

Circular chromatic numbers and fractional chromatic numbers of distance graphs

Gerard J. Chang* Lingling Huang* Xuding Zhu†

May 8, 1997 (revision October 1997)

Abstract

This paper studies circular chromatic numbers and fractional chromatic numbers of distance graphs $G(Z, D)$ for various distance sets D . In particular, we determine these numbers for those D sets of size two, for some special D sets of size three, for $D = \{1, 2, \dots, m, n\}$ with $1 \leq m < n$, for $D = \{q, q + 1, \dots, p\}$ with $q \leq p$, and for $D = \{1, 2, \dots, m\} - \{k\}$ with $1 \leq k \leq m$.

1 Introduction

Suppose S is a subset of a metric space \mathcal{M} with a metric δ , and D a subset of positive real numbers. The *distance graph* $G(S, D)$, with a *distance set* D , is the graph with vertex set S in which two vertices x and y are adjacent if and only if $\delta(x, y) \in D$. Distance graphs, first studied by Eggleton, Erdős and Skilton [7], were motivated by the well-known plane-coloring problem: What is the minimum number of colors needed to color all points of a Euclidean plane so that points at unit distances are colored with different colors. This problem is equivalent to determining the chromatic number of the distance graph $G(R^2, \{1\})$. It is well-known that the chromatic number

*Department of Applied Mathematics, National Chiao Tung University, Hsinchu 30050, Taiwan. Email: gchang,llhuang@math.nctu.edu.tw. Supported in part by the National Science Council under grant NSC86-2115-M009-002.

†Department of Applied Mathematics, National Sun Yat-sen University, Kaoshing 80424, Taiwan. Email: zhu@ibm18.math.nsysu.edu.tw. Supported in part by the National Science Council under grant NSC87-2115-M110-004.

of this distance graph is between 4 and 7 (see [12, 15]). However, the exact number of colors needed remains unknown.

For distance graphs on the real line R or the integer set Z , the problem of finding the chromatic numbers of $G(R, D)$ or $G(Z, D)$ for different D sets has been studied extensively (see [3, 4, 5, 6, 7, 8, 9, 10, 13, 14, 17, 18, 20, 21, 22]). Two recent papers [3, 14] related distance graphs to the T -coloring problem. Chromatic numbers and fractional chromatic numbers of distance graphs were used to derive bounds for T -spans of the corresponding T -colorings, and vice-versa. In this paper, we study circular chromatic numbers and fractional chromatic numbers of distance graphs $G(Z, D)$ for various D sets.

The circular chromatic number of a graph is a natural generalization of the chromatic number of a graph, introduced by Vince [16] under the name the “star chromatic number” of a graph. Suppose p and q are positive integers such that $p \geq 2q$. A (p, q) -coloring of a graph $G = (V, E)$ is a mapping c from V to $\{0, 1, \dots, p - 1\}$ such that $\|c(x) - c(y)\|_p \geq q$ for any edge xy in E , where $\|a\|_p = \min\{a, p - a\}$. The *circular chromatic number* $\chi_c(G)$ of G is the infimum of the ratios p/q for which there exist (p, q) -colorings of G .

Note that a $(p, 1)$ -coloring of a graph G is simply an ordinary p -coloring of G . Therefore, $\chi_c(G) \leq \chi(G)$ for any graph G . On the other hand, it has been shown [16] that for all graphs G , we have $\chi(G) - 1 < \chi_c(G)$. Therefore, $\chi(G) = \lceil \chi_c(G) \rceil$. In particular, two graphs with the same circular chromatic number also have the same chromatic number. However, two graphs with the same chromatic number may have different circular chromatic numbers. Thus $\chi_c(G)$ is a refinement of $\chi(G)$, and it contains more information about the structure of the graph. It is usually much more difficult to determine the circular chromatic number of a graph than to determine its chromatic number. The main results of this article determine the circular chromatic numbers of various distance graphs. These results maybe viewed as improvements on

previous results concerning the chromatic numbers of these distance graphs presented in [3, 4, 7, 13, 17, 21].

The fractional chromatic number of a graph is another well-known variation of the chromatic number. A *fractional coloring* of a graph G is a mapping c from $\mathcal{I}(G)$, the set of all independent sets of G , to the interval $[0, 1]$ such that $\sum_{x \in I \in \mathcal{I}(G)} c(I) \geq 1$ for all vertices x in G . The *fractional chromatic number* $\chi_f(G)$ of G is the infimum of the value $\sum_{I \in \mathcal{I}(G)} c(I)$ of a fractional coloring c of G .

