Circular perfect graphs

Xuding Zhu*
Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung, Taiwan 80424
Email: zhu@math.nsysu.edu.tw

June 2000

Abstract

For $1 \leq d \leq k$, let $K_{k/d}$ be the graph with vertices $0, 1, \cdots, k-1$, in which $i \sim j$ if $d \leq |i-j| \leq k-d$. The circular chromatic number $\chi_c(G)$ of a graph G is the minimum of those k/d for which G admits a homomorphism to $K_{k/d}$. The circular clique number $\omega_c(G)$ of G is the maximum of those k/d for which $K_{k/d}$ admits a homomorphism to G. A graph G is circular perfect if for every induced subgraph H of G we have $\chi_c(H) = \omega_c(H)$. In this paper we prove that if G is circular perfect then for every vertex x of G, $N_G[x]$ is a perfect graph. Conversely, we prove that if for every vertex x of G, $N_G[x]$ is a perfect graph and $G - N[x]$ is a bipartite graph with no induced P_4, then G is a circular perfect graph. In a companion paper, we apply the main result of this paper to prove an analogue of Hajós theorem for circular chromatic number for $k/d \geq 3$. Namely, we shall design a few graph operations and prove that for any $k/d \geq 3$, starting from the graph $K_{k/d}$, one can construct all graphs of circular chromatic number at least k/d by repeatedly applying these graph operations.

1 Introduction

Suppose $G = (V, E)$ and $H = (V', E')$ are graphs. A *homomorphism* of G to H is a mapping $f : V \to V'$ such that $f(x)f(y) \in E'$ whenever $xy \in E$.

*This research was partially supported by the National Science Council under grant NSC 89-2115-M-110-003

Homomorphism of graphs are studied as a generalization of graph colorings. A vertex coloring of a graph G with n-colors is equivalent to a homomorphism of G to K_n. We write $G \preceq H$ if there exists a homomorphism from G to H. Then $\preceq$ defines a partial order on the set of all finite graphs, which we denote by $(\mathcal{F}, \preceq)$.

We denote by $\mathcal{Z}_{\mathcal{G}}$ the set of complete graphs, i.e., $\mathcal{Z}_{\mathcal{G}} = \{K_1, K_2, \cdots, \}$. Then $\mathcal{Z}_{\mathcal{G}}$ forms an infinite increasing chain in $(\mathcal{F}, \preceq)$. Every graph $G \in \mathcal{F}$ admits a homomorphism to some member of the set $\mathcal{Z}_{\mathcal{G}}$, and contains some member of $\mathcal{Z}_{\mathcal{G}}$ as its subgraphs. The chromatic number $\chi(G)$ is the minimum n such that $G \preceq K_n$. The clique number $\omega(G)$ is the maximum n such that $K_n \preceq G$. We may view the set $\mathcal{Z}_{\mathcal{G}}$ as a representation of natural numbers, with K_n be a representation of the integer n. Then $\chi(G)$ is the least element of the set $\mathcal{Z}_{\mathcal{G}}$ which is "above" G in the order $\preceq$, and $\omega(G)$ is the maximum element of $\mathcal{Z}_{\mathcal{G}}$ which is "below" G in the order $\preceq$. In this sense, we may view the set $\mathcal{Z}_{\mathcal{G}}$ as a scale that measures a dimension of graphs.

Just as the set of natural numbers are extended to the set of rational numbers, we can "extend" the set $\mathcal{Z}_{\mathcal{G}}$ into a larger set. For those fractions k/d with $(k,d) = 1$ and $k \geq 2d$, we construct a graph $K_{k/d}$, which has vertices $\{0, 1, \cdots, k-1\}$ and edges $\{ij : d \leq |i-j| \leq k-d\}$. We shall denote by $\mathcal{Q}_{\mathcal{G}}$ the set $\{K_{k/d} : (k,d) = 1 \text{ and } k \geq 2d\} \cup \{K_1\}$. Note that $K_{k/1} = K_k$, and hence $\mathcal{Q}_{\mathcal{G}}$ is indeed an extension of $\mathcal{Z}_{\mathcal{G}}$. Moreover, the set $\mathcal{Q}_{\mathcal{G}}$ is also linearly ordered. It was shown in [2, 9] that if $k'/d' \geq 2$ and $k/d \geq 2$, then $k'/d' \leq k/d$ if and only if $K_{k'/d'} \preceq K_{k/d}$. Thus the set $\mathcal{Q}_{\mathcal{G}}$ together with the order $\preceq$ may be viewed as a representation of those rational numbers $r \geq 2$ or $r = 1$. The *circular chromatic number* $\chi_c(G)$ of a graph is the infimum of the rational numbers k/d for which $G \preceq K_{k/d}$. The circular chromatic number of graphs has been studied extensively in the literature (see [11] for a comprehensive survey on this subject). Similar to the definition of clique number, we define the circular clique number of a graph as follows:

Definition 1.1 *Suppose G is a graph. The* circular clique number $\omega_c(G)$ *of G to defined as*

$$\omega_c(G) = \sup\{k/d : K_{k/d} \text{ admits a homomorphism to } G\}.$$

It was shown in [9] that the infimum in this definition of $\chi_c(G)$ is always attained, and hence the infimum can be replaced by minimum. Therefore $\chi_c(G)$ is the least member of $\mathcal{Q}_{\mathcal{G}}$ which is above G in the order $\preceq$. We shall show in this paper that the supremum in the definition of $\omega_c(G)$ is also always attained, and hence can be replaced by the maximum. So $\omega_c(G)$ is the largest member of $\mathcal{Q}_{\mathcal{G}}$ which is below G in the order $\preceq$.

If the set $\mathcal{Z}_{\mathcal{G}}$ is considered as a scale that measures a dimension of graphs, then the set $\mathcal{Q}_{\mathcal{G}}$ is a refinement of that scale, just as the set of rational numbers provides a finer scale that measures the length of an object than that of integers. The chromatic number $\chi(G)$ of a graph G maybe regarded as an approximation of its circular chromatic number $\chi_c(G)$. The clique number of G maybe regarded as an approximation of its circular clique number.

It follows from the definition that

$$\omega(G) \leq \omega_c(G) \leq \chi_c(G) \leq \chi(G), \ \omega(G) = \lfloor \omega_c(G) \rfloor, \text{ and } \chi(G) = \lceil \chi_c(G) \rceil.$$

A graph G is *perfect* if for every induced subgraph H of G we have $\chi(G) = \omega(G)$. The following definition is a natural generalization of this concept to circular coloring.

Definition 1.2 *A graph G is called* circular perfect *if for every induced subgraph H of G we have $\chi_c(H) = \omega_c(H)$.*

This paper investigates the circular clique number of a graph and considers necessary and sufficient conditions for a graph to be circular perfect. First we discuss basic properties of the circular clique number of a graph. It is proved that the circular clique number $\omega_c(G)$ of a finite graph G is a rational number, and $\omega_c(G)$ is equal to the maximum k/d for which G contains $K_{k/d}$ as an induced subgraph.

For necessary conditions for a graph to be circular perfect, it is proved that for any circular perfect graph G and for any vertex x of G, $N_G[x]$ is a perfect graph. Here $N_G[x]$ denote the subgraph induced by the set $\{y \in V(G) : y \sim x, \text{ or } y = x\}$.

Conversely, we prove that if G is a graph for which $N_G[x]$ is perfect for any vertex x, then G is circular perfect, provided for every vertex x, $G - N_G[x]$ is a bipartite graph which contains no induced P_4.

In a companion paper [12], we shall use this result to prove an analogue of Hajós theorem for $k/d \geq 3$. Namely we shall prove that for $k/d \geq 3$, starting from $K_{k/d}$, one can construct all graphs G with $\chi_c(G) \geq k/d$ by repeatedly applying a few graph operations.

All the graphs in this paper are finite and simple. We write $x \sim_G y$ (or $x \sim y$ when the graph G is clear from the context) to mean that x is adjacent to y in G. For a vertex x of G, $N_G(x) = \{y \in V(G) : x \sim y\}$ denotes the set of neighbours of x, and $N_G[x] = N_G(x) \cup \{x\}$. We also use $N_G(x)$ and $N_G[x]$ to denote the subgraphs induced by these sets. When the graph G is clear from the context, we write $N(x)$ and $N[x]$ for $N_G(x)$ and $N_G[x]$. We denote by $G - N[x]$ the subgraph of G induced by $V(G) - N[x]$.

For two sets A, B, we write $A \subseteq B$ to mean that A is a subset of B, and write $A \subset B$ to mean that A is a proper subset of B.

Suppose a is a positive integer and b is an integer. We denote by "$b \pmod a$" the unique integer t such that $0 \leq t \leq a-1$ and $b \equiv t \pmod a$.

When we consider the graph $K_{k/d}$, we may view the vertices $0, 1, \cdots, k-1$ of $K_{k/d}$ as cyclically ordered to form a circle. We shall denote this cycle by C^k. For $a, b \in C^k$, we denote by $[a, b]_k$ the interval of the circle C^k from a to b. To be precise, if $a < b$ then $[a, b]_k = \{a, a+1, \cdots, b\}$, if $a > b$ then $[a, b]_k = \{a, a+1, \cdots, k-1, 0, 1, \cdots, b\}$. Similarly we define $(a, b)_k$, $[a, b)_k$ and $(a, b]_k$. When the integer k is clear from the context, we shall write $[a, b], [a, b), (a, b)$ for $[a, b]_k, [a, b)_k, (a, b)_k$.

