

Graphs of large girth with prescribed partial circular colourings

Zhishi Pan

and

Xuding Zhu*

Department of Applied Mathematics

National Sun Yat-sen University

Kaohsiung, Taiwan 80424

Email: zhu@math.nsysu.edu.tw

Abstract

This paper completes the constructive proof of the following result: Suppose $p/q \geq 2$ is a rational number, A is a finite set and f_1, f_2, \dots, f_n are mappings from A to $\{0, 1, \dots, p-1\}$. Then for any integer g , there is a graph $G = (V, E)$ of girth at least g with $A \subset V$, such that G has exactly n (p, q) -colourings (up to equivalence) g_1, g_2, \dots, g_n , and each g_i is an extension of f_i . A probabilistic proof of this result was given in [8]. A constructive proof of the case $p/q \geq 3$ was given in [7].

Keywords: circular chromatic number, girth, uniquely colourable

Mathematical Subject Classification: 05C15

1 Introduction

Suppose $G = (V, E)$ is a graph and $r \geq 2$ is a real number. An r -colouring of G is a mapping $f : V \rightarrow [0, r)$ such that for every edge xy of G , $1 \leq |f(x) - f(y)| \leq r - 1$.

The set $[0, r)$ of “colours” is viewed to form a circle C^r which is obtained from the interval $[0, r]$ by identifying 0 and r into a single point. For $a, b \in [0, r)$, let $[a, b)_r$

*This research was partially supported by the National Science Council under grant NSC91-2115-M-110-004

denote the interval of C^r from a to b along the “increasing” direction. To be precise, if $a < b$, then $[a, b]_r = \{t : a \leq t \leq b\}$. If $a > b$, then $[a, b]_r = \{t : a \leq t < r, \text{ or } 0 \leq t \leq b\}$. For two points $a, b \in C^r$, $|a - b|_r$ is the circular distance between a and b , and is defined as $|a - b|_r = \min\{|a - b|, r - |a - b|\}$. So an r -colouring of G is a mapping $f : V \rightarrow [0, r)$ such that for every edge xy of G , $|f(x) - f(y)|_r \geq 1$.

We say G is r -colourable if G has an r -colouring. The *circular chromatic number* $\chi_c(G)$ of G is defined as

$$\chi_c(G) = \inf\{r : G \text{ is } r\text{-colourable}\}.$$

If $r = k$ is an integer, then an ordinary k -colouring of G is an r -colouring as defined above, and conversely, for an r -colouring f defined as above, $g(x) = \lfloor f(x) \rfloor$ is an ordinary k -colouring of G . Therefore $\chi_c(G) \leq \chi(G)$ for any graph G . On the other hand, it is known [2, 10, 13] that for any graph G , $\chi_c(G) > \chi(G) - 1$, and hence $\chi(G) = \lceil \chi_c(G) \rceil$. So $\chi_c(G)$ is a refinement of $\chi(G)$.

An equivalent definition of the circular chromatic number is as follows: For a rational number $r = p/q \geq 2$, let $K_{p/q}$ be the graph with vertex set $\{0, 1, \dots, p-1\}$, in which two vertices i, j are adjacent if $q \leq |i - j| \leq p - q$. A graph G is called (p, q) -colourable if there is a homomorphism from G to $K_{p/q}$ (i.e., a mapping $f : V(G) \rightarrow \{0, 1, \dots, p-1\}$ such that $q \leq |f(x) - f(y)| \leq p - q$ for any edge xy of G). Then

$$\chi_c(G) = \inf\{p/q : G \text{ is } (p, q)\text{-colourable}\}.$$

Two (p, q) -colourings f, g of a graph G is *equivalent* if there is an automorphism σ of $K_{p/q}$ such that $f = \sigma \circ g$. A graph G is called *uniquely* (p, q) -colourable, if there is an onto homomorphism f from G to $K_{p/q}$, and all (p, q) -colourings of G are equivalent. It is known [11] that if G is uniquely (p, q) -colourable, then $\chi_c(G) = r$.