For any graph G , it is well-known that

$$\omega(G) \leq |G|/\alpha(G) \leq \chi_f(G) \leq \chi_c(G) \leq \lceil \chi_c(G) \rceil = \chi(G). \quad (*)$$

For simplicity, let $\omega(S, D)$, $\alpha(S, D)$, $\chi_f(S, D)$, $\chi_c(S, D)$ and $\chi(S, D)$ denote respectively, the clique number, the independence number, the fractional chromatic number, the circular chromatic number, and the chromatic number of a distance graph $G(S, D)$.

Chromatic numbers of distance graphs with distance sets $|D| \leq 2$ were determined by Chen, Chang and Huang [4] and Voigt [17]. Chromatic numbers of distance graphs with $|D| = 3$ were determined by Zhu [21]. In Section 2, we use a “multiplier method” to establish an upper bound for the circular chromatic number of a distance graph $G(Z, D)$ with an arbitrary distance set D . This upper bound is then used to determine the circular chromatic numbers and the fractional chromatic numbers of those distance graphs with distance sets D for $|D| = 2$, for some special D with $|D| = 3$, for $D = \{1, 2, \dots, m, n\}$ with $1 \leq m < n$, and for $D = \{q, q + 1, \dots, p\}$ with $q \leq p$. The chromatic number for $G(Z, D)$ with $D = \{q, q + 1, \dots, p\}$ was determined in [7, 13].

Chromatic numbers of distance graphs with distance sets of the form $D_{m,k} = \{1, 2, \dots, m\} - \{k\}$, with $1 \leq k \leq m$, were studied in [3, 7, 13, 14]. Partial results concerning chromatic numbers of such distance graphs were obtained in [7, 13, 14],

and a complete solution was recently obtained by Chang, Liu and Zhu [3]. The authors of [3] also obtained circular chromatic numbers of such distance graphs for some special values of m and k . In Section 3, we determine the circular chromatic numbers $\chi_c(Z, D_{m,k})$ for all integer pairs m, k .

2 Multiplier method for $\chi_f(Z, D)$ and $\chi_c(Z, D)$

In this section we use a “multiplier method” to establish an upper bound on $\chi_c(Z, D)$ for an arbitrary D set. We then use this upper bound to determine circular chromatic numbers for some D sets.

The multiplier method was used in [2] to study the density of D -sets, and was also used in [11] to study fractional chromatic numbers and circular chromatic numbers of circulant graphs. In taking distance graphs to be “infinite” circulant graphs, Theorem 2.2 below is parallel to a result in [11]. Half of the proof of Theorem 2.3 is parallel to an argument in [2].

Lemma 2.1 *Suppose D is a set of positive integers, and that p and r are positive integers. Let*

$$d_D(p, r) = \min\{|ri \bmod p|_p : i \in D\}.$$

If $d_D(p, r) \geq 1$, then $\chi_c(Z, D) \leq p/d_D(p, r)$.

Proof. It is straightforward to verify that the coloring defined as $c(i) = (ri \bmod p)$ for $i \in Z$ is a $(p, d_D(p, r))$ -coloring of the distance graph $G(Z, D)$. ■

Let $f_D = \inf\{p/d_D(p, r) : d_D(p, r) \geq 1\}$. The function is well-defined since $d_D(p, r)$ is always an integer between 0 and $\lfloor p/2 \rfloor$. Theorem 2.2 below follows from Lemma 2.1.

Theorem 2.2 *For any set D of positive integers, $\chi_c(Z, D) \leq f_D$.*

It is known [4, 17] that if D contains exactly two relatively prime integers, then $\chi(Z, D) = 2$ when the two integers are odd and $\chi(Z, D) = 3$ when the two integers have different parities. We first use f_D to determine $\chi_c(Z, D)$ and $\chi_f(Z, D)$ for D with $|D| = 2$.

Theorem 2.3 *If $D = \{a, b\}$ and $\gcd(a, b) = 1$, then*

$$\chi_f(Z, D) = \chi_c(Z, D) = f_D = (a + b) / \lfloor (a + b) / 2 \rfloor.$$

Proof. Suppose both a and b are odd. Since $2 \leq \omega(Z, D)$ and $d_D(2, 1) = 1$, the theorem follows from (*) and Theorem 2.2.

