An analogue of Hajós' Theorem for the circular chromatic number (II)

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Abstract

This paper designs a set of graph operations, and proves that for $2 \leq k/d < 3$, starting from $K_{k/d}$, by repeatedly applying these operations, one can construct all graphs G with $\chi_c(G) \geq k/d$. Together with the result proved in [20], where a set of graph operations were designed to construct graphs G with $\chi_c(G) \geq k/d$ for $k/d \geq 3$, we have a complete analogue of Hajós' Theorem for the circular chromatic number.

1 Introduction

The circular chromatic number (also known as the star chromatic number) of a graph is a natural generalization of the chromatic number of a graph. There are quite a few equivalent definitions of the circular chromatic number of a graph. Suppose $r \geq 2$ is a real number. For a graph $G = (V, E)$, an r-colouring f of G is a mapping $f : V \to [0, r)$ such that for every edge xy of G, $1 \leq |f(x) - f(y)| \leq r - 1$. For any real number r and for $0 \leq x < r$, let $|x|_r = \min\{|x|, r - |x|\}$. Then an r-colouring f of G is a

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mapping $f : V \to [0, r)$ such that $|f(x) - f(x)|_r \geq 1$ for every edge xy of G. The *circular chromatic number* $\chi_c(G)$ of a graph G with at least one edge is the infimum of those r for which G has an r-colouring. It can be verified [17] that in case $r = k/d$ is a rational number, then G is r-colourable if and only if there is a mapping $f : V \to \{0, 1, \cdots, k-1\}$ such that for any two adjacent vertices x and y of G, we have $d \leq |\phi(x) - \phi(y)| \leq k - d$. Such a mapping is called a (k, d)-colouring of G. An equivalent definition of the circular chromatic number is that $\chi_c(G)$ is equal to the infimum of the ratio k/d for which there exists a (k, d)-colouring of G. (See [10, 17] for the proof of the equivalence).

For $k \geq 2d$, let $K_{k/d}$ be the graph with vertex set $\{0, 1, \cdots, k-1\}$, in which i is adjacent to j if and only if $d \leq |i - j| \leq k - d$. (The graph $K_{k/d}$ is also denoted by G^k_d in the literature.) Then a (k, d)-colouring of a graph G is simply a *homomorphism* (i.e., edge-preserving vertex mapping) from G to $K_{k/d}$. Observe that if $d = 1$, then $K_{k/1}$ is the complete graph K_k, and a homomorphism from G to K_k is equivalent to a k-colouring of G. For any graph H, a homomorphism from a graph G to H is called an *H-colouring* of G. In the study of the circular chromatic number of graphs, the graphs $K_{k/d}$ play the same role as the complete graphs in the study of chromatic number [14, 17]. We call the graphs $K_{k/d}$ the *circular complete graphs.*

Many questions concerning the chromatic number of a graph have been studied in the framework of circular chromatic number. Quite a few classical results concerning the chromatic number of graphs have been generalized to results concerning the circular chromatic number. For example, it was shown in [11] that for any $r \geq 2$, there are graphs G of arbitrary large girth and with $\chi_c(G) = r$ (see [17] for a constructive proof). As a complement to the Four Colour Theorem, it was shown in [5, 8, 15, 16] that a rational number r is the circular chromatic number of a non-trivial planar graph if and only if $2 \leq r \leq 4$. In [4, 13], it is shown that if $n \geq 5$, then for any rational number $2 \leq r \leq n - 1$ there is a K_n-minor free graph G with $\chi_c(G) = r$.

Hajós' theorem is one of the classical results in the field of graph colouring. It says that all graphs G with $\chi(G) \geq k$ can be constructed from copies of K_k by repeatedly applying three operations: adding veritces and edges, identifying nonadajcent vertices, and applying Hajós' sum. Here the Hajós' sum of two graphs G and G' is the graph obtained from the vertex disjoint union of G and G' by deleting an edge xy from G, an edge $x'y'$ from G', identifying x and x', adding an edge joining y and y'. (This operation is actually the *series connection* of G and G', as defined in [7] for matroids).

The problem of finding an analogue of Hajós' theorem [3] for the circular chromatic number and for general H-colouring was discussed in [6, 12, 20, 17].