Given a graph G and a proper subgraph H of G. A homomorphism from G to H which fixes every vertex of H is called a *retraction*. If there is a retraction from G to H, then we say G *retracts* to H. A graph G is a *core* if G' does not retract to any of its proper subgraphs (or equivalently, G admits no homomorphism to any of its proper subgraphs.) A core of a graph G is a subgraph H of G which is a core and G retracts to H. It is well-known that each finite graph has a unique core (up to an isomorphism). If H is the core of G then G and H have the same circular chromatic number and the same circular clique number. Thus for the calculation of circular chromatic number and circular clique number, we may restrict to core graphs. It is easy to see that in a core graph, for two vertices x, y of G, none of $N(x)$ and $N(y)$ is a subset of the other.

2 Basic properties of circular clique number

By definition, $\omega_c(G)$ is equal to the supremum of those rational numbers k/d for which $K_{k/d}$ admits a homomorphism to G. A natural question is whether or not $\omega_c(G)$ is a rational number and if so, say $\omega_c(G) = k/d$, does $K_{k/d}$ admits a homomorphism to G ? This section answers this question in the affirmative. We shall prove that for any finite graph G, $\omega_c(G)$ is a rational number, and is equal to the maximum of those rational numbers k/d such that G contains $K_{k/d}$ as an induced subgraph.

Lemma 2.1 *Suppose $k \geq 2d$ and $(k, d) = 1$. If H is obtained from $K_{k/d}$ by adding an edge, then $\omega(H) \geq p/q > k/d$ for some $0 < p < k$ and $0 < q < d$.*

Proof. Because $K_{k/d}$ is vertex transitive, we may assume that $H = K_{k/d} + 0a$ for some $1 \leq a \leq d-1$. We shall show that $\omega_c(K_{k/d} + 0a) > k/d$.

We select a sequence of vertices $x_0, x_1, \cdots, x_{p-1}$ of $K_{k/d}$ as follows: Let $x_0 = 0, x_1 = a, x_2 = a + d \bmod k, \cdots$. In general, if we have chosen x_i ($i \geq 1$), then let $x_{i+1} = x_i + d \bmod k$, provided that $x_i + d \bmod k \notin [k-d+a+1, 0]$. In case $x_i + d \bmod k \in [k-d+a+1, 0]$, we let $p - 1 = i$, and we have completed the selection. For convenience, we let $x_p = x_0 = 0$, and assume that $x_{p-1} + d' = 0 \pmod k$ for an integer d'. Then $d \leq d' \leq 2d - a - 1$.

First we observe that the selection process stops in a finite number of steps. Otherwise, the sequence contains repeated vertex. Assume $i < j$ and $x_i = x_j$ is the first pair of equal vertices. Then $j - i < k$ because there are vertices of $K_{k/d}$ not in the sequence (for example those vertex in $[a+1, d]$). Moreover $i \neq 0$ (because if so, then $x_i = x_j = 0$ and by definition the sequence will stop at the step of considering x_j). Therefore $x_i = x_j = x_i + (j-i)d \bmod k$, i.e., $(j-i)d = tk$ for some integer t. This is impossible, as $(k, d) = 1$ and $j - i < k$.

Since

$$\begin{aligned}
x_0 = x_p &= x_{p-1} + d' \bmod k \\
&= x_{p-2} + d + d' \bmod k \\
&= x_{p-3} + 2d + d' \bmod k \\
&\quad \cdots\cdots \\
&= x_0 + a + d' + (p-2)d \bmod k,
\end{aligned}$$

we conclude that $a + d' + (p-2)d = qk$ for some integer q.

We may imagine that the vertices $x_0, x_1, \cdots, x_{p-1}$ are chosen in this order by a person walking along the circle C^k. After picking x_{p-1} and returns to 0, the person has walked around C^k exactly q times. Observe that after choosing vertex x_i, one traverses distance d (along the clockwise direction of the circle) to pick the next vertex x_{i+1}, except for the first step from x_0 to x_1, and the last step from x_{p-1} to $x_p = x_0$. It follows that for each x_i, the interval $[x_i, x_{i+1})$ contains exactly q of the x_j's, including the interval $[x_0, x_1)$ and the interval $[x_{p-1}, x_0)$, because $[a+1, d]$ contains no x_j's and $[x_{p-1}+d, x_0)$ contains no x_j's.

For each i the q x_j's contained in $[x_i, x_{i+1})$ form an independent set. Moreover, it is easy to see that x_j is not adjacent to x_i if and only if either $x_j \in [x_i, x_{i+1})$ or $x_j \in (x_{i-1}, x_i]$. Therefore, the set $x_0, x_1, \cdots, x_{p-1}$ induces a $K_{p/q}$. Hence $\omega(K_{k/d} + 0a) \geq p/q$.

Since $a + d' + (p-2)d = qk$, and $d \leq d' \leq 2d - a - 1$, we conclude that $pd > pd - 1 \geq a + d' + (p-2)d = qk$. Therefore $p/q > k/d$. It follows from the construction we know that $p < k$. This completes the proof of Lemma 2.1. $\blacksquare$

It can be proved that for the graph $G' = K_{k/d} + 0a$, $\omega_c(G') = p/q$ for the p, q defined in the proof above. However, we shall not need that.

Lemma 2.2 *Suppose f is a homomorphism of $K_{k/d}$ to G. If f is not an embedding then there is a fraction $\frac{p}{q} > \frac{k}{d}$, $p < k$ and $K_{p/q}$ admits a homomorphism to G.*

Proof. Assume that f is not an embedding, and without loss of generality assume that for some $k - d < i < k$ we have $f(i) = f(0)$. Let $a = i + d - k$. Then a is a neighbour of i, but $1 \leq a \leq d - 1$. Since f is a homomorphism, we know that $f(0)f(a) = f(i)f(a)$ is an edge of G. Therefore f is a homomorphism from $K_{k/d} + 0a$ to G. By Lemma 2.1, $\omega_c(K_{k/d} + 0a) \geq p/q > k/d$ for some $p < k$ and $q < d$. So $K_{p/q}$ admits a homomorphism to $K_{k/d} + 0a$, and hence admits a homomorphism to G. $\blacksquare$

Theorem 2.1 *If G is a finite graph then $\omega_c(G) = \frac{k}{d}$ for some integers $d \leq k \leq |V(G)|$. Moreover, if $\omega_c(G) = \frac{k}{d}$ and $(k, d) = 1$, then $K_{k/d}$ is an induced subgraph of G.*

Proof. By Lemma 2.2, to determine the circular chromatic number $\omega_c(G)$ of G, it suffices to consider those $K_{k/d}$ which admits an embedding to G. Therefore $\omega_c(G) = \frac{k}{d}$ for some integers $d \leq k \leq |V(G)|$, and $K_{k/d}$ is a subgraph of G. If $K_{k/d}$ is not an induced subgraph of G, then there is an integer $1 \leq a \leq d-1$ such that $K_{k/d} + 0a$ is a subgraph of G, but by Lemma 2.1 $\omega_c(K_{k/d} + 0a) > k/d$. $\blacksquare$

Theorem 2.2 *For any graph G we have*

$$\omega(G) \leq \omega_c(G) < \omega(G) + 1.$$

Proof. It follows from the definition that $\omega(G) \leq \omega_c(G)$. On the other hand, let $n = \lfloor k/d \rfloor$, then K_n admits a homomorphism to $K_{k/d}$. So if $\omega_c(G) = k/d$, then $\omega(G) \geq n > k/d - 1$. Therefore for any graph G we have $\omega(G) = \lfloor \omega_c(G) \rfloor$. $\blacksquare$

3 Necessary conditions for a graph to be circular perfect

In the study of circular chromatic number and circular clique number of a graph, the graphs $K_{k/d}$ play the same role as that of complete graphs in the study of chromatic number and clique number. We call the graph s$K_{k/d}$ the *circular complete graphs*. The concept of circular perfect graphs would be meaningless if some of the circular complete graphs are not circular perfect. However, unlike the concept of perfect graph, where the complete graphs are obviously perfect, it is not obvious that circular complete graphs $K_{k/d}$ are circular perfect. We shall first prove that the graphs $K_{k/d}$ are indeed circular perfect.

Theorem 3.1 *Suppose $k \geq 2d$ are positive integers with $gcd(k, d) = 1$. The graph $K_{k/d}$ is circular perfect.*

Proof. The vertex set of $K_{k/d}$ is $V = \{0, 1, \cdots, k-1\}$. Let H be a subgraph of $K_{k/d}$ induced by a subset S of V. Without loss of generality, we may assume that H contains no isolated vertices. We define a directed graph D with vertex set S as follows: For each $i \in S$, let t_i be the least positive integer such that $t_i \geq d$ and $i + t_i \bmod k \in S$. Then put a directed edge joining i to $i + t_i$. The resulting directed graph is D. It follows from the construction that each vertex of D has out-degree 1, so D contains a directed cycle. Moreover, since H contains no isolated vertex, so if $i \in S$, then there is a $j \in S$ such that $d \leq |j - i| \leq k - d$. Therefore each directed edge of D is an edge of H after the orientation be omitted. (In particular, D has no loops. Indeed it is not difficult to see that a loop in D would correspond to an isolated vertex in H). Let $D' = (i_0, i_1, i_2, \cdots, i_{p-1})$ be a directed cycle of D. Let $S' = \{i_0, i_1, i_2, \cdots, i_{p-1}\}$ and let H' be the subgraph of H induced by S'.