The existence of graphs of large girth with given circular chromatic number has been studied in a few papers. It was first asked by Abbott and Zhou [1] whether there exists, for any integer $n \geq 3$, a triangle free graph G with $\chi_c(G) = n$. Zhu [11] answered this question in affirmative, by showing that for any integer g and for any rational $r \geq 2$, there is a graph G of girth at least g which is uniquely r -colourable. Later on, Nešetřil and Zhu [8] proved the following much stronger result:

Theorem 1.1 *Suppose $p/q \geq 2$ is a rational number, A is a finite set and f_1, f_2, \dots, f_n are mappings from A to $\{0, 1, \dots, p-1\}$. Then for any integer g , there is a graph $G = (V, E)$ of girth at least g with $A \subset V$, such that G has exactly n non-equivalent (p, q) -colourings g_1, g_2, \dots, g_n , and each g_i is an extension of f_i .*

Theorem 1.1 is a generalization of a result of V. Müller [6] who proved this for k -colourings. The proofs in [11] and in [8] use the probabilistic method. A constructive

proof of Theorem 1.1 for $p/q \geq 3$ is given in [7]. For the case $2 < p/q < 3$, a constructive proof of Theorem 1.1 was missing. In this paper, we give a constructive proof of Theorem 1.1, for all $p/q \geq 2$.

2 Preliminaries

An undirected graph is viewed as a symmetric directed graph. Each edge $e = xy$ of G corresponds to a pair of opposite arcs $a = \vec{xy}$ and $a^{-1} = \vec{yx}$. We shall denote by $A(G)$ the arc set of G . For each arc a , a^{-1} is the opposite arc of a which corresponds to the same edge as a . All graphs considered in this paper are undirected graphs (and hence are symmetric directed graphs). A *rooted graph* is a pair (G, a) such that G is a graph and a is an arc of G . The arc a is called the root arc of G . The edge corresponding to a is called the *root edge* of G . A *walk* W of G is a sequence (x_0, x_1, \dots, x_k) of vertices of G such that for each i , $\vec{x_i x_{i+1}} \in A(G)$. For convenience, we also use W to denote the set of arcs in W , i.e., $W = \{\vec{x_i x_{i+1}} : i = 1, 2, \dots, k-1\}$. If $x_k = x_0$, then W is called a *closed walk*.

A *tension* of a graph G is a mapping $f : A(G) \rightarrow [0, r)$ such that for each closed walk W of G , $\sum_{a \in W} f(a) = 0$. In particular, for each arc a , $f(a^{-1}) = -f(a)$. Suppose $r > 2$ is a real number. An *r-tension* of G is a tension f of G such that for each arc a , $1 \leq |f(a)| \leq r - 1$. An *r-tension* of a graph G corresponds to an *r-colouring* of G . Indeed, if f is an *r-colouring* of G , then $\delta f : A(G) \rightarrow [0, r)$ defined as $\delta f(\vec{xy}) = f(y) - f(x)$ is an *r-tension*. Conversely, if g is an *r-tension* of G , then let x_0 be a fixed vertex of G , and for each vertex x of G , let W_x be a walk from x_0 to x . Then $f(x) = \sum_{a \in W_x} g(a) \pmod{r}$ is an *r-colouring* of G . So

$$\chi_c(G) = \inf\{r : G \text{ has an } r\text{-tension}\}.$$

If (G, a) is a rooted graph with root edge e , then a *rooted r-tension* of G is a tension of G such that for each arc $b \neq a, a^{-1}$, $1 \leq |f(b)| \leq r - 1$. The *r-label set* of (G, a) is defined as

$$L_T^r(G, a) = \{f(a) \pmod{r} : f \text{ is a rooted } r\text{-tension of } G\}.$$

Similar to the correspondence between *r-colouring* and *r-tension*, it is easy to see that the following is an equivalent definition of the label set:

$$L_T^r(G, a) = \{f(x) - f(y) \pmod{r} : f \text{ is an } r\text{-colouring of } G - e\}.$$

In case the root arc a is clear from the context, or is irrelevant, we write $L_T^r(G)$ for $L_T^r(G, a)$.

As an example, it is easy to verify that for any $r \geq 2$, $L_T^r(C_2) = [1, r - 1]_r$, where C_2 is the 2-cycle, i.e., the graph with two vertices and two parallel edges.

Definition 2.1 A rooted graph (H, a) is an r -superedge if $L_T^r(H) = [1, r - 1]_r$.

The concept of superedge is the main tool in [7] for the construction of the required large girth graphs. The reason that the proof in [7] only works for $r \geq 3$ is that r -superedges are constructed for $r \geq 3$ only. The main result of this paper is the following:

Theorem 2.1 For any integer g and for any rational number $r \geq 2$, there is an r -superedge (H, a) of girth at least g .