Suppose that a and b have different parities, i.e., $a + b$ is odd. Assume that $a + b = p$. Since $\gcd(p, b - a) = 1$, there exists a positive integer r such that $r(b - a) \equiv 1 \pmod{p}$. Since $r(b + a) \equiv 0 \pmod{p}$, it follows that $2rb \equiv -2ra \equiv 1 \pmod{p}$. Hence, $ra \equiv -rb \equiv (p - 1)/2 \pmod{p}$, which implies that $d_D(p, r) = (p - 1)/2$. Hence, according to Theorem 2.2, $\chi_c(Z, D) \leq f_D \leq 2p/(p - 1) = (a + b) / \lfloor (a + b) / 2 \rfloor$. On the other hand, it is easy to see that $G(Z, D)$ contains the odd cycle C_p . Thus, $2p/(p - 1) \leq p/\alpha(C_p) \leq \chi_f(C_p) \leq \chi_f(Z, D) \leq \chi_c(Z, D)$. This completes the proof of the theorem. ■

Note that precisely the same arguments in the first two lines of the proof above also give that $\chi_f(Z, D) = \chi_c(Z, D) = f_D = 2$ if D contains only odd integers.

We now consider circular chromatic numbers and fractional chromatic numbers of distance graphs $G(Z, D)$ with $|D| = 3$. Zhu [21] proved the following result for chromatic numbers, which provides a range for circular chromatic numbers.

Theorem 2.4 ([21]) *If $D = \{a, b, c\}$, where $a < b < c$ are relatively prime positive*

integers, then

$$\chi(Z, D) = \begin{cases} 2, & \text{if } a, b, c \text{ are odd,} \\ 4, & \text{if } a = 1 \text{ and } b = 2 \text{ and } c \equiv 0 \pmod{3}, \\ 4, & \text{if } a + b = c \text{ and } a \not\equiv b \pmod{3}, \\ 3, & \text{otherwise.} \end{cases}$$

Theorem 2.5 *If $D = \{a, a + 1, c\}$, with $a + 1 < c$, where $c + a = (2a + 1)k + r$, with $k \geq 1$ and $0 \leq r \leq 2a$, then*

$$\chi_f(Z, D) \leq \chi_c(Z, D) \leq f_D \leq \begin{cases} (c + a)/(ak), & \text{if } 0 \leq r \leq a, \\ (c + a + 1)/(ak + r - a), & \text{if } a + 1 \leq r \leq 2a. \end{cases}$$

Proof. Note that $ck \equiv -ak \pmod{c+a}$ and $c(k+1) \equiv -(a+1)(k+1) \pmod{c+a+1}$. Therefore, $d_D(c + a, k) = ak$ for all r , and $d_D(c + a + 1, k + 1) = ak + r - a$ when $a + 1 \leq r \leq 2a$. The theorem then follows. \blacksquare

Theorem 2.6 *If $D = \{a, a + 1, c\}$ with $a + 1 < c$ and $c + a \equiv 2a$ or $0 \pmod{2a + 1}$, then $\chi_f(Z, D) = \chi_c(Z, D) = f_D = 2 + 1/a$.*

Proof. Since $G(Z, D)$ contains the odd cycle C_{2a+1} , according to (*), $2 + 1/a = (2a + 1)/\alpha(C_{2a+1}) \leq \chi_f(C_{2a+1}) \leq \chi_f(Z, D)$. On the other hand, since $c + a \equiv 2a$ or $0 \pmod{2a + 1}$, it follows from Theorem 2.5 that $f_D \leq 2 + 1/a$. \blacksquare

Denote the subgraph of $G(Z, D)$ induced by $V_i = \{0, 1, \dots, i\}$ as G_i .

Theorem 2.7 *If $D = \{2, 3, c\}$, with $3 < c$, where $c + 2 = 5k + r$, with $k \geq 1$ and $0 \leq r \leq 4$, then*

$$\chi_f(Z, D) = \chi_c(Z, D) = f_D = \begin{cases} (c + 2)/2k, & \text{if } r = 1, 2, \\ (c + 3)/(2k + 1), & \text{if } r = 3, \\ 5/2, & \text{if } r = 4, 0. \end{cases}$$

Proof. The case in which $r = 4$ or 0 follows from Theorem 2.6. For the other cases, Theorem 2.5 implies that $f_D \leq (c + 2)/2k$ when $r = 1, 2$, and $f_D \leq (c + 3)/(2k + 1)$ when $r = 3$. Therefore it suffices to show that $\alpha(G_{c+1}) \leq 2k$ when $r = 1, 2$ and $\alpha(G_{c+2}) \leq 2k + 1$ when $r = 3$.