Recently, a set of graph operations was designed in [20], and it was proved there that for $k/d \geq 3$, starting from the graph $K_{k/d}$, by repeatedly applying these graph operations, one can construct all graphs G with $\chi_c(G) \geq k/d$. However, those graph operations do not work for the case $k/d < 3$. In this paper, we design another set of graph operations, and prove that for $2 < k/d < 3$, starting from the graph $K_{k/d}$, by repeatedly applying these graph operations, one can construct all graphs G with $\chi_c(G) \geq k/d$. Together with the result proved in [20], this gives a complete analogue of Hajós' theorem for the circular chromatic number.

There is a strengthening of Hajós' theorem, i.e., Ore's theorem, in which the Hajós' sum and the identification of nonadjacent vertices are combined into a single operation: Ore's operation, which is to take the Hajós' sum of G and G' and then identify some pairs of nonadjacent vertices, where each pair consists of one vertex from G and one vertex from G'. Ore's Theorem says that starting from copies of K_k, one can construct all graphs G with $\chi(G) \geq k$ by repeatedly applying the two operations: (1) adding vertices and edges and (2) the Ore operation defined above. Ore's theorem is very much similar to Hajós' theorem. However, there are some applications [1] of Ore's theorem where it cannot be replaced by Hajós' theorem. What we shall prove in this paper is indeed a generalization of Ore's theorem to circular chromatic number.

2 A set of graph operations

This section presents 5 graph operations which do not decrease the circular chromatic number. Operation 1 is an operation which was used in [20] as well. The other operations are new. For the completeness of this paper, we shall also include the short proof that Operation 1 does not decrease the circular chromatic number.

Operation 1

Given $2k+1$ $(k \geq 1)$ graphs $G_0, G_1, G_2, \cdots, G_{2k}$. For $i = 0, 1, 2, \cdots, 2k$, let $e_i = x_i y_i$ be an edge of G_i. Let $C_{2k+1} = (c_0, c_1, \cdots, c_{2k})$ be a cycle of length $2k+1$. Construct a new graph from the disjoint union of $G_0, G_1, G_2, \cdots, G_{2k}, C_{2k+1}$ as follows:

- delete the edges e_i, for $i = 0, 1, 2, \cdots, 2k$,
- identify all the x_i's into a single vertex and name it x,

- identify y_i and c_i for $i = 0, 1, 2, \cdots, 2k$.

We shall denote the resulting graph by $S_1(G_0, e_0, G_1, e_1, G_2, e_2, \cdots, G_{2k}, e_{2k})$, or simply by S_1 if the graphs G_i and edges e_i are clear from the context.

Let $f : S_1 \to Z$ be a mapping such that for each $j \in Z$, $f^{-1}(j)$ is an independent set of S_1, and moreover for each G_i, $|f^{-1}(j) \cap V(G_i)| \leq 1$. Then for each $j \in Z$, we identify all vertices of $f^{-1}(j)$ into a single vertex. We denote the resulting graph by S_1^*, and call S_1^* a graph obtained from $G_0, G_1, G_2, \cdots, G_{2k}$ by Operation 1. (Observe that from $G_0, G_1, G_2, \cdots, G_{2k}$, one can construct many graphs S_1^*. By an abuse of notation, we use S_1^* to denote any such a graph).

Note that the second step in the construction above is to identify some nonadjacent vertices. The first half of the operation corresponds to Hajós' sum, where the whole operation corresponds to an Ore operation.

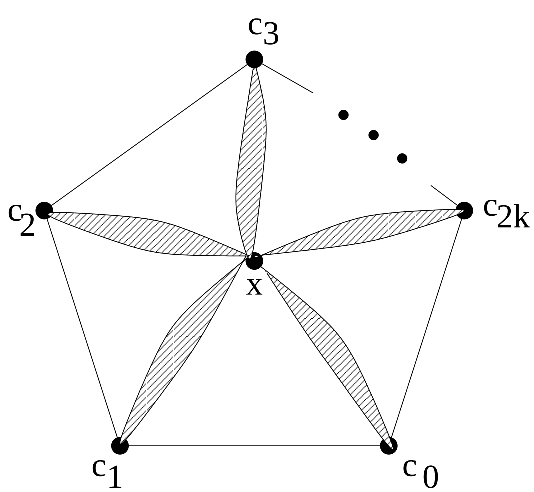

Figure 1: The graph S_1 (each shaded area represents a $G_i - e_i$)