Theorem 1.1 follows from Theorem 2.1 and the following result of [7]:

Lemma 2.1 [7] Suppose $r = p/q \geq 2$ is a rational, where $(p, q) = 1$. If there is an r -superedge (H, a) of girth at least g , then for any finite set A , for any mappings f_1, f_2, \dots, f_n from A to $\{0, 1, \dots, p-1\}$, there is a graph $G = (V, E)$ of girth at least g with $A \subset V$, such that G has exactly n non-equivalent (p, q) -colourings g_1, g_2, \dots, g_n , and each g_i is an extension of f_i .

The proof of Lemma 2.1 is a little bit technical, and the readers are referred to [7] for the details. Here we only give a sketch of how an superedge is used to construct such a graph.

Suppose (H, a) is an r -superedge, G is a graph and $e' = x'y'$ is an edge of G . To replace e' with superedge (H, a) means to delete the edge e' , add a (vertex disjoint) copy of $H - e$, where $e = xy$ is the root edge of H , and identify x with x' , y with y' .

To construct the graph G as required in Lemma 2.1, we take the categorical product $(K_{p/q})^n$ of n copies of $K_{p/q}$. The graph $(K_{p/q})^n$ has exactly n (p, q) -colourings, namely, the n projections π_i ($i = 1, 2, \dots, n$). Duplicate each vertex of $(K_{p/q})^n$ m times, where m is a sufficiently large integer. Denote the resulting graph by Q . There are exactly n (p, q) -colourings, say π'_i , of Q , where π'_i is an extension of π_i for $i = 1, 2, \dots, n$. When m is large enough, we can embed A into $V(Q)$ so that f_i is a restriction of π'_i for $i = 1, 2, \dots, n$. Now replace each edge of Q by a copy of the superedge (H, a) . Denote the resulting graph by Q' , which is a graph of large girth. The vertices of Q' are partitioned into two parts: the ‘‘old’’ vertices of G , and the ‘‘new’’ vertices that are added together with the copies of the superedge. The set \mathcal{C} of all (p, q) -colourings of Q' is partitioned into n subsets $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n$, and for each i , all the colourings in \mathcal{C}_i are extensions of π'_i . In other words, if we only look at the colours of the old vertices, there are exactly n (p, q) -colourings of Q' . However, there are probably many ways to colour the new vertices.

Next we build a graph R which has many layers V_1, V_2, \dots, V_t , where t is sufficiently large and $V_i = V(Q) \times \{i\}$. Edges are added between the vertices of two consecutive

layers in such a way that: (1) the resulting graph has large girth, and (2) if the i th layer is (p, q) -coloured as $f(x, i) = \pi'_j(x)$ for all $x \in V(Q)$, then it is uniquely extended to the $(i + 1)$ th layer as $f(x, i + 1) = \pi'_j(x)$ for all $x \in V(Q)$. Now identify the first layer of R with the set of old vertices of Q' , we obtain a graph R' . The graph R' has large girth, and if we only look at the colours of the vertex set of R , there are precisely n (p, q) -colourings of R' . The next goal is to fix the colours of the other vertices of R' . Fix n colourings g_1, g_2, \dots, g_n , of R' , where g_i is an extension of π'_i . Then by adding edges between new vertices of Q' and vertices of R , we can make sure that the colours of the new vertices of Q' are also fixed as in the n colourings g_1, g_2, \dots, g_n . Since R' has t layers, where t is sufficiently large, and vertices of one layer is only adjacent to vertices of the two adjacent layers, the edges can be added in such a way that no short cycles will be produced. So the resulting graph G satisfies the requirements of Lemma 2.1.

The remainder of this paper is devoted to the construction of r -superedges of large girth.

3 r -indicators

Definition 3.1 Suppose $r \geq 2$ is a real number. A rooted graph H with root edge e is called an r -indicator if $L_T^r(H) = \{0\}$.

In this section, we show that if there is an r -indicator H , then one can easily construct an r -superedge.

Given two rooted graphs (G, a) and (G', a') with root arcs $a = \vec{x}y$ and $a' = \vec{x}'y'$, respectively, the *series join* of (G, a) and (G', a') is the rooted graph (G'', a'') obtained from the vertex disjoint union of G and G' by deleting the root edges $e = xy$ and $e' = x'y'$, then identify y' and x into a single vertex z , and add an root edge $e'' = xy'$ with root arc \vec{xy}' . The following lemma is easy and is proved in [9].

