) = f(v_{n-3})$, we obtain a required (p, q)-colouring of C_n. Thus we can assume that $f(v_{n-3}) \notin L(v_{n-1})$. Similarly, we assume that $f(v_0) \notin L(v_{n-2})$.

As $f(v_0) \in N(f(v_{n-3})) \setminus L(v_{n-2})$ and $N(f(v_{n-3})) \cap L(v_{n-2}) \neq \emptyset$, it follows that $N(f(v_{n-3}))$ contains $[f(v_0), \ell(v_{n-2})]_p$ or $[\ell(v_{n-2}) + 2q, f(v_0)]_p$. By symmetry, we may assume that

$$[f(v_0), \ell(v_{n-2})]_p \subseteq N(f(v_{n-3})). \tag{1}$$

Then the interval $N(\ell(v_{n-2}))$ contains $f(v_{n-3}) \notin [\ell(v_{n-1}), \ell(v_{n-1}) + 2q]_p$, but does not contain $\ell(v_{n-1})$ (as by our assumption, $\ell(v_{n-2})$ is not adjacent to $\ell(v_{n-1})$). Since $N(\ell(v_{n-2})) \cap [\ell(v_{n-1}), \ell(v_{n-1}) + 2q]_p \neq \emptyset$, we conclude that

$$[\ell(v_{n-1}) + 2q, f(v_{n-3})]_p \subseteq N(\ell(v_{n-2})). \tag{2}$$

Without loss of generality, we may assume that $f(v_0) = 0$. It follows from (1) and (2) that

$$q = f(v_0) + q < \ell(v_{n-2}) + q \leq \ell(v_{n-1}) + 2q < f(v_{n-3}) \leq p - q.$$

Hence $\ell(v_{n-1}) + 2q \sim f(v_0)$.

By (1) and (2), we have $f(v_{n-3}) \sim \ell(v_{n-2})$ and $\ell(v_{n-2}) \sim \ell(v_{n-1}) + 2q$. Therefore f can be extended to a required colouring of C_n by letting $f(v_{n-2}) = \ell(v_{n-2})$ and $f(v_{n-1}) = \ell(v_{n-1}) + 2q$. □

Corollary 5.5 *If $n \geq 4$ is even, then $ch_{cc}(C_n) = 2$.*

If n is odd, it follows from a result of [13] that $ch_{cc}(C_n) \leq ch_c(C_n) = \frac{2n}{n-1}$. We conjectured that $ch_{cc}(C_n) = 2$ for all n in an earlier version of this paper, and the conjecture is confirmed by Liu [4]. It is recently proved by Pan and Zhu [5] that every 2-choosable graph is circular consecutive 2-choosable.

6 Bounds on $ch_{cc}(G)$ for special graphs

This section discusses the circular consecutive choosability of some other special classes of graphs. Lower and upper bounds are obtained for $ch_{cc}(G)$ for these graphs.

A *k-tuple colouring* of a graph G is an assignment of k distinct colours to each vertex of G so that adjacent vertices receive no colours in common. The *kth chromatic number* of G, denoted by $\chi_k(G)$, is the smallest number of colours needed to give G a k-tuple colouring. Clearly $\chi_1(G)$ is the ordinary chromatic number of G.

The *fractional chromatic number* of G, denoted by $\chi_f(G)$, is defined as $\inf\{\frac{\chi_k(G)}{k} : k = 1, 2, \dots\}$. It is well-known [7] that for finite graphs G, the infimum is always attained and $\chi_f(G)$ is always a rational. From the definition of $\chi_k(G)$ and $\chi_f(G)$, we have $\chi_k(G) \geq k\chi_f(G)$.

Suppose $G = (V, E)$ is a graph and m is a positive integer. We denote by $G[m]$ the graph obtained from G by replacing each vertex with an independent set of cardinality m. Namely $G[m]$ has vertex set $V \times \{0, 1, \cdots, m-1\}$ in which $(x, i)(y, j)$ is an edge if and only if $xy \in E$.

Theorem 6.1 *Let H be a graph with $\chi_f(H) = \chi_c(H)$. Then for any real number $r \in [\chi_c(H), 2\chi_c(H))$,*

$$ch^r_{cc}(H[m]) \geq r - \frac{r}{\chi_c(H)} - \frac{r}{m} + 1.$$

Proof. Let

$$\varepsilon = \frac{r}{m},\ \delta = \frac{r}{\chi_c(H)} + \frac{r}{m} - 1.$$

Then $\chi_f(H) = \chi_c(H) = \frac{r}{\delta - \varepsilon + 1}$.