Consider the graph G_{c+2} for $r = 1, 2, 3$. Decompose the vertex set $\{0, 1, \dots, c + 2\}$ into $k + 1$ subsets $I_i = \{5i, 5i + 1, \dots, 5i + 4\}$ for $0 \leq i \leq k - 1$, and $J = \{5k, \dots, 5k + r = c + 2\}$. Then, $J = \{c + 1, c + 2\}$ when $r = 1$, $J = \{c, c + 1, c + 2\}$ when $r = 2$, and $J = \{c - 1, c, c + 1, c + 2\}$ when $r = 3$. Suppose that G_{c+2} has an independent set S of size $2k + 2$. We may assume that $0 \in S$ and then $c \notin S$. Since every 5 consecutive vertices in G_{c+2} form a 5-cycle, we conclude that $|I_i \cap S| = |J \cap S| = 2$ for $0 \leq i \leq k - 1$. Then $c + 1 \in S$, and hence, $1 \notin S$.

Since $|I_0 \cap S| = 2$, 2 and 3 are not in S . We therefore conclude that $4 \in S$. In a general step, using the fact that $|I_i \cap S| = 2$ and $5(i - 1) - 1 \in S$, it is straightforward to derive that $5i - 1 \in S$. Therefore, $5k - 1 \in S$. Since $5k - 1 = c$ when $r = 1$, and $5k - 1$ is adjacent to $c + 1$ when $r = 2$ or 3 , we have contradictions. Hence, $\alpha(G_{c+2}) \leq 2k + 1$ for $r = 1, 2, 3$. Moreover, for the case in which $r = 1$ or 2 , any independent set S' of G_{c+2} of size $2k + 1$ that contains the vertex 0 does not contain the vertex $c + 1$. Hence, $c + 2 \in S'$ and $\alpha(G_{c+1}) \leq 2k$. This completes the proof of the theorem. ■

Theorem 2.8 *Suppose $D = \{a, b, a + b\}$, with $0 < a < b$ and $\gcd(a, b) = 1$. If $a \equiv b \pmod{3}$, then $\chi_f(Z, D) = \chi_c(Z, D) = f_D = 3$.*

Proof. Since $\gcd(a, b) = 1$ and $a \equiv b \pmod{3}$, we have $a, b, c \not\equiv 0 \pmod{3}$ and so $d_D(3, 1) = 1$. The theorem then follows from (*) and the fact that $\{0, a, a + b\}$ is a clique. ■

Theorem 2.9 *If $D = \{1, 2, \dots, m, n\}$, with $1 \leq m < n$, then*

$$\chi_f(Z, D) = \chi_c(Z, D) = f_D = \begin{cases} m + 1, & \text{if } n \not\equiv 0 \pmod{m + 1}, \\ m + 1 + 1/k, & \text{if } n = k(m + 1). \end{cases}$$

Proof. Suppose $n \not\equiv 0 \pmod{m + 1}$. Since $m + 1 \leq \omega(G)$ and $d_D(m + 1, 1) = 1$, the theorem follows from (*) and Theorem 2.2.

Suppose $n = k(m + 1)$. Since every independent set of G_n contains at most one vertex from any $m + 1$ consecutive vertices, and at most one vertex from $\{0, n\}$, $\alpha(G_n) = k$. Consequently, $m + 1 + 1/k = (n + 1)/\alpha(G_n) \leq \chi_f(G_n) \leq \chi_f(Z, D)$. Also, $f_D \leq (n + 1)/d_D(n + 1, k) = m + 1 + 1/k$. The theorem then follows. ■

Corollary 2.10 *If $D = \{1, 2, 3k\}$, where $k \geq 1$, then $\chi_f(Z, D) = \chi_c(Z, D) = 3 + 1/k$.*

This is one of the two cases covered by Theorem 2.4 in which we have $\chi(Z, D) = 4$. The other is that in which $D = \{a, b, c\}$, $a + b = c$ and $a \not\equiv b \pmod{3}$. We note that in this case, the chromatic number of $G(Z, D)$ is easily determined. However, the circular chromatic numbers of $G(Z, D)$ are still unknown, except for some special values of a and b .

We summarize the results for $D = \{a, b, c\}$ with $a < b < c$ and $\gcd(a, b, c) = 1$ in the following table.

Conditions of a, b, c		$\chi_f(Z, D), \chi_c(Z, D), f_D$	$\chi(Z, D)$
a, b, c are odd		2	2
$a = 1, b = 2, c = 3k$		$3 + \frac{1}{k}$ (Corollary 2.10)	4
$c = a + b, a \not\equiv b \pmod{3}$?	4
$c = a + b, a \equiv b \pmod{3}$		3 (Theorem 2.8)	3
$b = a + 1, c \equiv a \text{ or } a + 1 \pmod{2a + 1}$		$2 + \frac{1}{a}$ (Theorem 2.6)	
$a = 2, b = 3, c + 2 = 5k + r$	$r = 1, 2$	$\frac{c+2}{2k}$ (Theorem 2.7)	
	$r = 3$	$\frac{c+3}{2k+1}$ (Theorem 2.7)	
otherwise		?	