Lemma 3.1 If (G'', a'') is the series join of (G, a) and (G', a') , then for any $r \geq 2$,

$$L_T^r(G'') = L_T^r(G) + L_T^r(G') = \{t + t' \pmod{r} : t \in L_T^r(G), t' \in L_T^r(G')\}.$$

Theorem 3.1 Suppose (H, a) is an r -indicator. Let (H', a') be the series join of H and C_2 . Then (H', a') is an r -superedge. Moreover, if H has girth g , then H' has girth at least g .

Proof. If (H, a) is an r -indicator, then it follows from Lemma 3.1 that $L_T^r(H') = L_T^r(H) + L_T^r(C_2) = [1, r - 1]_r$. The moreover part of the theorem is trivial. \blacksquare

So to prove the existence of an r -superedge of girth at least g , it suffices to prove the existence of an r -indicator of girth at least g .

The following lemma shows that every r -indicator corresponds to a graph of circular chromatic number r .

Lemma 3.2 *If a rooted graph H with root edge $e = uv$ is an r -indicator, then $\chi_c(H - e) = r$.*

Proof. Since $L_T^r(H) \neq \emptyset$, it follows that $H - e$ is r -colourable. Assume to the contrary that $\chi_c(H - e) = r' < r$. Then $L_T^{r'}(H) \neq \emptyset$. Assume $t \in L_T^{r'}(H) \neq \emptyset$ and f is a rooted r' -tension of H with $f(a) = t$, where $a = u\vec{v}$ is the rooted arc. Let $D = D_f^+(H)$ be the orientation of G defined as $\vec{xy} \in D$ if and only if $f(\vec{xy}) > 0$. As f is a rooted tension, it is easy to see that D is acyclic. For a subset X of $V(G)$, denote by $[X, \overline{X}]$ the cut which consists of all the arcs \vec{xy} with $x \in X$ and $y \notin X$. Then there is a cut $[X, \overline{X}]$ of H which contains the arc a such that for any arc $\vec{xy} \in [X, \overline{X}]$, $f(\vec{xy}) \geq 0$. For any $0 \leq \epsilon \leq r - r'$, let f' be defined as

$$f'(\vec{xy}) = \begin{cases} f(\vec{xy}), & \text{if } \vec{xy}, \vec{yx} \notin [X, \overline{X}], \\ f(\vec{xy}) + \epsilon, & \text{if } \vec{xy} \in [X, \overline{X}], \\ f(\vec{xy}) - \epsilon, & \text{if } \vec{yx} \in [X, \overline{X}]. \end{cases}$$

Then f' is a rooted r -tension of H with $f'(a) = f(a) + \epsilon$. Thus $L_T^r(H)$ contains an interval of positive length, in contrary to the assumption that $L_T^r(H) = \{0\}$. ■

4 Construction of r -indicators of large girth

In this section, we present a method that construct an r -indicator of large girth, for each rational $r \geq 2$.

Definition 4.1 *Suppose $r \geq 2$ is a rational number. A rooted graph (H, a) is called an r -semi-indicator if $L_T^r(H) \neq \emptyset$ and $L_T^r(H) \cap [1, r - 1]_r = \emptyset$.*

Lemma 4.1 *For any integer $g \geq 3$, for any $r \geq 2$, there is an r -semi-indicator (H, a) of girth at least g .*

Proof. It is well-known [3, 5] that one can construct a graph G of girth at least g such that $\chi(G) > r + 1$. As $\chi_c(G) > \chi(G) - 1$, we have $\chi_c(G) > r$. By deleting edges from G , if necessary, we obtain a graph H and an edge e of H such that (i) $\chi_c(H) > r$, and (ii) $\chi_c(H - e) \leq r$. Let a be an arc corresponding to e . It follows from definition that $L_T^r(H) \neq \emptyset$ and $L_T^r(H) \cap [1, r - 1]_r = \emptyset$. ■

Suppose (H, a) is an r -semi-indicator, where $a = \vec{xy}$ is the root arc. Let $\tau(H) = \max\{|f(a)|_r : f \text{ is a rooted } r\text{-tension of } H\}$. It follows from the definition of an r -semi-indicator that $0 \leq \tau(H) < 1$.