Lemma 1 *If $k \geq 1$ and $\chi_c(G_i) \geq r \geq 2$ for $i = 0, 1, \cdots, 2k$, then $\chi_c(S_1^*) \geq \chi_c(S_1) \geq r$.*

Proof. Since identifying nonadjacent vertices does not decrease the circular chromatic number, it suffices to prove that $\chi_c(S_1) \geq r$. We refer to Fig. 1 for the names of the vertices. Assume to the contrary that for some $r' < r$, there is an r'-circular colouring of S_1. Since $\chi_c(G_i) \geq r$ for each i, we conclude that $|f(x) - f(c_i)|_{r'} < 1$. Without loss of generality, we assume that $f(x) = 1$. Then for each i, $0 < f(c_i) < 2$. Without loss of generality, we may assume that $0 < f(c_0) \leq 1$. Then since $|f(c_i) - f(c_{i+1})|_r \geq 1$ for $i = 0, 1, \cdots, 2k-2$, by induction, we conclude that $0 < f(c_{2j}) < 1$ and $1 < f(c_{2j+1}) < 2$. Therefore $0 < f(c_0), f(c_{2k}) < 1$. But this is a contradiction, as c_0c_{2k} is an edge (which implies that $|f(c_0) - f(c_{2k})|_{r'} \geq 1$). ∎

To be consistent with the labels in [20], our next operation is called Operation 4.

Operation 4

Suppose $k \geq 3$, $G_0, G_1, \cdots, G_{4k-1}$ are graphs and for $i = 0, 1, \cdots, 4k-1$, $e_i = x_iy_i$ is an edge of G_i. Let $C_{2k} = (c_0, c_1, \cdots, c_{2k-1})$ be a cycle of length $2k$. Construct a new graph from the disjoint union of $G_0, G_1, \cdots, G_{4k-1}$ and C_{2k} as follows:

- delete the edge e_i from G_i, for $i = 0, 1, \cdots, 4k-1$,
- indentify $x_0, x_1, \cdots, x_{2k-1}$ into a single vertex x, and for $i = 0, 1, \cdots, 2k-1$, identify y_i with c_i,
- for $i = 2k, \cdots, 4k-1$, identify x_i with c_{i-2k} and y_i with c_{i-2k+3}, where the addition is modulo $2k$.

We shall denote the resulting graph by $S_4(G_0, e_0, \cdots, G_{4k-1}, e_{4k-1})$ or simply S_4 if the G_i's and e_i's are clear from the context.

Let $f : S_4 \to Z$ be a mapping such that for each $j \in Z$, $f^{-1}(j)$ is an independent set of S_4, and moreover for each G_i, $|f^{-1}(j) \cap V(G_i)| \leq 1$. Then for each $j \in Z$, we identify all vertices of $f^{-1}(j)$ into a single vertex. We denote the resulting graph by S_4^*, and call S_4^* a graph obtained from $G_0, G_1, G_2, \cdots, G_{4k-1}$ by Operation 4.

Lemma 2 *If $\chi_c(G_i) \geq r$ for $i = 0, 1, \cdots, 4k-1$, then $\chi_c(S_4^*) \geq \chi_c(S_4) \geq r$.*

Proof. Assume to the contrary that there is an $r' < r$ and an r'-colouring f of S_4. Similarly as in the proof of Lemma 1, we shall frequently use the fact that $|f(x_i) - f(y_i)|_{r'} < 1$ for $i = 0, 1, \cdots, 4k-1$ (note that x_i, y_i will have different names in the graph S_4).

Without loss of generality, we may assume that $f(x) = 1$. Since $\chi_c(G_i) \geq r$ for each i, $|f(c_i) - f(x)|_{r'} < 1$, for $i = 0, 1, \cdots, 2k-1$. So $0 < f(c_i) < 2$. Without loss of generality, we assume that $0 < f(c_0) \leq 1$. Since c_ic_{i+1} is an edge of S_4, we know that $|f(c_i) - f(c_{i+1})|_{r'} \geq 1$. Then by induction we