Now

$$\begin{aligned} i_1 &= i_0 + t_{i_0} \bmod k, \\ i_2 &= i_1 + t_{i_1} \bmod k \ = i_0 + t_{i_0} + t_{i_1} \bmod k, \\ &\cdots\cdots \\ i_0 &= i_0 + t_{i_0} + t_{i_1} + \cdots + t_{i_{p-1}} \ k. \end{aligned}$$

It follows that

$$t = t_{i_0} + t_{i_1} + t_{i_2} + \cdots + t_{i_{p-1}} \bmod k = 0.$$

Assume $t = kq$ for some integer q. Intuitively, we view the vertices of S' as been selected by a person traversing the circle C^k, in the order $i_0, i_1, \cdots, i_{p-1}$. When the person returns to i_0 after picking i_{p-1}, he has traversed the circle C^k q times.

It follows from the choice of t_j that if $i_j \in [i_{j'}, i_{j'+1})$ then $i_{j+1} \in [i_{j'+1}, i_{j'+2})$. Therefore, for each $j \in \{0, 1, \cdots, p-1\}$, the interval $[t_{i_j},\ t_{i_{j+1}})$ contains q vertices of S'. Again by the choice of t_j, these q vertices form an independent set of H'. Moreover, each pair of nonadjacent vertices is contained in such an independent set. Therefore H' is isomorphic to $K_{p/q}$. So $\omega_c(H) \geq \omega_c(H') = p/q$.

We claim that H admits a homomorphism to H'. This would imply $\chi_c(H) \leq p/q$, and hence $\chi_c(H) = \omega_c(H) = p/q$. As H is an arbitrary induced subgraph of $K_{k/d}$, we then conclude that $K_{k/d}$ is circular perfect. So it remains to prove H admits a homomorphism to H'. Define a mapping $f : S \to S'$ by letting $f(x) = i_j$, where $i_j \in S'$ is the unique vertex such that $[i_j, x] \cap S' = \{i_j\}$. Now we shall show that f is a homomorphism. Assume xy is an edge of H. Then $y = x + s \mod k$ for some $d \leq s \leq k - d$. Assume $f(x) = i_j$ and $f(y) = i_{j'}$. It follows from the choice of t_j that $y \notin [i_j, i_{j+1})$. Therefore $i_{j'} \notin [i_j, i_{j+1})$ (for otherwise $(i_{j'}, y]$ would contain i_{j+1}). Similarly, $i_j \notin [i_{j'}, i_{j'+1})$. Therefore i_j is adjacent to $i_{j'}$. So f is indeed a homomorphism. ∎

Theorem 3.2 *If G is circular perfect then for any induced subgraph H of G we have $\chi(H) - \omega(H) \leq 1$.*

Proof. If G is circular perfect and H is an induced subgraph of G then $\chi_c(H) = \omega_c(H)$. Since $\chi(H) < \chi_c(H) + 1$ and $\omega(H) > \omega_c(H) - 1$, so $\chi(H) - \omega(H) < 2$. But $\chi(H) - \omega(H)$ is an integer, so $\chi(H) - \omega(H) \leq 1$. ∎

Theorem 3.3 *If G is circular perfect then for every vertex x of G, $N[x]$ induces a perfect graph.*

Proof. Assume to the contrary that $N[x]$ is not perfect for some vertex x. Then $N(x)$ is not perfect. Let H be a subgraph of $N(x)$ for which $\omega(H) \leq \chi(H) - 1$. Let $H' = H \cup \{x\}$. It is proved in [10] that if a graph contains a universal vertex then its circular chromatic number is equal to its chromatic number. Therefore $\chi_c(H') = \chi(H') = \chi(H) + 1$, as H' contains a universal vertex. However, $\omega(H') = \omega(H) + 1 \leq \chi(H') - 1$. As $\omega_c(H') < \omega(H') + 1$, it follows that $\omega_c(H') < \chi_c(H')$, contrary to our assumption that G is circular perfect. ∎

This necessary condition is not sufficient. If G is the Petersen graph then for each vertex x, $N[x]$ is a star. So the above condition is satisfied. Yet $\omega_c(G) = 5/2$ and $\chi_c(G) = 3$.

4 A sufficient condition for a graph to be circular perfect

The main result of this paper is the following sufficient condition for a graph to be circular perfect.

Theorem 4.1 *Suppose G is a graph such that for every vertex x of G, $N[x]$ is a perfect graph and $G - N[x]$ is a bipartite graph with no induced P_4. Then G is circular perfect.*

In this paper, by P_4, we mean a path with 4 edges. It is easy to see that if G satisfies the condition of Theorem 4.1, then any induced subgraph of G satisfies that condition. Therefore to prove Theorem 4.1, it suffices to prove the following:

Theorem 4.2 *Suppose G is a graph and for every vertex x of G, $N[x]$ is a perfect graph and $V - N[x]$ is a bipartite graph which contains no induced P_4. Then $\chi_c(G) = \omega_c(G)$.*

The condition in Theorem 4.2 is not a necessary condition for a graph to be circular perfect. It is easy to construct circular perfect graphs G which contain vertices x such that $G - N[x]$ is not bipartite. However, the following example shows that these conditions are very tight in some sense. Let G be the graph as depicted in Figure 4 below. It is easy to verify that $\chi_c(G) = 8/3$ and $\omega_c(G) = 5/2$. So G is not circular perfect. But G "almost satisfies" the condition of Theorems 4.2. For each vertex x of G, $N[x]$ is a star, and $G - N[x]$ is either a P_3 or a P_4.

5 Proof of Theorem 4.2

To prove Theorem 4.2 we assume to the contrary that G is a connected core graph satisfies the condition of Theorem 4.2 but $\chi_c(G) > \omega_c(G)$. We shall derive a contradiction. The proof is quite long and complicated. We

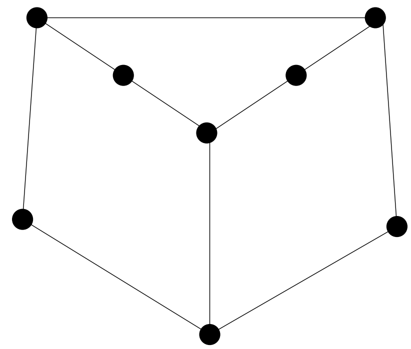

Figure 1: A non-circular perfect graph G

shall divide the argument into five subsections. Each subsection derives some properties of G, and all these properties together show that the graph G does not exist.

In the remaining of this section, G is a fixed core graph which is a counterexample to Theorem 4.2. The vertex set of G is V and the edge set of G is E. In this section and the next section, $N[x]$ (respectively, $N(x)$) means $N_G[x]$ (respectively, $N_G(x)$). For each vertex $x \in V$, we shall denote by H_x the subgraph $V - N[x]$. For a subset X of V, let $N(X) = \cup_{x \in X} N(x)$.

For two vertices $u, v \in V - N[x]$, we frequently need to compare the neighbourhood of u, v in $G - N[x]$. We shall use the following notation:

- $u \leq_x v$ means $N(u) \cap (V - N[x]) \subseteq N(v) \cap (V - N[x])$;
- $u <_x v$ means $N(u) \cap (V - N[x]) \subset N(v) \cap (V - N[x])$;
- $u =_x v$ means $N(u) \cap (V - N[x]) = N(v) \cap (V - N[x])$.

5.1 Connectness of H_x

In this subsection, we prove that H_x is a connected bipartite graph. First we note that H_x contains no isolated vertex, because if y is an isolated vertex of H_x, then $N(y) \subseteq N(x)$, contrary to the assumption that G is a core.

Lemma 5.1 *For each $x \in V$, if H_x is not connected then each component of H_x is a single edge.*

Proof. Assume to the contrary that H_x has a component Q which has at least three vertices. Then Q contains a vertex a which is adjacent to two vertices b, c. Let uv be an edge of another component of H_x.

If there is a vertex $w \in N(x)$ which is adjacent to b but not to c, then w is either not adjacent to u or not adjacent to v (or not adjacent to both), for otherwise w, u, v induce a triangle in H_c. Without loss of generality, assume that w is not adjacent u. Then c, a, b, w, x induce a P_4 in H_u, contrary to our assumption.

Thus we may assume that every vertex $w \in N(x)$ adjacent to b is also adjacent to c, and similarly, every vertex $w \in N(x)$ adjacent to c is also adjacent to b. Since G is a core, there is a vertex w in Q which is adjacent to b but not to c, and a vertex w' in Q adjacent to c but not to b. Then w, b, a, c, w' induce a P_4 in H_x, contrary to our assumption. ∎

Lemma 5.2 *For each $x \in V$, the graph H_x is connected.*

Proof. Assume to the contrary that for some vertex $x \in V$, H_x is not connected. By Lemma 5.1, each component of H_x is a single edge. Let ab and uv be two components of H_x.

It is easy to see that $N(a) \cap N(b) = N(u) \cap N(v)$, for otherwise, say there is a vertex $w \in N(u) \cap N(v) - N(a)$, then u, v, w induce a triangle in H_a. Let $Z = N(a) \cap N(b) = N(u) \cap N(v)$.