Lemma 4.2 *If $r = p/q$ is a rational number and $(p, q) = 1$, then for any r -semi-indicator (H, a) , $\tau(H) = \frac{i}{q}$ for some non-negative integer $i \leq q$.*

Proof. Assume $\tau(H) = t$ and $a = \vec{xy}$. Let $e = xy$. By definition, there is an r -colouring f of $H - e$ such that $f(x) = 0$ and $f(y) = t$. Let $g : V(G) \rightarrow \{0, 1, \dots, p-1\}$ be defined as $g(v) = \lceil f(v)q \rceil$. It is straightforward to verify that g is a (p, q) -colouring of $H - e$. Then $f'(v) = g(v)/q$ is an r -colouring with $f'(x) = 0$ and $f'(y) = g(y)/q \geq f(y)$. By definition of $\tau(H)$, we conclude that $f'(y) = f(y)$. ■

It follows from the definition that an r -indicator is an r -semi-indicator (H, a) with $\tau(H) = 0$. Suppose f is a rooted r -tension of H with $f(a) \geq 0$. By definition of $\tau(H)$, we have $|f(a)|_r \leq \tau(H)$. Let

$$\xi(f) = \begin{cases} 0, & \text{if } f(a) = 0; \\ 1, & \text{if } 0 < f(a) \leq \tau(H); \\ -1, & \text{if } r - \tau(H) \leq f(a) < r. \end{cases}$$

Note that $f(a) = \xi(f)|f(a)|_r \pmod{r}$.

Suppose A is a subset of $[0, r)$ and $s \geq 2$ is an integer. Let $sA = \{t_1 + t_2 + \dots + t_s \pmod{r} : t_i \in A\}$.

Given an r -semi-indicator (H, a) of girth at least g for which $\tau(H) > 0$. Let s be the least integer such that $s\tau(H) \geq 1$. Then $s \geq 2$. Let $\delta = 1 - (s-1)\tau(H) > 0$. Let k be the largest integer such that $k(r-2) < \delta$.

Let (Q, b) be the series join of s copies of (H, a) . Denote by (H_i, a_i) the i th copy of (H, a) in (Q, b) . Assume that e is the root edge of Q , and e_i is the root edge of H_i for $i = 1, 2, \dots, s$. Figure 1 below is an illustration of (Q, b) , where the thick edge indicates the root edge of (Q, b) , and the dotted lines indicate the (deleted) root edges of the copies of H .

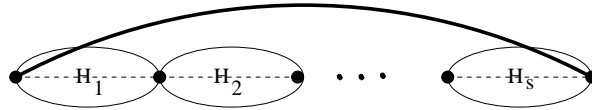


Figure 1: The rooted graph (Q, b)

It follows from Lemma 3.1 that $L_T^r(Q) = sL_T^r(H)$. Let (Q', b') be the parallel join of (Q, b) and (C_{2k+2}, d) , where d is an arbitrary arc of C_{2k+2} . As C_{2k+2} is the series join of $2k+1$ copies of C_2 , and $L_T^r(C_2) = [1, r-1]_r$, it follows from Lemma

3.1 that $L_T^r(C_{2k+2}) = [1 - k(r - 2), r - 1 + k(r - 2)]_r$, because $k(r - 2) < \delta < 1$. Thus $L_T^r(Q') \neq \emptyset$ if and only if $sL_T^r(H) \cap [1 - k(r - 2), r - 1 + k(r - 2)]_r \neq \emptyset$. Figure 2 below is an illustration of (Q', b') for the case $k = 2$. Note that in the case as depicted in Figure 2, Q' can be viewed as obtained from Q by adding a path of length 5 connecting the two end vertices of the root edge. In the general case, we add a path of length $2k + 1$ joining the two end vertices of Q . In particular, if $k = 0$, we simply add an edge joining the two end vertices of the root edge.

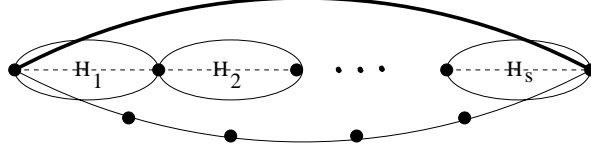


Figure 2: The rooted graph (Q', b')

Lemma 4.3 *Suppose f is a rooted r -tension of (Q', b') . Let f_i be the restriction of f to H_i . Then either for all i , $\xi(f_i) = 1$, or for all i , $\xi(f_i) = -1$.*

Proof. By Lemma 3.1,

$$f(b') = \sum_{i=1}^s f_i(a_i) \pmod{r} = \sum_{i=1}^s \xi(f_i) |f(a_i)|_r \pmod{r}.$$