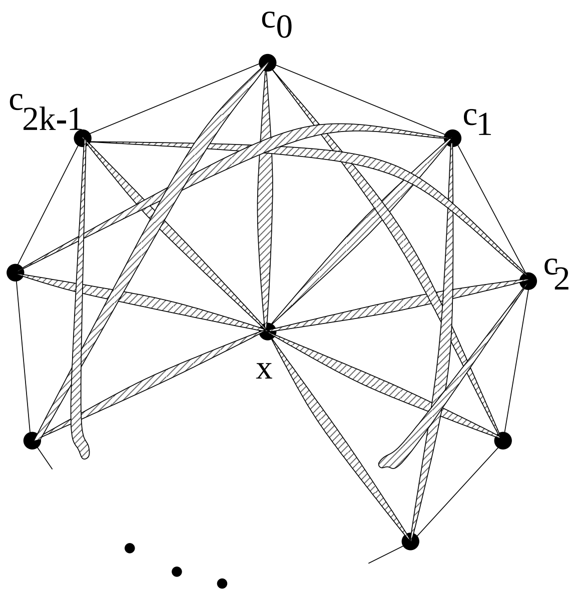

Figure 2: The graph S_4 (each shaded area represents a $G_i - e_i$)

can prove that $0 < f(c_{2j}) < 1$ and $1 < f(c_{2j+1}) < 2$ for $j = 0, 1, \cdots, k-1$. Without loss of generality, we may assume that $f(c_2) \leq f(c_0)$.

As c_2c_3 is an edge of S_4, we have $|f(c_3) - f(c_2)|_{r'} \geq 1$. As $c_0 = x_{2k}$ and $c_3 = y_{2k}$, we have $|f(c_3) - f(c_0)|_{r'} < 1$. So either $f(c_3) - f(c_0) < 1$ or $f(c_3) - f(c_0) > r' - 1$ (note that $f(c_3) > f(c_0)$). But $f(c_3) - f(c_0) > r' - 1$ would imply $f(c_3) - f(c_2) > r' - 1$, contrary to the fact that $|f(c_3) - f(c_2)|_{r'} \geq 1$. Therefore we conclude that $f(c_3) - f(c_0) < 1$. As $f(c_1) - f(c_0) > 1$, we know that $f(c_1) > f(c_3)$.

Similarly as above, we have $|f(c_3) - f(c_4)|_{r'} \geq 1$ and $|f(c_1) - f(c_4)|_{r'} < 1$. So either $f(c_1) - f(c_4) < 1$ or $f(c_1) - f(c_4) > r' - 1$. But $f(c_1) - f(c_4) < 1$ would imply that $f(c_3) - f(c_4) < 1$, contrary to the fact that $|f(c_3) - f(c_4)|_{r'} \geq 1$. Hence we have $f(c_1) - f(c_4) > r' - 1$. As $f(c_1) - f(c_2) \leq r' - 1$. Therefore $f(c_4) < f(c_2)$.

Repeat this argument (or by induction), we can prove that $f(c_{2j+2}) < f(c_{2j})$ and $f(c_{2j+1}) < f(c_{2j-1})$ for $j = 0, 1, \cdots, k-1$. Therefore $f(c_{2k-1}) < f(c_3)$. However, $f(c_3) - f(c_0) < 1$. So $0 < f(c_{2k-1}) - f(c_0) < 1$, contrary to the fact that c_0c_{2k-1} is an edge of S_4. ∎

Operation 5

Suppose $k \geq 1$, $G_0, G_1, \cdots, G_{4k+2}$ are graphs, and for $i = 0, 1, \cdots, 4k+2$, $e_i = x_iy_i$ is an edge of G_i. Let $P_{2k+1} = (p_0, p_1, \cdots, p_{2k+1})$ be a path of length $2k+1$. Construct a new graph from the disjoint union of $G_0, G_1, \cdots, G_{4k+2}$

and P_{2k+1} as follows:

- delete edge e_i from G_i, for $i = 0, 1, \cdots, 4k+2$,

- identify $x_0, x_1, \cdots, x_{2k+1}$ into a single vertex x, and for $i = 0, 1, \cdots, 2k+1$, identify y_i with p_i,

- add two vertices u, v, connect each of v, u to x by an edge, connect u to p_2 by an edge, and connect v to p_{2k+1} by an edge,

- for $i = 0, 1, \cdots, 2k-2$, identify p_i with x_{i+2k+2}, and identify p_{i+3} with y_{i+2k+2},

- identify x_{4k+1} with u, y_{4k+1} with p_0, and identify x_{4k+2} with v, y_{4k+2} with p_{2k-1}.