At least one of a, b is adjacent to some vertex w in $N(x) - Z$, for otherwise, we have a retraction mapping that fixes every other vertices, send a to u and b to v, contrary to the assumption that G is a core. Similarly, at least one of u, v is adjacent to some vertex w' in $N(x) - Z$. Assume $w \sim a$ and $w' \sim u$ (it is possible that $w = w'$).

If each of w is not adjacent to any of u, v, then b, a, w, x, w' induce a P_4 in H_v. Thus we may assume that $w \in N(x) - Z$ is adjacent to a and u.

If there is a vertex $w'' \in N(x) - Z$ which is adjacent to b but not to v, then w'', b, a, w, x induce a non-bipartite graph of H_v. Thus we may assume that every vertex in $N(x)$ adjacent to b is adjacent to v, and by symmetry, assume that every vertex in $N(x)$ adjacent to v is adjacent to b. Similarly, any vertex z of $N(x)$ adjacent to a is adjacent to u. But then we have a retraction of G which fixes every other vertex, and send a to u, b to v. ∎

Since H_x is a connected bipartite graph, there is a unique partition of the vertices of H_x into two parts. We shall denote these two parts by A_x and B_x, and write H_x as $H_x = (A_x \cup B_x, E_x)$, where E_x is the edge set of H_x.

Lemma 5.3 *For each $x \in V$, the graph H_x contains no induced $2K_2$ ($2K_2$ denotes the disjoint union of two copies of K_2).*

Proof. Assume to the contrary that H_x contains two edges ab and uv that induce a $2K_2$. Assume that $a, u \in A_x$ and $b, v \in B_x$. Let P be a shortest path connecting a and u in H_x. If P has length 2, say $P = (a, w, u)$, then b, a, w, u, v induce a P_4 in H_x. Otherwise P contains an induced P_4 in H_x. ∎

Corollary 5.1 *For any two vertices u, u' of A_x, u, u' are $\leq_x$-comparable, i.e., one of the sets $N_{H_x}(u)$ and $N_{H_x}(u')$ contains the other. Similarly, for any two vertices v, v' of B_x, v, v' are $\leq_x$-comparable.*

Proof. Assume to the contrary that there are vertices $u, u' \in A_x$ such that none of $N_{H_x}(u)$ and $N_{H_x}(u')$ contains the other. Let $v \in N_{H_x}(u) - N_{H_x}(u')$ and $v' \in N_{H_x}(u') - N_{H_x}(u)$. Then uv and $u'v'$ induce a $2K_2$ in H_x, contrary to Lemma 5.3. ∎

We call $u \in H_x$ a *$\leq_x$-minimum vertex* of H_x if $u \leq_x u'$ for all u' in the same part as u, and we call $u \in H_x$ a *$\leq_x$-maximum vertex* of H_x if $u' \leq_x u$ for all u' in the same part as u. By Corollary 5.1, each of A_x and B_x contains at least one $\leq_x$-maximum vertex and one $\leq_x$-minimum vertex of H_x. (Note that a $\leq_x$-maximum vertex could be a $\leq_x$-minimum vertex.)

For further discussion, we partition the vertex set V of G into three subsets T_1, T_2, T_3 which are defined as follows:

- $T_1 = \{x \in V : H_x$ is a complete bipartite graph $\}$;
- $T_2 = \{x \in V : H_x$ contains two vertices $u \in A_x, v \in B_x$, $u \not\sim v$ and there is a vertex $w \in N(x)$ such that w is adjacent to exactly one of $u, v\}$;
- $T_3 = V - (T_1 \cup T_2)$.

5.2 The set T_2 is empty

The goal of this subsection is to prove that $T_2 = \emptyset$. Thus we assume $T_2 \neq \emptyset$, and we shall derive a contradiction. In this subsection, let $x \in T_2$ be a fixed vertex. Let $u \in A_x, v \in B_x, w \in N(x)$ be vertices such that $u \not\sim v$, $w \sim u$ and $w \not\sim v$.

Lemma 5.4 *If $a \in A_x$ and $b \in B_x$ and $a \not\sim b$ then $N(x) \subseteq N(a) \cup N(b)$.*

Proof. Let $u' \in A_x$ and $v' \in B_x$ be the $\leq_x$-maximum vertices of H_x in A_x and B_x, respectively. By Corollary 5.1, $B_x \subseteq N(u')$ and $A_x \subseteq N(v')$.

Assume to the contrary of this lemma that there are $a \in A_x, b \in B_x$ and $w' \in N(x)$ such that $a \not\sim b$, $w' \not\sim a$ and $w' \not\sim b$. (Note that a could be u and b could be v). First we observe that $u' \sim w'$, for otherwise $w'x$ and $u'b$ induce a $2K_2$ in H_a, contrary to Lemma 5.3. Similarly $v' \sim w'$. If $v' \sim w$ then u, w, v' induce a triangle in H_v, contrary to our assumption. Thus we assume $v' \not\sim w$, and hence $w \neq w'$.

If $u \sim w'$ then $u \sim b$ (for otherwise u, w', v' induce a triangle in H_b) and $v \sim w'$ (for otherwise u, w', v' induce a triangle in H_v). Then $a \not\sim v$ for otherwise av, ub induce a $2K_2$ in H_x. But then u', v, w' induce a triangle in H_a.

Thus we assume that $u \not\sim w'$. Then $v \not\sim w'$ for otherwise u', v, w' induce a triangle in H_u. Now v', u, w, x, w' induce a nonbipartite subgraph of H_v, contrary to our assumption. This completes the proof of Lemma 5.4. ∎

Corollary 5.2 *If $a \in A_x$, $b \in B_x$ and $a \not\sim b$ then there is a vertex $w \in N(x)$ such that $w \sim a$ and $w \not\sim b$. Similarly, there is a vertex $w \in N(x)$ such that $w \sim b$ and $w \not\sim a$.*

Proof. Since $N(x) \not\subseteq N(a)$ (for otherwise G is not a core), there is a vertex $w \in N(x) - N(a)$. By Lemma 5.4, $w \sim b$. ∎

Lemma 5.5 *Suppose $a, a' \in A_x$ and $a <_x a'$. Then $N(x) \cap N(a') \subset N(x) \cap N(a)$.*

Proof. If $N(x) \cap N(a') = N(x) \cap N(a)$ then $N(a) \subset N(a')$, contrary to the assumption that G is a core. If there is a vertex $w \in N(x)$ such that $w \in N(a') - N(a)$, then let $b \in B_x$ such that $a \not\sim b$ and $a' \sim b$. By Lemma 5.4, $w \in N(b)$. Hence b, w, a' induce a triangle in H_a. Therefore $N(x) \cap N(a') \subset N(x) \cap N(a)$. ∎

Lemma 5.6 *If $a, a' \in A_x$ and $a =_x a'$, then $N_{H_x}(a) = B_x$.*

Proof. Assume to the contrary that $a, a' \in A_x$, $a =_x a'$ and there is a vertex $b \in B_x$ such that $a \not\sim b$ and $a' \not\sim b$. Let a'' be a $\leq_x$-maximum vertex in A_x (so $N_{H_x}(a'') = B_x$).

Now there is a vertex $w \in N(x)$ such that $w \sim a$ and $w \not\sim a'$, for otherwise $N(a) \subseteq N(a')$, contrary to the assumption that G is a core. Similarly there is a vertex $w' \in N(x)$ such that $w' \sim a'$ and $w' \not\sim a$. By Lemma 5.5, $w \not\sim a''$ and

$w' \not\sim a''$. Then a, w, x, w', a' induce either a P_4 (if $w \not\sim w'$) or a nonbipartite graph (if $w \sim w'$) in $H_{a''}$, contrary to our assumption. ∎

It follows from Lemma 5.6 that for every $x \in T_2$, there is a unique $\leq_x$-minimum vertex in A_x and a unique $\leq_x$-minimum vertex in B_x.

Lemma 5.7 *Suppose $a \in A_x$ is a $\leq_x$-minimum vertex of H_x. Let $R = N(x) - N(a)$. Then the following are true:*

1. *$a \in T_2$.*

2. *One part of H_a is equal to $(A_x - \{a\}) \cup R$ and the other part of H_a is equal to $(B_x - N(a)) \cup \{x\}$. We shall let $A_a = (A_x - \{a\}) \cup R$ and $B_a = (B_x - N(a)) \cup \{x\}$.*

3. *All the vertices in R are $\leq_x$-maximum vertices of A_a. Moreover, for $u, u' \in A_x - \{a\}$, $u \leq_x u'$ if and only if $u \leq_a u'$.*

4. *x is a $\leq_x$-minimum vertex of B_a. Moreover, for $u, u' \in B_x - N(a)$, $u \leq_x u'$ if and only if $u \leq_a u'$.*

Proof. Assume $a \in A_x$ is a $\leq_x$-minimum vertex. Let $b \in B_x$ be a $\leq_x$-minimum vertex, and let $a' \in A_x$ and $b' \in B_x$ be the $\leq_x$-maximum vertices of H_x in A_x and B_x, respectively. Since $x \in T_2$, H_x is not complete and hence $a \not\sim b$. Since a', b' are the $\leq_x$-maximum vertices, so $a' \sim b$, $a \sim b'$. By Corollary 5.2, there is a vertex $w \in N(x)$ such that $w \sim b$ and $w \not\sim a$.