If there are indices i, j such that $\xi(f_i) = 1$ and $\xi(f_j) = -1$, or there is an index with $\xi(f_i) = 0$, then

$$\left| \sum_{i=1}^s \xi(f_i) |f(a_i)|_r \right| \leq (s - 1)\tau(H) < 1 - k(r - 2),$$

contrary to the fact that $f(b') \in L_T^r(C_{2k+2}) = [1 - k(r - 2), r - 1 + k(r - 2)]_r$. ■

Let Q'' be the graph constructed as follows: Take g copies of $Q' - e'$, say $Q'_1 - e'_1, Q'_2 - e'_2, \dots, Q'_g - e'_g$, where e'_i is the root edge of Q'_i . Recall that each copy of $Q' - e'$ consists of s copies of H . Let $H_{i,j}$ denotes the j th copy of H in $Q'_i - e'_i$, and $a_{i,j} = \overrightarrow{x_{i,j}y_{i,j}}$ be the root arc of $H_{i,j}$ and let $e_{i,j}$ denote the root edge of $H_{i,j}$. For $i = 1, \dots, g - 1$, identify $H_{i,s}$ with $H_{i+1,1}$. Figure 3 is an illustration of the construction of Q'' .

Lemma 4.4 *The graph Q'' is r -colourable if and only if $Q' - e'$ is r -colourable. Moreover, if f is an r -tension of Q'' , then either for all i, j , $\xi(f_{i,j}) = 1$, or for all i, j , $\xi(f_{i,j}) = -1$, where $f_{i,j}$ is the restriction of f to $H_{i,j}$.*

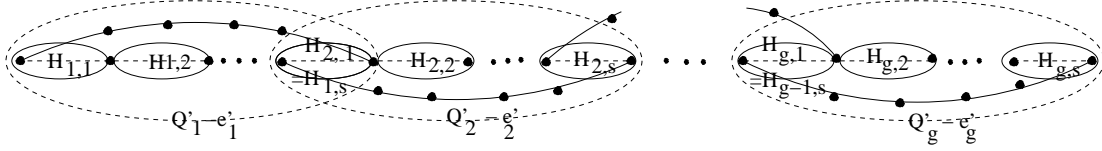


Figure 3: The graph Q''

Proof. As $Q' - e'$ is a subgraph of Q'' , so if Q'' is r -colourable, then $Q' - e'$ is r -colourable. Conversely, if Q' is r -colourable, then $sL_T^r(H) \cap [1 - k(r - 2), r - 1 + k(r - 2)]_r \neq \emptyset$, and hence there exist $t_1, t_2, \dots, t_s \in L_T^r(H)$ such that

$$t_1 + t_2 + \dots + t_s \in [1 - k(r - 2), r - 1 + k(r - 2)]_r.$$

Let f_j be a rooted r -tension of H with $f_j(a) = t_j$. Let $g_{i,j}$ be a rooted r -tension on $H_{i,j}$ defined as

$$g_{i,j} = \begin{cases} f_j, & \text{if } i \text{ is odd;} \\ f_{s+1-j}, & \text{if } i \text{ is even.} \end{cases}$$

It is straightforward to verify that the union of $g_{i,j}$ ($i = 1, 2, \dots, g, j = 1, 2, \dots, s$) is an r -tension of Q'' , i.e., Q'' is r -colourable.

By applying Lemma 4.3 to each copy of Q' in Q'' , we obtain the moreover part. \blacksquare

Lemma 4.5 *If (H, a) is an r -semi-indicator of girth at least g and with $\tau(H) > 0$, then there is an r -semi-indicator (H', a') of girth at least g for which $\tau(H') < \tau(H)$.*

Proof. Assume (H, a) is an r -semi-indicator of girth at least g for which $\tau(H) > 0$. Let s be the least integer such that $s\tau(H) \geq 1$. Let $\delta = 1 - (s - 1)\tau(H) > 0$. Let k be the largest integer such that $k(r - 2) < \delta$. We consider two cases.

Case 1. $sL_T^r(H) \cap [1 - k(r - 2), r - 1 + k(r - 2)]_r \neq \emptyset$.