We shall denote the resulting graph by $S_5(G_0, e_0, \cdots, G_{4k+2}, e_{4k+2})$ or simply S_5 if the G_i's and e_i's are clear from the context.

Let $f : S_5 \to Z$ be a mapping such that for each $j \in Z$, $f^{-1}(j)$ is an independent set of S_5, and moreover for each G_i, $|f^{-1}(j) \cap V(G_i)| \leq 1$. Then for each $j \in Z$, we identify all vertices of $f^{-1}(j)$ into a single vertex. We denote the resulting graph by S_5^*, and call S_5^* a graph obtained from $G_0, G_1, G_2, \cdots, G_{4k+2}$ by Operation 5.

Lemma 3 *If $\chi_c(G_i) \geq r$ for $i = 0, 1, \cdots, 4k+2$, then $\chi_c(S_5^*) \geq \chi_c(S_5) \geq r$.*

Proof. Assume to the contrary that there is an $r' < r$ and an r'-colouring f of S_5.

Without loss of generality, we may assume that $f(x) = 1$. Since $\chi_c(G_i) \geq r$ for each i, $|f(p_i) - f(x)|_{r'} < 1$, for $i = 0, 1, \cdots, 2k+1$. So $0 < f(p_i) < 2$ for $i = 0, 1, \cdots, 2k+1$. As $p_i p_{i+1}$ is an edge, we have $|f(p_i) - f(p_{i+1})| \geq 1$. Without loss of generality, we assume that $0 < f(p_0) \leq 1$. Then similarly as the proof of Lemma 2, we can prove that $0 < f(p_{2j}) < 1$ and $1 < f(c_{pj+1}) < 2$ for $j = 0, 1, \cdots, k$.

First we consider the case that $f(p_2) \leq f(p_0)$. Since ux is an edge of S_5, either $f(u) = 0$ or $f(u) \geq 2$. As $0 < f(p_2) < 1$ and up_2 is an edge of S_5, we conclude that $f(u) \neq 0$. But $|f(u) - f(p_0)|_{r'} < 1$. Therefore

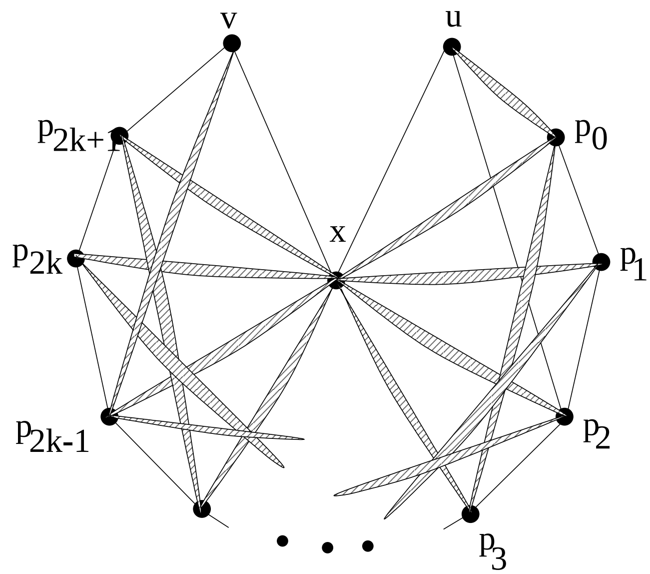

Figure 3: The graph S_5 (each shaded area represents a $G_i - e_i$)

$f(u) - f(p_0) > r' - 1$. But this implies that $f(u) - f(p_2) > r' - 1$, contrary to the fact that up_2 is an edge of S_5.

Assume now that $f(p_2) > f(p_0)$. Similarly as in the proof of Lemma 2, one can show that $f(p_{2j+2}) > f(p_{2j})$ and $f(p_{2j+1}) > f(p_{2j-1})$ for $j = 0, 1, \cdots, k$. In particular, $f(p_{2k+1}) > f(p_{2k-1})$. As vx is an edge of S_5, either $f(v) = 0$ or $f(v) \geq 2$. On the other hand, we must have $|f(v) - f(p_{2k+1})|_{r'} \geq 1$ and $|f(v) - f(p_{2k-1})|_{r'} < 1$.