Now $x, w, b, a' \in H_a$, and x, b is in one part of H_a and w, a' is in the other part of H_a. Assume that $a', w \in A_a$ and $x, b \in B_a$. Since $x \not\sim a'$, H_a is not a complete bipartite graph.

(1): There is a vertex $w' \in N(x)$ such that $w' \sim a$ and $w' \not\sim a'$, for otherwise we have $N(a) \subseteq N(a')$, contrary to the assumption that G is a core. Since $x \in B_a$, $a' \in A_a$, $w' \in N(a)$, $x \not\sim a'$, $x \sim w'$ and $a' \not\sim w'$, by definition, $a \in T_2$.

(2): It is easy to see that

$$V - N[a] = (B_x - N(a)) \cup \{x\} \cup R \cup (A_x - \{a\}).$$

As $x \in B_a$ is adjacent to every vertex of R, $b \in B_a$ is adjacent to a' and $w \in R$, a' is adjacent every vertex of $B_x - N(a)$, we conclude that $A_a = (A_x - \{a\}) \cup R$ and $B_a = (B_x - N(a)) \cup \{x\}$.

(3): It follows from Lemma 5.4 that for every $v \in B_a$, $R \subseteq N(v)$. So for every $w'' \in R$, $B_a \subseteq N(w'')$, hence each vertex of R is a $\leq_x$-maximum vertex of A_a. If $u \in A_x - \{a\}$, then $N_{H_a}(u) = N_{H_x}(u) - N(a)$. Therefore if $u, u' \in A_x - \{a\}$, then $u \leq_x u'$ if and only if $u \leq_a u'$.

(4): Since every vertex of $B_x - N(a)$ is adjacent to a', and x is not adjacent to a'. By Corollary 5.1, $x <_a u$ for any $u \in B_x - N(a)$, i.e., x is a $\leq_x$-minimum vertex of B_a. If $u \in B_x - N(a)$ then by Lemma 5.4, $N_{H_a}(u) = (N_{H_x}(u) - \{a\}) \cup R$. Therefore for $u, u' \in B_x - N(a)$, $u \leq_x u'$ if and only if $u \leq_a u'$. ∎

Lemma 5.8 *The set T_2 is empty.*

Proof. Assume to the contrary that $T_2 \neq \emptyset$. We define a graph Q as follows: $V(Q) = T_2$ and uv is an edge of Q if and only if u is a $\leq_x$-minimum vertex of H_v (or equivalently v is a $\leq_x$-minimum vertex of H_u). Since for each $u \in T_2$, H_u contains two $\leq_x$-minimum vertices (one in A_x and one in B_x), Q is a 2-regular graph. Let

$$C = (x_0, x_1, x_2, \cdots, x_{k-1})$$

be a cycle which is a connected component of Q. Arbitrarily assign a direction of traversal to the cycle C, say $x_0 \to x_1 \to x_2 \to \cdots$. For any x_i, the two neighbours x_{i-1} and x_{i+1} are the two $\leq_x$-minimum vertices of x_i. It follows from Lemma 5.7 that we can properly label the two parts of H_{x_i} so that $x_{i+1} \in A_{x_i}$ and $x_{i-1} \in B_{x_i}$, for all i (where summation and subtraction in the index are modulo k). It follows from (3) and (4) of Lemma 5.7 that for each i, there is a $d_i \geq 2$ such that

$$A_{x_i} = \{x_{i+1}, x_{i+2}, \cdots, x_{i+d_i-1}\}.$$

Indeed, x_{i+1} is the $\leq_x$-minimum vertex of A_{x_i}. Now x_{i+2} is the $\leq_x$-minimum vertex of $A_{x_{i+1}}$ and by Lemma 5.7 (3), x_{i+2} is the $\leq_x$-minimum vertex of $A_{x_i} - \{x_{i+1}\}$, provided $A_{x_i} - \{x_{i+1}\} \neq \emptyset$. In general, if

$$A_{x_i} - \{x_{i+1}, x_{i+2}, \cdots, x_{i+t}\} \neq \emptyset,$$

then by Lemma 5.7 (3), x_{i+t+1} is the $\leq_x$-minimum vertex of $A_{x_{i+t}}$ which is the $\leq_x$-minimum vertex of

$$A_{x_i} - \{x_{i+1}, x_{i+2}, \cdots, x_{i+t}\}.$$

Similarly, there is a d'_i such that $B_{x_i} = \{x_{i-1}, x_{i-2}, \cdots, x_{i-d'_i+1}\}$.

By (2) of Lemma 5.7, $A_{x_{i+1}} = (A_{x_i} - \{x_{i+1}\}) \cup R$, where $R = N(x_i) - N(x_{i+1})$. Since $R \neq \emptyset$ (for otherwise $N(x_{i+1}) \subseteq N(x_i)$, contrary to the assumption that G is a core), we conclude that $|A_{x_{i+1}}| \geq |A_{x_i}|$ for all i (again summation in the index is modulo k). So

$$|A_{x_0}| \leq |A_{x_1}| \leq |A_{x_2}| \leq \cdots \leq |A_{x_{k-1}}| \leq |A_{x_0}|.$$

Therefore $|A_{x_i}| = |A_{x_j}|$ for all $0 \leq i, j \leq k-1$. So $d_i = d_j$ for all $0 \leq i, j \leq k-1$. Let $d = d_i$.

Similarly by (2) of Lemma 5.7, $B_{x_{i+1}} = (B_{x_i} - N(x_{i+1})) \cup \{x_i\}$. As $N(x_{i+1}) \cap B_{x_i} \neq \emptyset$, we conclude that $|B_{x_{i+1}}| \leq |A_{x_i}|$ for all i (again summation in the index is modulo k). So

$$|B_{x_0}| \geq |B_{x_1}| \geq |B_{x_2}| \geq \cdots \geq |B_{x_{k-1}}| \geq |B_{x_0}|.$$

Therefore $|B_{x_i}| = |B_{x_j}|$ for all $0 \leq i, j \leq k-1$. So $d'_i = d'_j$ for all $0 \leq i, j \leq k-1$. Because $\Sigma_{i=0}^{k-1} d_i = \Sigma_{i=0}^{k-1} d'_i$, which is equal to the number of nonedges of the subgraph of G induced by $x_0, x_1, \cdots, x_{k-1}$, we conclude that $d'_i = d$ for all i.

Now for any $x_i \in C$, any vertex x of G not adjacent to x either belong to A_{x_i} or belong to B_{x_i}. So $x = x_j \in C$ for some j, and $d \leq |i - j| \leq k - d$. Therefore, the subgraph of G generated by $x_0, x_1, \cdots, x_{k-1}$ is isomorphic to $K_{k/d}$. Moreover, if $y \in V(G) - C$ then y is adjacent to every vertex of C.

Now if $V(G) = C$, then $G = K_{k/d}$, contrary to our assumption. Assume $V(G) - C \neq \emptyset$ and $y \in V(G) - C$. Then $C \subset N[y]$. If $K_{k/d}$ is a core then $K_{k/d}$ is not a perfect graph (as $d \geq 2$), contrary to our assumption that $N[y]$ induces a perfect graph. If $K_{k/d}$ is not a core then let f be a retraction on the subgraph induced by $x_0, x_1, \cdots, x_{k-1}$, then f can be extended to a retraction of G by fixing every other vertices of G, contrary to the assumption that G is a core. ∎

5.3 Structures of H_x for $x \in T_3$ and of a hypergraph

By Lemma 5.8, $V = T_1 \cup T_3$. If $x \in T_1$, then H_x is a complete bipartite graph. In this section, we shall show that if $x \in T_3$, the structure of H_x is also very simple. Then we build a hypergraph and discuss properties of this hypergraph.

Lemma 5.9 *If $x \in T_3$, then there exist an unique $u \in A_x$ and $v \in B_x$ such that $u \not\sim v$.*

Proof. Assume to the contrary that there are two vertices $u, u' \in A_x$ such that each of them is not adjacent to some vertices of B_x. Let $v \in B_x$ be a $\leq_x$-minimum vertex of B_x. Then by Corollary 5.1, $u \not\sim v$ and $u' \not\sim v$. Assume that $u \leq_x u'$, i.e., $N_{H_x}(u) \subseteq N_{H_x}(u')$. By the definition of T_3, we have

$$N(u) \cap N(x) = N(v) \cap N(x) = N(u') \cap N(x).$$

Then $N(u) \subseteq N(u')$, contrary to the assumption that G is a core. ∎

So for $x \in T_3$, the graph H_x is equal to a complete bipartite graph minus one edge.