Let Q'' be the graph constructed as above. Let (H', a') be the rooted graph constructed as follows: Take two copies of Q'' , say Q''_1 and Q''_2 . The copies of H in Q''_1 and Q''_2 are denoted by $H_{i,j}^1$ and $H_{i,j}^2$, respectively. The root arc of $H_{i,j}^l$ is $a_{i,j}^l = \overrightarrow{x_{i,j}^l y_{i,j}^l}$. Identify $H_{1,1}^1$ with $H_{1,1}^2$ and identify $y_{g,s}^1$ and $y_{g,s}^2$. Add an edge e' joining $x_{g,s}^1$ and $x_{g,s}^2$. The root arc is $a' = \overrightarrow{x_{g,s}^1 x_{g,s}^2}$. Figure 4 is an illustration of the construction of H' .

By Lemma 4.4, Q'' is r -colourable. Hence $H' - e'$ is r -colourable, implying that $L_T^r(H', a') \neq \emptyset$. Let f be an arbitrary rooted r -tension of H' . Let $f_{i,j}^l$ be the restriction of f to $H_{i,j}^l$. By the moreover part of Lemma 4.4, either for all l, i, j , $\xi(f_{i,j}^l) = 1$, or for all l, i, j , $\xi(f_{i,j}^l) = -1$. This implies that

$$|f(a')| = ||f(a_{g,s}^2)|_r - |f(a_{g,s}^1)|_r|.$$

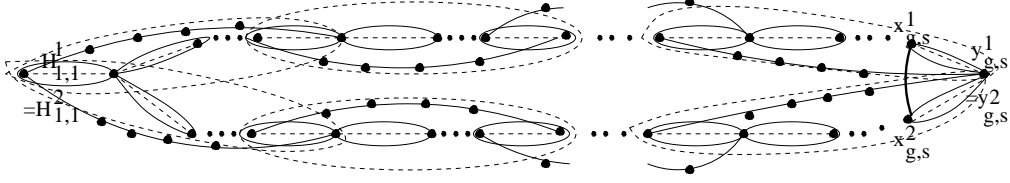


Figure 4: The rooted graph (H', a') for Case 1

As

$$0 < ||f(a_{g,s}^2)|_r - |f(a_{g,s}^1)|_r| < \tau(H),$$

we conclude that $|f(a')|_r < \tau(H)$. So $\tau(H') < \tau(H)$.

Case 2. $sL_T^r(H) \cap [1 - k(r - 2), r - 1 + k(r - 2)]_r = \emptyset$.

Let $\delta' = 1 - k(r - 2) - (s - 1)\tau(H) = \delta - k(r - 2)$ (note that it is possible that $k = 0$, in which case $\delta' = \delta$). By the maximality of k , $\delta' \leq r - 2$. Moreover,

$$L_T^r(H) \cap [\delta', r - 2 + \delta'] = \emptyset,$$

for otherwise, there exists $t \in L_T^r(H) \cap [\delta', (2k + 1)(r - 2) + \delta']$, which implies that

$$(s - 1)\tau(H) + t \in sL_T^r(H) \cap [1 - k(r - 2), r - 1 + k(r - 2)]_r,$$

contrary to the assumption that $sL_T^r(H) \cap [1 - k(r - 2), r - 1 + k(r - 2)]_r = \emptyset$.

Furthermore, $sL_T^r(H) \cap [1 - k(r - 2), r - 1 + k(r - 2)]_r = \emptyset$ implies that $(2k + 1)(r - 2) + \delta' < \tau(H)$. In particular, $2\delta' \leq (r - 2) + \delta' < \tau(H)$.

Let (H', a') be constructed as follows: Take g copies of $H - e$, say $H_1 - e_1, H_2 - e_2, \dots, H_g - e_g$, where $e_i = x_i y_i$. Identify x_1, x_2, \dots, x_g into a single vertex x . For $i = 1, 2, \dots, g - 1$, connect y_i to y_{i+1} by a path of length 2. Finally, add an edge e' connecting y_1 and y_g , and the root arc is $a' = \overrightarrow{y_1 y_g}$. Figure 5 is an illustration of the construction of H' .