If $f(v) = 0$, then $f(p_{2k-1}) > r' - 1$ which implies that $f(p_{2k+1}) > r' - 1$, contrary to the fact that $|f(v) - f(p_{2k+1})|_{r'} \geq 1$.

If $f(v) \geq 2$ then $|f(v) - f(p_{2k+1})|_{r'} \geq 1$ implies that $f(v) - f(p_{2k+1}) > 1$ (as $f(p_{2k+1}) < 2$). Hence $f(v) - f(p_{2k-1}) \geq 1$. Therefore $|f(v) - f(p_{2k-1})|_{r'} \geq 1$ (as $f(p_{2k-1}) > 1$, so $f(v) - f(p_{2k-1}) < r' - 1$). This is a contradiction. ∎

Operation 6

Suppose $k \geq 2$, $G_0, G_1, \cdots, G_{4k}$ are graphs, and for $i = 0, 1, \cdots, 4k$, $e_i = x_i y_i$ is an edge of G_i. Let $P_{2k} = (p_0, p_1, \cdots, p_{2k})$ be a path of length $2k$. Construct a new graph from the disjoint union of $G_0, G_1, \cdots, G_{4k}$ and P_{2k} as follows:

- delete the edges e_i, for $i = 0, 1, \cdots, 4k$,
- identify $x_0, x_1, \cdots, x_{2k}$ into a single vertex x, and for $i = 0, 1, \cdots, 2k$,

identify y_i with p_i,

- add two vertices u, v, connect each of v, u to x by an edge, connect u to p_0 by an edge, and connect v to p_{2k} by an edge,

- for $i = 0, 1, \cdots, 2k-3$, identify c_i with x_{i+2k+1}, and identify c_{i+3} with y_{i+2k+1},

- identify x_{4k-1} with u, y_{4k-1} with p_2, and identify x_{4k} with v, y_{4k} with p_2.

We shall denote the resulting graph by $S_6(G_0, e_0, \cdots, G_{4k}, e_{4k})$ or simply S_6 if the G_i's and e_i's are clear from the context.

Let $f : S_6 \to Z$ be a mapping such that for each $j \in Z$, $f^{-1}(j)$ is an independent set of S_6, and moreover for each G_i, $|f^{-1}(j) \cap V(G_i)| \leq 1$. Then for each $j \in Z$, we identify all vertices of $f^{-1}(j)$ into a single vertex. We denote the resulting graph by S_6^*, and call S_6^* a graph obtained from $G_0, G_1, G_2, \cdots, G_{4k}$ by Operation 6.

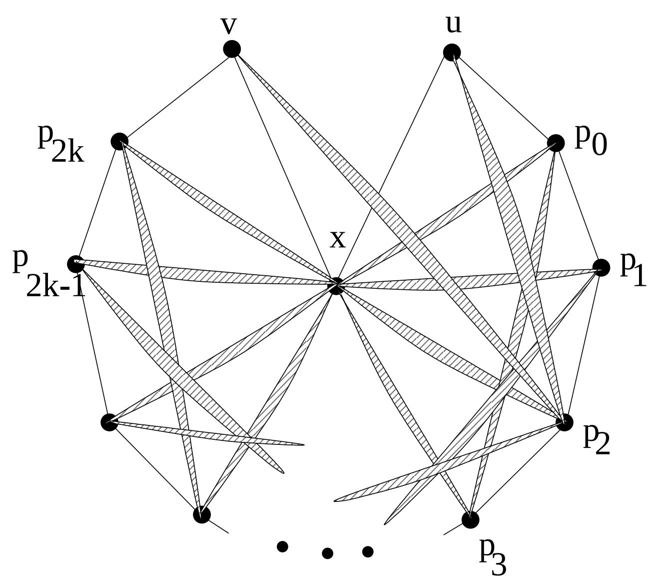

Figure 4: The graph S_6 (each shaded area represents a $G_i - e_i$)

Lemma 4 *If $\chi_c(G_i) \geq r$ for $i = 0, 1, \cdots, 4k$, then $\chi_c(S_6^*) \geq \chi_c(S_6) \geq r$.*

Proof. Assume to the contrary that there is an $r' < r$ and an r'-colouring f of S_6.