Lemma 5.10 *Suppose $a \in T_3$, $b \in A_a, c \in B_a$ are the unique vertices such that $b \not\sim c$. Then $b, c \in T_3$, $a \in A_b$ and $c \in B_b$ are the unique vertices such that $a \not\sim c$; $a \in A_c$ and $b \in B_c$ are the unique vertices such that $a \not\sim b$. Moreover, $A_a - \{b\} \neq \emptyset$, $B_a - \{c\} \neq \emptyset$, and if $y \in A_a - \{b\}$ or $y \in B_a - \{c\}$ then $y \in T_1$.*

Proof. Since $a \in T_3$, by the definition of T_3,

$$N(a) \cap N(b) = N(a) \cap N(c).$$

Hence $N(a) - N(b) = N(a) - N(c)$. Let

$$\begin{aligned} X_c &= A_a - \{b\}, \quad X_b = B_a - \{c\}, \\ X_a &= N(a) - N(b) = N(a) - N(c), \\ Y &= V - (X_b \cup X_c \cup X_a \cup \{a, b, c\}). \end{aligned}$$

Then the four sets X_b, X_c, X_a, Y are pairwise disjoint, and $N(a) = X_a \cup Y$, $N(b) = X_b \cup Y$ and $N(c) = X_c \cup Y$.

The set X_a is an independent set, for otherwise if xy is an edge in X_a, then x, y, a induce a triangle in H_b (as well as in H_c).

By Lemma 5.9, $X_c \cup X_b$ induces a complete bipartite graph, with X_b, X_c as the two parts. If there is a vertex $w \in X_a$ and a vertex $u \in X_b$ such that w is not adjacent to u then ub, xw is a $2K_2$ in H_c, contrary to Lemma 5.3. Therefore $X_b \cup X_a$ also induces a complete bipartite graph. Similarly $X_c \cup X_a$ induces a complete bipartite graph. Hence $X_c \cup X_b \cup X_a$ induces a complete tripartite graph with X_c, X_b, X_a as the three parts. The adjacency of the sets X_a, X_a, X_c, Y and a, b, c is as illustrated in Fig. 2 below.

For $A_a - \{b\} \neq \emptyset$, for otherwise c is an isolated vertex of H_a. If $y \in A_a - \{b\} = X_c$, H_y is a complete bipartite graph, with one part consists of $\{a, b\}$, the other part is equal to $(V - N[y]) \cap Y$. Similarly if $y \in B_a - \{c\}$ then H_y is also a complete bipartite graph. This completes the proof of Lemma 5.10. ∎

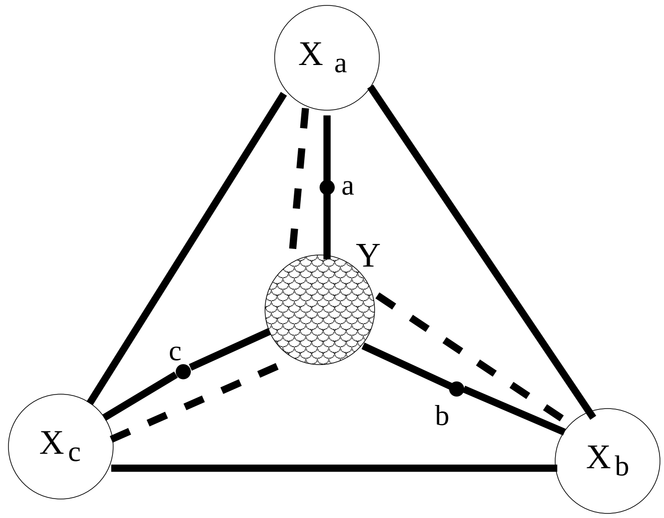

Figure 2: The adjacency of the sets X_a, X_a, X_c, Y and a, b, c, where a thick line indicates complete adjacency of the two sets, and a dotted line indicates partial adjacency.

We call the triple a, b, c as in Lemma 5.10 a T_3*-triple*. It follows from Lemma 5.10 that the vertices of T_3 are partitioned into T_3-triples.

Now we construct a hypergraph $\mathcal{H}$ as follows: The vertex set of $\mathcal{H}$ is V. The edge set of $\mathcal{H}$ is $\mathcal{E} = \{E_{x,1}, E_{x,2} : x \in V\}$, where for each $x \in V$, $E_{x,1} = A_x \cup \{x\}$ and $E_{x,2} = B_x \cup \{x\}$. The hypergraph $\mathcal{H}$ has no multiple edges, i.e., if $E_{x,j} = E_{y,i}$ then $E_{x,j}, E_{y,i}$ is counted as one hyperedge.

Observe that each hyperedge of $\mathcal{H}$ is an independent set of G. If $y \not\sim x$, then either $y \in E_{x,1}$ or $y \in E_{x,2}$, i.e., each pair of nonadjacent vertices is contained in a hyperedge.

A *cycle* of $\mathcal{H}$ of length m is a sequence

$$x_0, E_0, x_1, E_1, x_2, E_2, \cdots, x_{m-1}, E_{m-1}$$

of distinct vertices and hyperedges such that $x_i, x_{i+1} \in E_i$ for $i = 0, 1, \cdots, m-2$, and $x_0, x_{m-1} \in E_{m-1}$. A cycle of length m is also called an m-cycle.

Assume a, b, c is a T_3-triple. Let X_a, X_b, X_c be the sets as defined in the proof of Lemma 5.10. Then

$$\begin{aligned} E_{a,1} &= X_b \cup \{a, c\} = E_{c,2}, \\ E_{a,2} &= X_c \cup \{a, b\} = E_{b,1}, \\ E_{b,2} &= X_a \cup \{b, c\} = E_{c,1}, \end{aligned}$$

These three hyperedges form a 3-cycle

$$a, E_{a,1}, c, E_{c,1}, b, E_{b,1}.$$

We call this 3-cycle a *special triangle* of $\mathcal{H}$ corresponds to the T_3-triple a, b, c.

Lemma 5.11 *For each vertex $x \in V$, there are exactly two hyperedges containing x.*

Proof. For $x \in V$, $E_{x,1}, E_{x,2}$ are two hyperedges containing x. Assume to the contrary that E' is another hyperedge containing x. Without loss of generality, assume that $E' = E_{y,2}$ for some vertex y. Since $y \not\sim x$, so $y \in E_{x,1} \cup E_{x,2}$. Without loss of generality, we may assume that $y \in E_{x,1}$. Furthermore, without loss of generality, we assume that $E_{x,1} - E_{y,2} \neq \emptyset$. Let $w \in E_{x,1} - E_{y,2}$. Since $w, y \in E_{x,1}$, we know that $w \not\sim y$. Therefore $w \in E_{y,1}$. Let $w' \in E_{y,2}$ such that $w \sim w'$ (the vertex w' exists because H_y is connected). Then $w' \not\sim x$, for otherwise w', x would be in different parts of H_y and hence w, x are in the same part of H_y. Also $w' \notin E_{x,1}$ (because $w \in E_{x,1}$). So H_x is not a complete graph, and hence $x \in T_3$. Moreover, the T_3-triple containing x is x, y, w'. But as shown in the paragraph preceding this lemma, in this case, we should have $E_{x,1} = E_{y,2}$. ∎

Lemma 5.12 *Two distinct hyperedges of $\mathcal{H}$ have at most one common vertex.*

Proof. Assume to the contrary that there are two hyperedges containing both x, y. By Lemma 5.11, there are exactly two hyperedges containing x, and exactly two hyperedges containing y. Thus $\{E_{x,1}, E_{x,2}\} = \{E_{y,1}, E_{y,2}\}$. It follows that $N(x) = N(y)$, contrary to the assumption that G is a core. ∎

Lemma 5.12 show that $\mathcal{H}$ has no cycles of length 2. We have observed above that each T_3-triple corresponds to a special triangle in $\mathcal{H}$. Now we shall show that there are no other triangles in $\mathcal{H}$.

Lemma 5.13 *Each 3-cycle of $\mathcal{H}$ is a special triangle corresponds to a T_3-triple.*

Proof. Let a, E_1, b, E_2, c, E_3 be a 3-cycle of $\mathcal{H}$. Then E_1, E_3 are the two hyperedges containing a, E_1, E_2 are the two hyperedges containing b, E_2, E_3 are the two hyperedges containing c. Let $X_c = E_1 - \{a, b\}$, $X_a = E_2 - \{a, c\}$ and $X_b = E_3 - \{a, c\}$. We have $X_c \neq \emptyset$, for otherwise $N(b) \subseteq N(c)$. Similarly $X_a, X_b \neq \emptyset$. Now by definition, a, b, c is a T_3-triple, and the 3-cycle a, E_1, b, E_2, c, E_3 is the corresponding special triangle. ∎

5.4 More on the structure of $\mathcal{H}$

This subsection investigates further the structure of the hypergraph $\mathcal{H}$. We shall prove that $\mathcal{H}$ contains no odd cycles other than the special triangles, and that any vertex $x \in T_3$ is not contained in any cycle other than the special triangles.

Lemma 5.14 *The hypergraph $\mathcal{H}$ contains no odd cycles other than the special triangles correspond to T_3 triples.*

Proof. Assume the lemma is not true. Let $C = (x_1, E_1, x_2, E_2, \cdots, x_{2k+1}, E_{2k+1})$ be a shortest odd cycle of $\mathcal{H}$, which is not a special triangle. By Lemma 5.13, $k \geq 2$. For each vertex x_i, the hyperedges E_{i-1} and E_i are the only two hyperedges of $\mathcal{H}$ containing x_i. Therefore the subgraph Q of G induced by the vertices $x_1, x_2, \cdots, x_{2k+1}$ is the complement of the odd cycle C_{2k+1}. If G contains a vertex x which is not contained in the union $\cup_{i+1}^{2k+1} E_i$, then x is adjacent to all x_i $(i = 1, 2, \cdots, 2k+1)$; i.e., $Q \subseteq N[x]$. This is in contrary to our assumption that $N[x]$ induces a perfect graph.