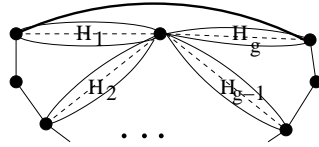


Figure 5: The rooted graph (H', a') for Case 2

It is obvious that H' has girth at least g and $H' - e'$ is r -colourable. Let f be an arbitrary r -colouring of $H' - e'$. We need to prove that $|f(y_1) - f(y_g)|_r < \tau(H)$. Without loss of generality, we may assume that $f(x) = 0$. Then $f(y_i) \in L_T^r(H)$ for

$i = 1, 2, \dots, g$. Note that $L_T^r(H) \cap [\delta', r - 2 + \delta] = \emptyset$. Let

$$\begin{aligned} B_1 &= L_T^r(H) \cap (r - \delta', \delta')_r, \\ B_2 &= L_T^r(H) \cap ((2k + 1)(r - 2) + \delta', \tau(H)]_r, \\ B_3 &= L_T^r(H) \cap [r - \tau(H), r - (2k + 1)(r - 2) - \delta')_r. \end{aligned}$$

Then for each $i = 1, 2, \dots, g$, there is an index $\sigma(i) \in \{1, 2, 3\}$ such that $f(y_i) \in B_{\sigma(i)}$. Since y_i and y_{i+1} is connected by a path of length 2, it is easy to verify that $|f(y_i) - f(y_{i+1})| \in [2, r - 2]_r$. It follows that for any $j \in \{1, 2, 3\}$, if $f(y_i) \in B_j$, then $f(y_{i+1}) \in B_j$, i.e., $\sigma(i) = \sigma(i + 1)$. Thus if $f(y_1) \in B_j$ then $f(y_g) \in B_j$. If $j = 1$, then

$$|f(y_1) - f(y_g)|_r < 2\delta' < \tau(H).$$

If $j = 2, 3$, then it is easy to see that $|f(y_1) - f(y_g)|_r < \tau(H)$. ■

Corollary 4.1 *For any rational number $r = p/q \geq 2$ and for any integer g , there is an r -indicator of girth at least g .*

Proof. By Lemma 4.1, there is an r -semi-indicator. By Lemma 4.2, there is an r -semi-indicator (H, a) of girth at least g such that $\tau(H)$ is minimum among all r -semi-indicators of girth at least g . By Lemma 4.5, $\tau(H) = 0$. So (H, a) is an r -indicator. ■

Combining Corollary 4.1 and Theorem 3.1, we have the following corollary.

Corollary 4.2 *For any rational $r \geq 2$ and for any integer g , there is an r -superedge (H, a) of girth at least g .*

Combining Corollary 4.1 and Lemma 3.2, we have the following corollary.

Corollary 4.3 *For any rational $r \geq 2$ and for any integer g , there is a graph G of girth at least g such that $\chi_c(G) = r$.*

As mentioned in Section 1, Corollary 4.3 was first proved in [11] by the probabilistic method. Note that Corollary 4.3 also follows from Theorem 1.1, as uniquely r -colourable graphs G have $\chi_c(G) = r$.

References

- [1] H. L. Abbott and B. Zhou, *The star chromatic number of a graph*, J. Graph Theory **17** (1993), 349-360.

- [2] J. A. Bondy and P. Hell, *A note on the star chromatic number*, J. Graph Theory **14** (1990), 479-482.
- [3] P. Erdős, *Graph theory and probability*, Can. J. Math. 11(1959), 34-38.
- [4] B. Larose and C. Tardif, *Strongly rigid graphs and projectivity*, Multiple-Valued Logic. 7(2001),339-361.
- [5] V. Müller, *On colorable critical and uniquely colorable critical graphs*, In: Recent Advances in Graph Theory (ed. M. Fiedler), Academia, Prague, 1975.
- [6] V. Müller, *On coloring of graphs without short cycles*, Discrete Math., 26(1979), 165-179.
- [7] J. Nešetřil and X. Zhu, *Construction of sparse graphs with prescribed circular colorings*, Discrete Mathematics 233(2001), 277-291.
- [8] J. Nešetřil and X. Zhu, *On Sparse Graphs with Given Colorings and Homomorphisms*, Journal of Combinatorial Theory Series B, to appear.
- [9] Z. Pan and X. Zhu, *The circular chromatic number of series-parallel graphs of large odd girth*, Discrete Mathematics 245(2002), 235-246.
- [10] A. Vince, *Star chromatic number*, J. Graph Theory 12 (1988), 551-559.
- [11] X. Zhu, *Uniquely H -colorable graphs with large girth*, J. Graph Theory, **23** (1996), 33-41.
- [12] X. Zhu, *Construction of uniquely H -colorable graphs*, J. Graph Theory, 30(1999), 1-6.
- [13] X. Zhu, *Circular chromatic number, a survey*, Discrete Mathematics 229(2001), 371-410.