Theorem 2.11 *If $D = \{q, q + 1, \dots, p\}$, with $q \leq p$, then $\chi_f(Z, D) = \chi_c(Z, D) = f_D = 1 + p/q$.*

Proof. Since $d_D(p + q, 1) = q$, we conclude that $f_D \leq (p + q)/q$. On the other hand, it is quite obvious that $\alpha(G_{p+q-1}) = q$. Hence, $\chi_f(Z, D) = \chi_c(Z, D) = f_D = 1 + p/q$. ■

Theorem 2.12 *If $D = [1, r]$, where r is any real number greater than or equal to 1, then $\chi_f(R, D) = \chi_c(R, D) = 1 + r$.*

Proof. We first consider the case in which $r = p/q$ is rational. Let $D' = \{q, q + 1, \dots, p\}$. It is then straightforward to verify that each connected component of $G(R, D)$ is isomorphic to $G(Z, D')$. According to Theorem 2.11, $\chi_f(R, D) = \chi_c(R, D) = 1 + r$.

When r is irrational, then let $(r_i : i = 1, 2, \dots)$ and $(r'_i : i = 1, 2, \dots)$ be sequences of rational numbers such that $r'_i \leq r \leq r_i$ for each i and $\lim_{i \rightarrow \infty} r'_i = \lim_{i \rightarrow \infty} r_i = r$. The above argument then shows that $1 + r'_i \leq \chi_f(R, D) \leq \chi_c(R, D) \leq 1 + r_i$ for each i . Therefore, $\chi_f(R, D) = \chi_c(R, D) = 1 + r$. ■

It was shown by Eggleton et al. [10] (Theorem 2) that if a prime distance graph has a proper k -coloring, then it has a periodic k -coloring. The proof in fact shows that any k -colorable distance graph has a periodic k -coloring. We remark that an argument parallel to the proof of Theorem 2 of [10] shows that if a distance graph $G(Z, D)$ has a (p, q) -coloring, then it has a periodic (p, q) -coloring. Also we note that a (p, q) -coloring derived by the multiplier method is always a periodic (p, q) -coloring.

3 Circular chromatic number $\chi_c(Z, D_{m,k})$

As mentioned in the introduction, Chang, Liu and Zhu [3] determined the chromatic number and the fractional chromatic number of the distance graph $G(Z, D_{m,k})$, where

$D_{m,k} = \{1, 2, \dots, m\} - \{k\}$ and $1 \leq k \leq m$. They also determined the circular chromatic number of $G(Z, D_{m,k})$ for some pairs of integers m and k

Let $m+k+1 = 2^r m'$ and $k = 2^s k'$, where m' and k' are both odd. The following table shows their results.

Conditions of m, k, r, s		$\chi_f(Z, D_{m,k})$	$\chi_c(Z, D_{m,k})$	$\chi(Z, D_{m,k})$
$2k > m$		k	k	k
$2k \leq m$	$r > s$	$\frac{m+k+1}{2}$	$\frac{m+k+1}{2}$	$\frac{m+k+1}{2}$
	$r = 0$			$\gcd(m+k+1, k) = 1$
			$\gcd(m+k+1, k) \neq 1$	$\frac{m+k+3}{2}$
$1 \leq r \leq s$?	

The circular chromatic numbers $\chi_c(Z, D_{m,k})$ remain unknown for those pairs of integers m, k corresponding to the question mark in the table above. In this section, we shall fill in the unknown part of the table above by showing that $\chi_c(Z, D_{m,k}) = \frac{m+k+2}{2}$ when $2k \leq m$ and $r \leq s$ and $\gcd(m+k+1, k) \neq 1$.

The following lemma was proven in [16] and is used frequently in our proofs.

Lemma 3.1 ([16]) *If G has a circular chromatic number $\frac{p}{q}$ (where p and q are relatively prime), then $p \leq |V(G)|$, and any (p, q) -coloring c of G is an onto mapping from $V(G)$ to $\{0, 1, \dots, p-1\}$.*

As in the preceding section, we denote the subgraph of $G(Z, D_{m,k})$ induced by $V_i = \{0, 1, \dots, i\}$ as G_i . We shall first derive a lower bound for $\chi_c(Z, D_{m,k})$.