Let $l : V(H[m]) \to [0, r)$ be defined as

$$l((v, j)) = j\varepsilon,\ j = 0, 1, \dots, m-1.$$

Assume to the contrary that $ch^r_{cc}(H[m]) < r - \frac{r}{\chi_c(H)} - \frac{r}{m} + 1 = r - \delta$. Then for some $\delta' > \delta$, there is a circular r-colouring which is compatible with $(l, r - \delta')$. Since $\chi_c(H) \leq r < 2\chi_c(H)$, we have $0 \leq \delta - \varepsilon < 1$. Without loss of generality, we may assume that $\delta' - \varepsilon < 1$.

If $m = 1$ then $r - \frac{r}{\chi_c(H)} - \frac{r}{m} + 1 < 0$ and the theorem holds trivially. Thus we assume that $m \geq 2$. Let f be a circular r-colouring of $H[m]$ which is compatible with $(l, r - \delta')$.

First we claim that for each vertex v of H, there are two vertices (v, j_1) and (v, j_2) of $H[m]$ such that $|f((v, j_1)) - f((v, j_2))|_r \geq \delta' - \varepsilon$. Let $V_v = \{(v, 0), (v, 1), \cdots, (v, m-1)\}$. If the claim is not true, then for some $v \in V(H)$, $f(V_v)$ is contained in an interval $[a, b]_r$ of length less than $\delta' - \varepsilon$. Without loss of generality, we may assume that $0 \leq a < b < r - \varepsilon$. Let j be the smallest index such that $j\varepsilon > b$. Since $a \leq f((v, j)) \leq b < j\varepsilon = l((v, j))$, the length of $[f((v, j)) - l((v, j))]_r$ is equal to $f((v, j)) - l((v, j)) + r$. As $f((v, j)) \geq a$, $b - a < \delta' - \varepsilon$ and $b \geq (j - 1)\varepsilon$, we have

$$f((v, j)) - l((v, j)) + r \geq a - b - (j\varepsilon - b) + r \geq r - (b - a) - \varepsilon > r - \delta'.$$

This is in contrary to the assumption that f is compatible with $(l, r - \delta')$.

For each $v \in V(H)$, let (v, j_1) and (v, j_2) be two vertices such that $|f((v, j_1)) - f((v, j_2))|_r \geq \delta' - \varepsilon$.

Let $A_v = \cup_{i=0}^{m-1}[f((v, i)), f((v, i)) + 1)_r$. If for all i, j, $[f((v, i)), f((v, i)) + 1)_r \cap [f((v, j)), f((v, j)) + 1)_r \neq \emptyset$, then A_v is a single interval. Since $\delta' - \varepsilon < 1$ and since there are j_1 and j_2 such that $|f((v, j_1)) - f((v, j_2))|_r \geq \delta' - \varepsilon$, we conclude that the total length of A_v is at least $1 + \delta' - \varepsilon$. Otherwise, A_v has total length at least $2 > 1 + \delta' - \varepsilon$.

If $u, v \in V(H)$ are adjacent, then for any $j, j' \in \{0, 1, \cdots, m - 1\}$, $|f((v, j)) - f((u, j'))|_r \geq 1$. Therefore $A_v \cap A_u = \emptyset$. Let q be a positive integer and let $k = \lfloor (\delta' - \varepsilon)q \rfloor + q - 1$. For each vertex v of H, let $\phi(v) = \{i : i/q \in A_v\}$. Since A_v is either an interval of length at least $1 + \delta' - \varepsilon$ or contains two disjoint intervals of length 1, we conclude that $|\phi(v)| \geq k$. Let $n = \lfloor rq \rfloor + 1$. Then $\phi(v) \subseteq \{0, 1, \cdots, n-1\}$ for all $v \in V(H)$. Hence ϕ gives a k-tuple n-colouring of H. Therefore

$$\chi_f(H) \leq n/k = \frac{\lfloor rq \rfloor + 1}{\lfloor (\delta' - \varepsilon)q \rfloor + q - 1} \leq \frac{rq + 1}{(\delta' - \varepsilon + 1)q - 2}.$$

By letting q approach infinity, we obtain the following contradiction:

$$\chi_f(H) \leq \lim_{q \to \infty} \frac{rq + 1}{(\delta' - \varepsilon + 1)q - 2} = \frac{r}{\delta' - \varepsilon + 1} < \frac{r}{\delta - \varepsilon + 1} = \chi_f(H).$$