Without loss of generality, we may assume that $f(x) = 1$. Since $\chi_c(G_i) \geq r$ for each i, $|f(p_i) - f(x)|_{r'} < 1$, for $i = 0, 1, \cdots, 2k$. So $0 < f(p_i) < 2$ for $i = 0, 1, \cdots, 2k$. Similarly to the proof of Lemma 2, we may assume that $f(p_0) < 1$, which then implies that $0 < f(c_{2j}) < 1$ and $1 < f(c_{2j+1}) < 2$ for $j = 0, 1, \cdots, k$. Use the same argument as in the proof of Lemma 2, we can show that either for all j,

$$1 > f(p_{2j}) > f(p_{2j-2}) > 0, \quad \text{and} \quad 2 > f(p_{2j+1}) > f(p_{2j-1}) > 1$$

or for all j,

$$0 < f(p_{2j}) < f(p_{2j-2}) < 1, \quad \text{and} \quad 1 < f(p_{2j+1}) < f(p_{2j-1}) < 2.$$

First we consider the case that $f(p_2) > f(p_0)$. Since ux is an edge of S_6, either $f(u) = 0$ or $f(u) \geq 2$. As $0 < f(p_0) < 1$ and up_0 is an edge of S_6, we conclude that $f(u) \neq 0$. But $|f(u) - f(p_2)|_{r'} < 1$. Therefore $f(u) - f(p_2) > r' - 1$. This implies that $f(u) - f(p_0) > r' - 1$, contrary to the fact that up_0 is an edge of S_6.

Assume now that $f(p_2) < f(p_0)$. Then $0 < f(p_{2k}) < f(p_2) < 1$. As vx is an edge of S_6, we know that $f(v) \geq 2$ (similarly as above $f(v) \neq 0$). On the other hand, we must have $|f(v) - f(p_{2k})|_{r'} \geq 1$, which implies that $f(v) - f(p_{2k}) \leq r' - 1$. Therefore

$$1 \leq f(v) - f(p_2) < f(v) - f(p_{2k}) \leq r' - 1,$$

i.e., $|f(v) - f(p_2)|_{r'} \geq 1$, contrary to the assumption that $\chi_c(G_{4k}) \geq r$. ∎

Operation 7

Suppose $k \geq 2$, $G_0, G_1, \cdots, G_{4k}$ are graphs, and for $i = 0, 1, \cdots, 4k$, $e_i = x_i y_i$ is an edge of G_i. Let $P_{2k} = (p_0, p_1, \cdots, p_{2k})$ be a path of length $2k$. Construct a new graph from the disjoint union of $G_0, G_1, \cdots, G_{4k}$ and P_{2k} as follows:

- delete the edges e_i, for $i = 0, 1, \cdots, 4k$,
- for $i = 0, 1, \cdots, 2k$, identify x_i into a single vertex x, and identify y_i with p_i,
- add two vertices u, v, connect each of v, u to x by an edge, connect each of v, u to p_2 by an edge,

- for $i = 0, 1, \cdots, 2k-3$, identify c_i with x_{i+2k+1}, and identify c_{i+3} with y_{i+2k+1},

- identify x_{4k-1} with u, y_{4k-1} with p_0, and identify x_{4k} with v, y_{4k} with p_{2k}.

We shall denote the resulting graph by $S_7(G_0, e_0, \cdots, G_{4k}, e_{4k})$ or simply S_7 if the G_i's and e_i's are clear from the context.

Let $f : S_7 \to Z$ be a mapping such that for each $j \in Z$, $f^{-1}(j)$ is an independent set of S_7, and moreover for each G_i, $|f^{-1}(j) \cap V(G_i)| \leq 1$. Then for each $j \in Z$, we identify all vertices of $f^{-1}(j)$ into a single vertex. We denote the resulting graph by S_7^*, and call S_7^* a graph obtained from $G_0, G_1, G_2, \cdots, G_{4k}$ by Operation 7.