Lemma 3.2 *Suppose $2k \leq m$. Let $m+k+1 = 2^r m'$ and $k = 2^s k'$, where r and s are non-negative integers and m' and k' are odd integers. If $1 \leq r \leq s$, then $\chi_c(G_{m+2k-1}) > \frac{m+k+1}{2}$.*

Proof. Since $m+k+1$ is even and $\chi_c(G_{m+2k-1}) > \chi(G_{m+2k-1}) - 1$, it suffices to show that $\chi(G_{m+2k-1}) > \frac{m+k+1}{2}$. Assume to the contrary that $\chi(G_{m+2k-1}) \leq \frac{m+k+1}{2}$, and that c is a $\frac{m+k+1}{2}$ -coloring of G_{m+2k-1} .

For each integer i with $0 \leq i \leq k-2$, consider the subgraph of G_{m+2k-1} induced by the $m+k+1$ vertices $\{i, i+1, \dots, i+m+k\}$. This graph has an independence number 2. Therefore, each of the $\frac{m+k+1}{2}$ colors is used at most, and thus exactly, twice in this subgraph. Consequently, vertices i and $i+m+k+1$ have the same colors for all $0 \leq i \leq k-2$. Therefore, for each $j \in S := \{0, 1, \dots, m+k\}$, the only possible vertices in S having the same color as j are $j+k$ and $j-k$.

Consider the *circulant graph* $C(m+k+1, k)$, with vertex set S and in which vertex i is adjacent to vertex j if and only if $j \equiv i+k$ or $i-k \pmod{m+k+1}$. It follows from the discussion in the preceding paragraph that two vertices x and y in S have the same color only if xy is an edge of the circulant graph $C(m+k+1, k)$. Since the intersection of each color class with S contains exactly two vertices, the coloring induces a perfect matching of $C(m+k+1, k)$. However, $C(m+k+1, k)$ is the disjoint union of d cycles of length $\frac{m+k+1}{d}$, where $d = \gcd(m+k+1, k)$. Since $C(m+k+1, k)$ has a perfect matching, each cycle has an even length. This implies that $r > s$, contrary to the assumption $r \leq s$. Hence, $\chi(G_{m+2k-1}) > \frac{m+k+1}{2}$. \blacksquare

Lemma 3.3 *Suppose $2k \leq m$. If $m+k+1$ is odd and $\gcd(m+k+1, k) \neq 1$, then $\chi_c(G_{m+k}) > \frac{m+k+1}{2}$, and hence, $\chi_c(G_{m+2k-1}) > \frac{m+k+1}{2}$.*

Proof. First, it is clear that $\chi_c(G_{m+k}) \geq \frac{m+k+1}{\alpha(G_{m+k})} = \frac{m+k+1}{2}$. Suppose $\chi_c(G_{m+k}) = \frac{m+k+1}{2}$. Since $m+k+1$ and 2 are relatively prime, every $(m+k+1, 2)$ -coloring c of G_{m+k} is onto and hence is one-to-one; i.e., there exists an ordering $x_0, x_1, x_2, \dots, x_{m+k}$ of V_{m+k} such that $c(x_i) = i$ for $0 \leq i \leq m+k$. Therefore, $X = (x_0, x_1, \dots, x_{m+k}, x_0)$ is a cycle in the complement G' of G_{m+k} .

Let $m = ak + b$, where $0 \leq b < k$. Since all vertices of $\{k-1, k, \dots, m+1\}$ are of degree two in G' , the following paths must be on the cycle X :

$$P_i : i, k+i, 2k+i, \dots, ak+i, (a+1)k+i \quad \text{for } 0 \leq i \leq b;$$

$$P_j : j, k + j, 2k + j, \dots, ak + j \quad \text{for } b + 1 \leq j \leq k - 1.$$

For each vertex u , let $N(u) = \{v \in V_{m+k} : uv \in E(G')\}$. Since $N(k - 1) = \{2k - 1, m + k\}$ and $m + k = (a + 1)k + b$, we have that $P_b P_{k-1}$ is a path of the cycle X . Since $N(k - 2) = \{2k - 2, m + k - 1, m + k\}$ and vertex $m + k$ is on the path $P_b P_{k-1}$, we have that $P_{b-1} P_{k-2}$ is a path of the cycle X . Continuing this process, we have that $P'_t = P_{b+1+t} P_t$, where the index $b + 1 + t$ is taken modulo k , is a path of the cycle X for $0 \leq t \leq k - 1$. Since $\gcd(m + k + 1, k) \neq 1$, we have $\gcd(b + 1, k) \neq 1$. Therefore, these paths P'_t form at least 2 disjoint cycles, contrary to our assumption that X is a cycle. Thus, the coloring c does not exist and $\chi_c(G_{m+k}) > \frac{m+k+1}{2}$.