Thus we assume that $V = \cup_{j=1}^{2k+1} E_j$. If each edge E_i contains exactly two vertices, then G is the complement of the odd cycle C_{2k+1}, i.e., $G = G_2^{2k+1}$. So G is circular perfect, contrary to our assumption. Thus there is an edge E_i which contains more than two vertices.

Claim: *There is a vertex $y \in E_i$ and $y \neq x_i, x_{i+1}$ and y is not contained in any E_j for $j \neq i$.*

Suppose E_i contains more than two vertices. Let $y \in E_i$ such that $y \neq x_i, x_{i+1}$. If y is not contained in any other E_j then we are done. Thus we assume that $y \in E_j$ for some $j \neq i$. Since $E_i \cap E_{i+1} = \{x_{i+1}\}$ and $E_{i-1} \cap E_i = \{x_i\}$ so $j \neq i-1, i+1$. Without loss of generality, we assume that $j > i+1$.

Now
$$C' = (y, E_i, x_{i+1}, E_{i+1}, \cdots, x_j, E_j)$$
and
$$C'' = (y, E_j, x_{j+1}, E_{j+1}, \cdots, E_{2k+1}, x_1, E_1, x_2, \cdots, E_i)$$
are two cycles of $\mathcal{H}$. The sum of the lengths of C' and C'' is equal to the length of C plus 2. Therefore, one of C', C'' is an odd cycle. Without loss of generality, we assume that C' is of odd length. Since the length of C'' is at least 4, we know that the length of C' is smaller than the length of C. By the choice of C, we conclude that C' is a special triangle corresponds to

the T_3-triple y, x_{i+1}, x_{i+2}. Therefore E_{i+1} also contains a third vertex z. By Lemma 5.10, $z \in T_1$. Now use the same argument as above, we can conclude that z is not contained in any other E_j (as z cannot be contained in a special triangle). This proves the Claim.

Assume now that $y \in E_i$ and $y \neq x_i, x_{i+1}$ and $y \notin E_j$ for $j \neq i$. Let E' be the other hyperedge containing y, and let $z \in E'$ be another vertex of E'. (Note that each vertex of G is contained in two hyperedges and each hyperedge contains at least two vertices). Since $V = \cup_{j=1}^{2k+1} E_j$, we assume that $z \in E_j$ for some j.

Without loss of generality, we assume that $j > i+1$. Now

$$C' = (y, E_i, x_{i+1}, E_{i+1}, \cdots, x_j, E_j, z, E')$$

and

$$C'' = (y, E', z, E_j, x_{j+1}, E_{j+1}, \cdots, E_{2k+1}, x_1, E_1, x_2, \cdots, E_i)$$

are two cycles of $\mathcal{H}$. The sum of the lengths of C' and C'' is equal to the length of C plus 4. Therefore, one of C', C'' is an odd cycle. Without loss of generality, we assume that C' is of odd length.

If C' is a special triangle, then E' contains a third vertex w. In this case, we shall use the vertex w to play the role of z. (Note that in this case $w \in T_1$ by Lemma 5.10 and hence the corresponding C' cannot be a special triangle). Therefore we may assume that C' is not a special triangle.

The sum of the lengths of C' and C'' is equal to the length of C plus 4. Since the length of C'' is at least 4, by the minimality of the length of C we conclude that C'' is of length 4 and C' is of the same length as C. So $j = i+2$.

We shall show that E' contains only two vertices. Assume to the contrary that E' contains a third vertex w, and assume that $w \in E_t$. Similar to the argument in the proof of the Claim above, we can show that w, E_t together with two other vertices and two hyperedges of C' form a special triangle. Without loss of generality, we may assume that $t = i+1$ and $y, E_i, x_{i+1}, E_{i+1}, w, E'$ is a special triangle. Then we have a 5-cycle

$$x_i, E_i, x_{i+1}, E_{i+1}, w, E', z, E_{i-2}, x_{i-1}, E_{i-1}, x_i.$$

By the minimality of C we conclude that C and C' are all 5-cycles.

For simplicity, we let $i = 3$. Then the hyperedges of the cycle C together with hyperedge E' and the named vertices are as depicted in Fig. 3 below.

If E_2 or E_5 contains a third vertex, then the same discussion as above would derive a contradiction, because by Lemma 5.10, $x_1, x_2, x_3, x_5 \in T_1$, so

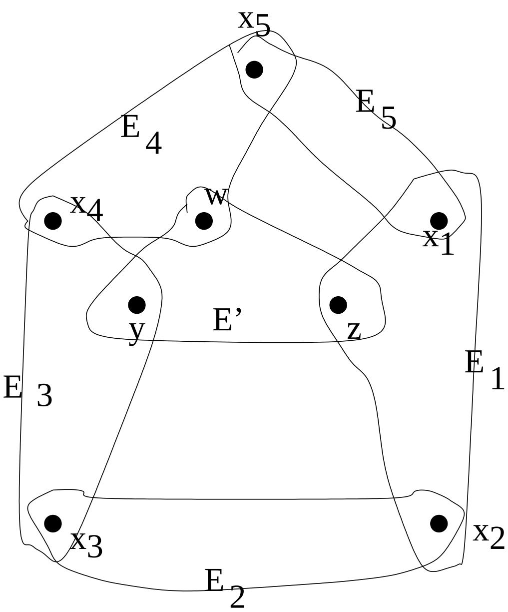

Figure 3: The hyperedges of C and E'

these vertices cannot be contained in a special triangle. Thus each of E_2 and E_5 contains two vertices. So the vertices of G are covered by E_1, E_3, E_4. This implies that G is 3-colorable (as each of the hyperedge is an independent set of G). On the other hand, z, x_3, x_5 induce a triangle in G, contrary to the assumption that G is a core. This contradiction shows that E' indeed contains only two vertices y and z.

By interchanging the roles of E' and E_{i-1}, C' and C, the same argument shows that E_{i-1} contains only two vertices x_{i-1} and x_i. Now it is straightforward to verify that the mapping f which sends y to x_i, sends x_{i-1} to z, and fixes every other vertex is a retraction of G, contrary to the assumption that G is a core. This completes the proof of Lemma 5.14. ∎

Corollary 5.3 *If $a \in T_3$ then a is contained in no cycle other than the special triangle.*

Proof. Assume to the contrary that a is contained in a cycle

$$C = (a, E_1, x_1, E_2, x_2, \cdots, x_{k-1}, E_k)$$

which is not a special triangle. Then $k \geq 4$ and by Lemma 5.14, k is even. The two edges E_1 and E_k together with another hyperedge E' form a special triangle.

We divide the remaining discussion into two cases.

Case 1: One of x_1, x_{k-1} is in the same T_3 triple as a.

Without loss of generality, we assume that x_1 is in the same T_3 triple as a. Let z be the other vertex of the T_3-triple containing a and x_1. Then the special triangle corresponds to a, x_1, z is

$$a, E_1, x_1, E_2, z, E_k.$$

The cycle C and the special triangle are as depicted in Fig. 4 below.

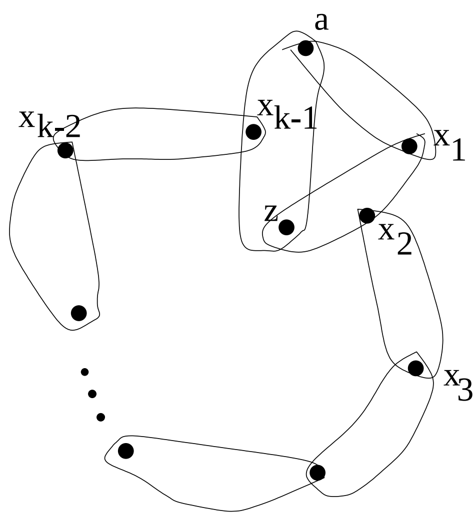

Figure 4: A depiction of the cycle C and the special triangle for Case 1

Then

$$x_{k-1}, E_k, z, E_2, x_2, E_3, \cdots, E_{k-1}$$

is an odd cycle of $\mathcal{H}$, which is not a special triangle, contrary to Lemma 5.14.

Case 2: None of x_1, x_{k-1} is in the same T_3 triple as a.

Assume the T_3 triple containing a is a, b, c. By Lemma 5.10, each of the hyperedges E_1 and E_k contains two vertices of the triple. Without loss of generality, we assume that E_1 contains a, b, and E_k contains a, c. Let E' be the other hyperedge of the special triangle. Then E' contains b, c. The cycle C and the special triangle are as depicted in Fig. 5 below.

Therefore

$$b, E_1, x_1, E_2, x_2, \cdots, x_{k-1}, E_k, c, E'$$

is a cycle of $\mathcal{H}$ whose length is equal to the length of C plus 1, contrary to Lemma 5.14. ∎

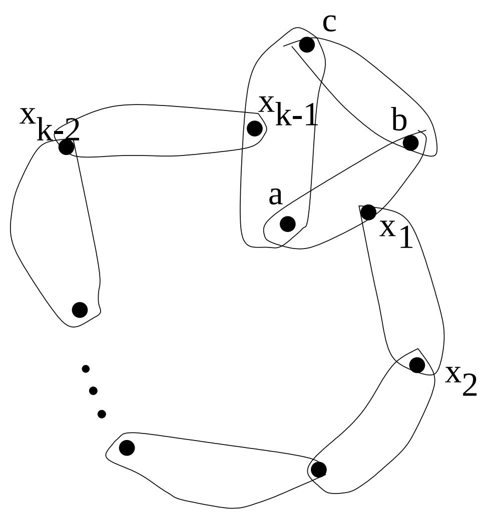

Figure 5: A depiction of the cycle C and the special triangle for Case 2

5.5 The final contradiction

In this section, by applying Hall's theorem, Lemma 5.14 and Corollary 5.3, we derive a final contradiction, and complete the proof of Theorem 4.2.