□

Corollary 6.2 *Let H be a graph with $\chi_f(H) = \chi_c(H)$. Then for any real number $r \in [\chi_c(H), 2\chi_c(H))$,*

$$\lim_{m \to \infty} ch^r_{cc}(H[m]) = r - \frac{r}{\chi_c(H)} + 1.$$

Proof. Let m be a positive integer. By Theorem 6.1, we have

$$ch^r_{cc}(H[m]) \geq r - \frac{r}{\chi_c(H)} - \frac{r}{m} + 1.$$

By Lemma 3.7,

$$ch^r_{cc}(G) \leq \frac{r}{\chi_c(G)} ch^{\chi_c(G)}_{cc}(G) - \frac{r}{\chi_c(G)} + 1.$$

Since $ch^{\chi_c(G)}_{cc}(G) \leq \chi_c(G)(1 - \frac{1}{|V(G)|})$, we have

$$ch^r_{cc}(H[m]) \leq r - \frac{r}{\chi_c(H)} - \frac{r}{mn} + 1.$$

Thus $\lim_{m \to \infty} ch^r_{cc}(H[m]) = r - \frac{r}{\chi_c(H)} + 1$. □

Corollary 6.3 *For any positive integers n, m, for any real number $n \leq r < 2n$,*

$$ch^r_{cc}(K_n[m]) \geq r - \frac{r}{n} - \frac{r}{m} + 1.$$

By Theorem 4.4, we have

$$ch_{cc}^r(K_n[m]) \le r - \frac{r}{nm} - \frac{r}{n} + 1.$$

So there is a small gap between the upper and lower bounds for $ch_{cc}^r(K_n[m])$. We have not been able to determine the exact value of $ch_{cc}^r(K_n[m])$ for all n, m, r. In the following, we determine the value for the case $n = 2$ and the case $n \ge 3$ but $m = 2kn-1$ for some positive integer k.

Theorem 6.4 *For any real number $r \in [2, 4)$ and any positive integer m,*

$$ch_{cc}^r(K_{m,m}) = \frac{r}{2} - \frac{r}{2m} + 1.$$

Proof. If $m = 1$ then, by Theorem 5.2, the theorem is true. Thus we assume that $m \ge 2$ and assume the vertices of $K_{m,m}$ are $\{v_{i,j} : i = 1, 2, j = 0, 1, \cdots, m-1\}$, where $v_{1,j}$ is adjacent to $v_{2,j'}$ for $0 \le j, j' \le m-1$. By Theorem 4.4, since $\chi_c(K_{m,m}) = 2$, we have $ch_{cc}^r(K_{m,m}) \le \frac{r}{2} - \frac{r}{2m} + 1$. It remains to show that $ch_{cc}^r(K_{m,m}) \ge \frac{r}{2} - \frac{r}{2m} + 1$. Let $\varepsilon = \frac{r}{m}$ and $\delta = \frac{r}{2} + \frac{r}{2m} - 1$. Then $r = 2 + 2\delta - \varepsilon$, $\delta - \varepsilon < 1$ and $\varepsilon \le 2\delta$.

Let

$$\begin{aligned} l(v_{1,j}) &= j\varepsilon, & j &= 0, 1, \ldots, m-1; \\ l(v_{2,j}) &= j\varepsilon + \delta + 1, & j &= 0, 1, \ldots, m-1. \end{aligned}$$

We shall show that for any real number $\delta' > \delta$, $K_{m,m}$ has no circular r-colouring compatible with $(l, r - \delta')$. Assume to the contrary that f is an r-colouring of $K_{m,m}$ which is compatible with $(l, r-\delta')$. Let $A_i = \cup_{j=0}^{m-1}[f(v_{i,j}), f(v_{i,j})+1)_r$. Then $A_1 \cap A_2 = \emptyset$, and the sum of the total lengths of A_1 and A_2 is at most r. As each A_i has total length at least 1, and $r < 4$, we conclude that at least one of A_1, A_2, say A_1, has total length less than 2. This implies that A_1 is a single interval. Assume $A_1 = [a, b)_r$. Observe that $l(v_{1,j})$ $(j = 0, 1, \cdots, m-1)$ partition the circle $S(r)$ into m intervals of length ε. Thus by symmetry, we may assume that $1 \le b < 1 + \varepsilon$. Since $l(v_{1,1}) = \varepsilon$ and