Since G_{m+k} is a subgraph of G_{m+2k-1} , we conclude that $\chi_c(G_{m+2k-1}) > \frac{m+k+1}{2}$. ■

Theorem 3.4 *Suppose $2k \leq m$. Let $m+k+1 = 2^r m'$ and $k = 2^s k'$, where r and s are non-negative integers and m' and k' are odd integers. If $r \leq s$ and $\gcd(m+k+1, k) \neq 1$, then $\chi_c(Z, D_{m,k}) \geq \frac{m+k+2}{2}$.*

Proof. Suppose $\chi_c(G_{m+2k-1}) = \frac{p}{q}$, where p and q are relatively prime. Then, $p \leq |V_{m+2k-1}| = m + 2k$ and $\frac{p}{q} > \frac{m+k+1}{2}$ according to Lemmas 3.2 and 3.3. If $q \geq 3$, then $p > \frac{q}{2}(m + k + 1) \geq \frac{3}{2}(m + k + 1) > m + 2k$, a contradiction. Hence, $q \leq 2$ and so $\chi_c(Z, D_{m,k}) \geq \frac{p}{q} \geq \frac{m+k+2}{2}$. ■

Now we give an $(m + k + 2, 2)$ -coloring of $G(Z, D_{m,k})$ to show that $\chi_c(Z, D_{m,k}) \leq \frac{m+k+2}{2}$. We first give an $(m + k + 2, 2)$ -coloring of G_{m+k} that is a variation of the coloring given in Theorem 2.1 after a shift operation. It is then extended to an $(m + k + 2, 2)$ -coloring of $G(Z, D_{m,k})$.

Lemma 3.5 *If $2k \leq m$, then G_{m+k} has an $(m + k + 2, 2)$ -coloring c such that $c(x) = c(x - k) + 1$ for $k \leq x \leq m + k$.*

Proof. Suppose $m + k + 1 = dm'$ and $k = dk'$, where $\gcd(m + k + 1, k) = d$. Since $\gcd(m', k') = 1$, there exists an integer n such that $nk' \equiv 1 \pmod{m'}$. Let $a_i = in \pmod{m'}$ for $0 \leq i \leq m' - 1$. Consider the mapping c from V_{m+k} to $\{0, 1, \dots, dm' - 1 = m + k\}$ defined by $c(x) = a_i + jm'$, where $x = id + (d - 1 - j)$, with $0 \leq i \leq m' - 1$ and $0 \leq j \leq d - 1$.

For any edge xy in G_{m+k} , we shall prove that $\|c(x) - c(y)\|_{m+k+2} \geq 2$. Suppose to the contrary that $c(x) = c(y)$, or $c(x) = c(y) + 1$, or $c(x) + 1 = c(y)$. Let $x = i_1d + (d - 1 - j_1)$ and $y = i_2d + (d - 1 - j_2)$. For the case in which $c(x) = c(y)$, we have $a_{i_1} = a_{i_2}$ and $j_1 = j_2$, which imply $i_1 = i_2$ and $x = y$, a contradiction to xy being an edge. For the case in which $c(x) = c(y) + 1$, either (1) $a_{i_1} = a_{i_2} + 1$ and $j_1 = j_2$, or (2) $a_{i_1} = 0$ and $a_{i_2} = m' - 1$ and $j_1 = j_2 + 1$. In subcase (1), we have $i_1 \equiv i_2 + k' \pmod{m'}$. Thus, $x - y = k$ or $y - x = m + 1$, a contradiction. In subcase (2), we have $i_1 = 0$ and $i_2 = m' - k'$. Thus, $y - x = m + 2$, a contradiction. Similarly, it is impossible that $c(x) + 1 = c(y)$. This completes the proof of the lemma. \blacksquare

Theorem 3.6 *If $2k \leq m$, then $\chi_c(Z, D_{m,k}) \leq \frac{m+k+2}{2}$.*

Proof. Let c be the coloring of G_{m+k} given in Lemma 3.5. Consider the mapping c' of $G(Z, D_{m,k})$ defined by

$$c'(x) = \begin{cases} c(x), & \text{for } 0 \leq x \leq m + k, \\ (c'(x - k) + 1) \bmod (m + k + 2), & \text{for } m + k + 1 \leq x, \\ (c'(x + k) - 1) \bmod (m + k + 2), & \text{for } 0 > x. \end{cases}$$