Let $\mathcal{H}'$ be the hypergraph obtained from $\mathcal{H}$ by deleting all those hyperedges of special triangles, and then deleting the isolated vertices, i.e., those vertices in T_3. Let $Q_1, Q_2, Q_3, \cdots, Q_m$ be the connected components of $\mathcal{H}'$.

Lemma 5.15 *There is a connected component Q_i of $\mathcal{H}'$ such that there is at most one hyperedge $E' \in \mathcal{H} - \mathcal{H}'$ which intersects the vertices of Q_i.*

Proof. Let $R^1, R^2, \cdots, R^k$ be the special triangles of $\mathcal{H}$. By Corollary 5.3, none of the vertices in T_3 is contained in any cycle other than the special triangles. We build a graph T with vertex set $\{Q_1, Q_2, Q_3, \cdots, Q_m, R^1, R^2, \cdots, R^k\}$, and join Q_iR^j by an edge if there is a hyperedge of $\mathcal{H}$ (which is one of the deleted hyperedge) containing vertices from both Q_i and R^j, then T is a forest. Since each R^j is adjacent to at least three of Q_j's (cf. the proof of Lemma 5.10), all the leaves of T are the Q_j's.

Let Q_i be a leave vertex of T. Then there is at most one R^j which is adjacent to Q_i. Therefore, only the hyperedges of the special triangle R^j may intersect the vertices of Q_i. Among the three hyperedges of R^j, only one intersects the vertices of Q_i, for otherwise there is a cycle other than the special triangle containing the vertices of T_3. This completes the proof of Lemma 5.15. ∎

Without loss of generality, assume that there is at most one hyperedge E' in $\mathcal{H} - \mathcal{H}'$ intersects the vertices of Q_1.

Let F be the graph defined as follows: The vertices of F are the hyperedges of $Q_1 \cup \{E'\}$ (if E' does not exists, then the vertices of F are the hyperedges of Q_1); $E_i \sim_F E_j$ if and only if $E_i \cap E_j \neq \emptyset$ (as subsets of V).

Since each vertex of G is contained in exactly two hyperedges of $\mathcal{H}$ and hence is contained in at most two hyperedges of $\mathcal{H}'$, each cycle of F corresponds to a cycle of $\mathcal{H}$ of the same length. By Lemma 5.14, F contains no odd cycles, because the hypergraph induced by the hyperedges $Q_1 \cup \{E'\}$ contains no special triangles. Hence F is bipartite. Assume $V(F) = A \cup B$, where A, B are the two parts of F, and $E' \in A$ if E' exists.

For each subset $B' \subseteq B$, let $N_F(B') = \{E_i \in A : E_i$ adjacent to E_j in F for some $E_j \in B'\}$. We say $B' \subseteq B$ is *critical* if $|N_F(B') - \{E'\}| < |B'|$ and $|N_F(B'') - \{E'\}| \geq |B''|$ for any proper subset B'' of B'.

Lemma 5.16 *The set B contains a critical subset.*

Proof. Assume to the contrary that B contains no critical subsets. By Hall's theorem, F has a matching M that saturates B, and do not saturate $E' \in A$. Assume the edges of the matching M are

$$E_{i_1}E_{j_1}, E_{i_2}E_{j_2}, \cdots, E_{i_m}E_{j_m},$$

where $E_{j_1}, E_{j_2}, \cdots, E_{j_m}$ are the hyperedges in B. Let $\{x_t\} = E_{i_t} \cap E_{j_t}$ for $t = 1, 2, \cdots, m$. Since M is a matching of F, the vertices $x_1, x_2, \cdots, x_m$ induce a copy of K_m in G. Also observe that x_t is not contained in E', and hence in the graph G, x_t is adjacent to all vertices not in Q_1.

Let f be the mapping which retracts E_{j_t} to x_t for $t = 1, 2, \cdots, m$, and fixes every other vertex of G. We now show that f is a retraction of G. Assume that xy is an edge of G. If $x, y \in Q_1$, then x, y is not contained in the same hyperedge (as each hyperedge is an independent set of G). So $f(x) = x_t$ and $f(y) = x_{t'}$ for some $t \neq t'$. Therefore $f(x)f(y)$ is an edge of G. If $x, y \notin Q_1$, then $f(x) = x$, $f(y) = y$ and $f(x)f(y)$ is an edge of G. Assume now that $x \in Q_1$ and $y \notin Q_1$. Then $f(y) = y$ and $f(x) = x_t$ for some t. As observed above, x_t is adjacent to every vertex not in Q_1, so yx_t is an edge of G. This proves that f is a retraction, contrary to the assumption that G is a core. ∎

Let B be a critical subset B'. Then $|N_F(B') - \{E'\}| = |B'| - 1$ and hence $|N_F(B')| \leq |B'|$. So there exists a subset B'' of B' such that $|N_F(B'')| = |N_F(B')| = |B''|$. By applying Hall's theorem, we conclude that there is a matching of F that saturates both B'' and $N_F(B'')$. Let M be such a matching, and assume

$$E_{i_1}E_{j_1}, E_{i_2}E_{j_2}, \cdots, E_{i_m}E_{j_m}$$

are the edges of M, where $E_{j_1}, E_{j_2}, \cdots, E_{j_m}$ are the hyperedges in B'' and $E_{i_1}, E_{i_2}, \cdots, E_{i_m}$ are the hyperedges in $N_{F'}(B'')$. Let $\{x_t\} = E_{i_t} \cap E_{j_t}$ for $t = 1, 2, \cdots, m$. Then $x_1, x_2, \cdots, x_m$ induce a K_m in G.

Let $Z = \cup_{l=1}^{m} E_{j_l}$. Let f be the mapping which sends $E_{i_t} \cap Z$ to x_t, and fixes every other vertex of G. We now show that f is a homomorphism. The restriction of f to Z is obviously a homomorphism, as adjacent vertices in Z are sent to distinct x_t's, and $x_1, x_2, \cdots, x_m$ induce a complete graph. The restriction of f to $V - Z$ is the identity mapping, so it is a homomorphism. Now let $x \in Z$ and $y \in V - Z$. We need to prove that if $x \sim_G y$ then $f(x) \sim_G f(y)$.

Note that by Lemma 5.11, each vertex of G is contained in exactly two hyperedges. So each vertex of E_{j_t} is contained in some hyperedge E^* of $\mathcal{H}$ which intersects E_{j_t}. Since $N_F(B'') = \{E_{i_1}, E_{i_2}, \cdots, E_{i_m}\}$, $E^* = E_{i_l}$ for some l. Therefore, $Z = \cup_{l=1}^{m} E_{j_l} \subseteq \cup_{l=1}^{m} E_{i_l}$.

Assume $x \in E_{i_t} \cap Z$. Then $f(x) = x_t$. Since y is adjacent to x in G, so $y \notin E_{i_t}$. The two hyperedges containing x_t are E_{i_t} and E_{j_t}. Since $y \notin E_{i_t}, E_{j_t}$, so $x_t \sim_G y$.

This completes the proof of Theorem 4.2. ∎

References

[1] B. Bauslaugh and X. Zhu, *Circular coloring of infinite graphs*, Bulletin of The Institute of Combinatorics and its Applications, 24(1998), 79-80.

[2] J. A. Bondy and P. Hell, *A note on the star chromatic number*, J. Graph Theory 14 (1990), 479-482.

[3] W. Deuber and X. Zhu, *Circular coloring of weighted graphs*, J. Graph Theory, 23(1996), 365-376.

[4] G. Hajós, *Über eine Konstruktion nicht n-färbbarer Graphen*, Wiss. Zeitschrift Univ. Halle, Math.-Nat. 10 (1961), 113-114.

[5] T. R. Jensen and B. Toft, **Graph coloring problems**, Wiley 1995.

[6] S. Liaw, Z. Pan and X. Zhu, *Construction of of K_n-minor free graphs with given circular chromatic numbers*, Discrete Math., to appear.

[7] D. Moser, *The star-chromatic number of planar graphs*, J. Graph Theory 24(1997), 33-43.

[8] J. Nešetřil, *Homomorphism structure of classes of graphs*, Combinatorics, Probability and computing 8 (1999), 177-184.

[9] A. Vince, *Star chromatic number*, J. Graph Theory 12 (1988), 551-559.

[10] X. Zhu, *Star chromatic numbers and products of graphs*, J. Graph Theory 16 (1992), 557-569.

[11] X. Zhu, *The circular chromatic number, a survey*, Discrete Math. Vol. 229 (1-3) (2001), 371-410.

[12] X. Zhu, *An analogue of Hajós theorem for the circular chromatic number*, Proceedings of the American Mathematical Society, 129(2001), 2845-2852.