$[f(v_{1,1}) - l(v_{1,1})]_r \leq r - \delta'$, and since A_1 has total length less than 2, we conclude that

$$f(v_{1,1}) \in (r-1, r-\delta'+\varepsilon]_r.$$

(Note that in case $\delta' < \varepsilon$, $[r-\delta'+\varepsilon]_r = \varepsilon - \delta'$.) This implies that for any j, $1 \leq b \leq f(v_{2,j}) \leq r - \delta' + \varepsilon - 1$. In particular,

$$1 \leq f(v_{2,0}) \leq r - \delta' + \varepsilon - 1.$$

As $l(v_{2,0}) = 1 + \delta > r - \delta' + \varepsilon - 1$, we conclude that $f(v_{2,0}) < l(v_{2,0})$ and hence

$$[f(v_{2,0}) - l(v_{2,0})]_r = f(v_{2,0}) - l(v_{2,0}) + r \geq 1 - (1+\delta) + r = r - \delta > r - \delta'.$$

This is in contrary to the assumption that f is compatible with $(l, r-\delta')$. □

As a consequence of Corollary 4.3 and Theorem 6.4, we have the following corollary.

Corollary 6.5 *For any $m \geq 1$,*

$$ch_{cc}(K_{m,m}) = 3 - \frac{2}{m}.$$

Observe that $ch^4_{cc}(K_{m,m}) \leq ch^2_{cc}(K_{m,m}) = 2 - \frac{1}{m}$ and $\lim_{r \to 4^-} ch^r_{cc}(K_{m,m}) = 3 - \frac{2}{m}$. Thus $ch^r_{cc}(G)$ need not be continuous as a function of r.

Theorem 6.6 *Let n and m be two positive integers such that $m = 2kn - 1$ for some integer $k \geq 2$. Then for any real number r in $[n, 2nm/(m+1))$,*

$$ch^r_{cc}(K_n[m]) = r - \frac{r}{n} - \frac{r}{nm} + 1.$$

Proof. By Theorem 4.4, $ch^r_{cc}(K_n[m]) \leq r - \frac{r}{n} - \frac{r}{nm} + 1$. Let $\varepsilon = \frac{r}{m}$ and $\delta = \frac{r}{n} + \frac{r}{nm} - 1$. Then $r = n\delta - \varepsilon + n$ and $\delta + 1 = 2k\varepsilon$. The condition that $r < 2nm/(m+1)$ implies that $\delta < 1$. Let

$$l(i,j) = j\varepsilon + \delta, \quad j = 0, 1, \ldots, m-1, \ i = 0, 1, \cdots, n-1.$$

We shall prove that for any circular r-colouring f of $K_n[m]$, there is a vertex (i,j) such that $[f(i,j)-l(i,j)]_r \geq r-\delta$. For $i=0,1,\cdots,n-1$, let $A_i = \cup_{j=0}^{m-1}(f(i,j), f(i,j)+1]_r$, and let $I_i = \{j : 0 \leq j \leq m-1, j\varepsilon \in A_i\}$. Since $I_i \subseteq \{0,1,\cdots,m-1\}$ and $I_i \cap I_j = \emptyset$ for $i \neq j$, there is an index i such that $|I_i| \leq m/n < 2k$. As $1+\delta = 2k\varepsilon$ and $\delta < 1$, we conclude that $1 > k\varepsilon$ and hence for any interval X of $S(r)$ of length 1, $|X \cap \{j\varepsilon : j = 0,1,\cdots,m-1\}| \geq k$. Observe that A_i is the union of intervals of $S(r)$ of length 1. Since $|I_i| < 2k$, we conclude that A_i cannot contains two disjoint intervals of length 1. So all the intervals in A_i intersects, and hence A_i is a single interval. Assume $A_i = (a,b]_r$. Without loss of generality, assume that $0 \leq a < \varepsilon$. Since $|I_i| \leq 2k-1$, then $b < 2k\varepsilon = \delta + 1$. So $0 \leq f(i,j) \leq b-1 < \delta$ for $j = 0,1,\cdots,m-1$. Then $[f(i,0)-l(i,0)]_r = f(i,0)-l(i,0)+r \geq r-\delta$. □

As $\chi_c(K_n[m]) = n$, by Lemma 3.7, $ch_{cc}(K_n[m]) \leq 2n-1-\frac{2}{m}$. Let $r_0 = 2nm/(m+1)$. By Theorem 6.6, for $m = 2kn-1$ ($k \geq 2$), we have

$$ch_{cc}(K_n[m]) \geq \lim_{r \to r_0^-} ch^r_{cc}(K_n[m]) = 2n-1-\frac{2n}{m+1}.$$

When m is large, the lower bound and the upper bound for $ch_{cc}(K_n[m])$ are arbitrarily close. In this sense, the upper bound in Lemma 3.7 is best possible.

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