We now show that c' is a proper $(m + k + 2, 2)$ -coloring of $G(Z, D_{m,k})$ by induction. According to Lemma 3.5, c' is proper in G_{m+k} . Suppose c' is proper in G_{x-1} for $x \geq m + k + 1$. Let xy be an edge in G_x ; i.e., $y = x - i$ for some $i \in D_{m,k}$. First, $c'(y)$ is not equal to $c'(x) \bmod (m + k + 2)$ or $(c'(x) - 1) \bmod (m + k + 2)$, since $c'(y) \equiv c'(x) - 2 \pmod{m + k + 2}$ when $i = 2k$, and $y = x - i$ is adjacent to $x - k$ in G_{x-1} when $i \neq 2k$, where $c'(x - k) = (c'(x) - 1) \bmod (m + k + 2)$. Also,

$c'(y - k)$ is not equal to $c'(x) \bmod (m + k + 2)$, since $x - k$ is adjacent to $y - k$ in G_{x-1} and $c'(x - k) = (c'(x) - 1) \bmod (m + k + 2)$. Hence, $c'(y)$ is not equal to $(c'(x) + 1) \bmod (m + k + 2)$. By induction, c' is proper for non-negative vertices in $G(Z^+, D_{m,k})$. Similar arguments work for negative vertices. This completes the proof of the theorem. ■

Combining Theorems 3.4 and 3.6 and results in [3], we have

Theorem 3.7 *Suppose $2k \leq m$. Let $m+k+1 = 2^r m'$ and $k = 2^s k'$, where r and s are non-negative integers and m' and k' are odd integers. If $r \leq s$ and $\gcd(m+k+1, k) \neq 1$, then $\chi_c(Z, D_{m,k}) = \frac{m+k+2}{2}$; otherwise, $\chi_c(Z, D_{m,k}) = \frac{m+k+1}{2}$.*

Acknowledgements. The authors thank a referee for raising the problem of the existence of periodic (p, q) -colorings for distance graphs, which we answer at the end of Section 2.

References

- [1] J. A. Bondy and P. Hell, *A note on the star chromatic number*, J. Graph Theory 14 (1990) 479-482.
- [2] D. G. Cantor and B. Gordon, *Sequences of integers with missing differences*, J. Combin. Theory (A) 14 (1973) 281-287.
- [3] G. J. Chang, D. D.-F. Liu and X. Zhu, *Distance graphs and T-coloring*, manuscript (1997).
- [4] J. Chen, G. J. Chang and K. Huang, *Integral distance graphs*, J. Graph Theory 25 (1997) 287-294.
- [5] W. Deuber and X. Zhu, *The chromatic number of distance graphs*, Discrete Math. 165/166 (1997) 195-204.
- [6] R. B. Eggleton, *New results on 3-chromatic prime distance graphs*, Ars Combinatoria 26B (1988) 153-180.
- [7] R. B. Eggleton, P. Erdős and D. K. Skilton, *Colouring the real line*, J. Combinatorial Theory, Series B 39 (1985) 86-100.

- [8] R. B. Eggleton, P. Erdős and D. K. Skilton, *Research problem 77*, Discrete Math. 58 (1986) 323-324.
- [9] R. B. Eggleton, P. Erdős and D. K. Skilton, *Update information on research problem 77*, Discrete Math. 69 (1988) 105-106.
- [10] R. B. Eggleton, P. Erdős and D. K. Skilton, *Colouring prime distance graphs*, Graphs and Combinatorics 6 (1990) 17-32.
- [11] G. Gao and X. Zhu, *Star extremal graphs and the lexicographic product*, Disc. Math. 152 (1996) 147-156.
- [12] H. Hadwiger, H. Debrunner and V. Klee, *Combinatorial geometry in the plane*, Holt Rinehart and Winston, New York (1964).
- [13] A. Kemnitz and H. Kolberg, *Coloring of integer distance graphs*, Tech. Report, Institute für Mathematik, Technische Universität Braunschweig (1996).
- [14] D. D.-F. Liu, *T-coloring and chromatic number of distance graphs*, manuscript (1997).
- [15] L. Moser and W. Moser, *Solution to problem 10*, Canad. Math. Bull. 4 (1961) 187-189.
- [16] A. Vince, *Star chromatic number*, J. Graph Theory 12 (1988) 551-559.
- [17] M. Voigt, *Colouring of distance graphs*, Ars Combinatoria (to appear).
- [18] M. Voigt and H. Walther, *Chromatic number of prime distance graphs*, Discrete Appl. Math. 51 (1994) 197-209.
- [19] X. Zhu, *Star chromatic numbers and products of graphs*, J. Graph Theory 16 (1992) 557-569.
- [20] X. Zhu, *Colouring the distance graphs*, manuscript (1995).
- [21] X. Zhu, *Distance graphs on the real line*, manuscript (1996).
- [22] X. Zhu, *Non-integral distance graphs on the real line*, manuscript (1996).