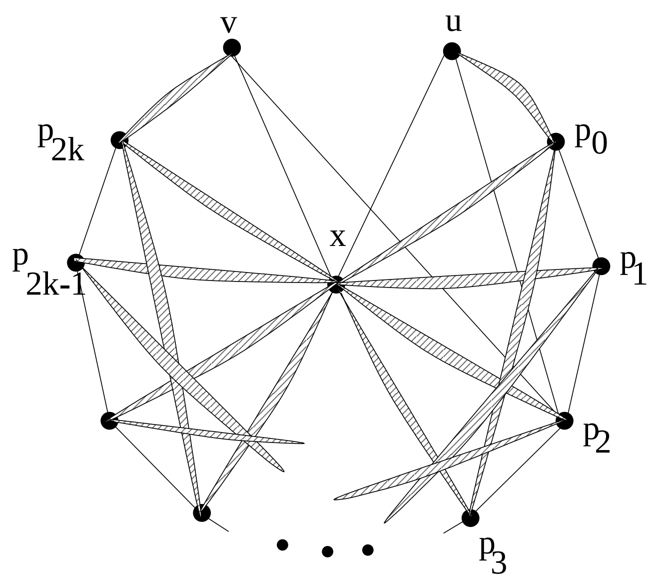

Figure 5: The graph S_7 (each shaded area represents a $G_i - e_i$)

Lemma 5 *If $\chi_c(G_i) \geq r$ for $i = 0, 1, \cdots, 4k$, then $\chi_c(S_7^*) \geq \chi_c(S_7) \geq r$.*

Proof. Operation 7 is very similar to Operation 6. The proof of Lemma 5 is also similar to the proof of Lemma 4. We shall omit some details.

Assume f is an r'-colouring (for some $r' < r$) of S_7 with $f(x) = 1$ and $f(p_0) < 1$. Similarly, we have either for all j,

$$1 > f(p_{2j}) > f(p_{2j-2}) > 0, \text{ and } 2 > f(p_{2j+1}) > f(p_{2j-1}) > 1$$

or for all j,

$$0 < f(p_{2j}) < f(p_{2j-2}) < 1, \quad \text{and} \quad 1 < f(p_{2j+1}) < f(p_{2j-1}) < 2.$$

Consider the case that $f(p_2) < f(p_0)$. Since u is adjacent to both p_2 and x, we conclude that $f(u) \geq 2$ and $f(u) - f(p_2) \leq r' - 1$. This implies that $1 \leq f(u) - f(p_0) \leq r' - 1$, contrary to the assumption $\chi_c(G_{4k-1}) \geq r > r'$. The other case is similar, and omitted. ∎

3 Circular perfect graphs

To prove that the graph operations introduced in this paper suffice to construct all graphs with circular chromatic number at least k/d (for $2 \leq k/d < 3$), we need some results concerning circular perfect graphs.

Definition 1 *Suppose G is a graph. The* circular clique number $\omega_c(G)$ *of G to defined as*

$$\omega_c(G) = \sup\{k/d : K_{k/d} \text{ admits a homomorphism to } G\}.$$

An equivalent definition of the circular clique number is that $\omega_c(G)$ is equal to the maximum of those k/d for which $K_{k/d}$ is an induced subgraph of G [18].

Definition 2 *A graph G is called* circular perfect *if for every induced subgraph H of G we have $\chi_c(H) = \omega_c(H)$.*

In [19], the author proved a sufficient condition for a graph to be circular perfect. To state this sufficient condition, we need the definition of "well-linked" paths.

Suppose $u, v \in V - N[x]$, we shall use the following notation:

- $u \leq^x v$ means $N(u) \cap N(x) \subseteq N(v) \cap N(x)$;

- $u =^x v$ means $N(u) \cap N(x) = N(v) \cap N(x)$.

Definition 3 *Given an induced path $P_n = (p_0, p_1, \cdots, p_n)$ of $G - N[x]$, we say P_n is* well-linked *with respect to x if one of the following holds:*

1. *For all* $i < j$ *of even parity (respectively, odd parity)* $p_i \leq^x p_j$ *and for all* $i' < j'$ *of odd parity (respectively, even parity),* $p_{j'} \leq^x p_{i'}$*;*

2. *For all odd (respectively, even) indices* i, i', $p_i =^x p_{i'}$. *There is an even (respectively, odd) index* j_0 *such that for all even (respectively, odd)* $i \leq j_0$, $p_i =^x p_{j_0}$, *for all even (respectively, odd)* $i > j_0$ *and* i, j_0, $p_i =^x p_{j_0+2}$.

The following lemma gives another definition of well-linked paths, by characterizing those paths which are not well-linked.

Lemma 6 *[19] An induced path* $P = (p_0, p_1, \cdots, p_n)$ *of* H_x *is not well-linked with respect to* x *if and only if one of the following hold:*

1. *there are three indices* $i < j < k$ *of the same parity such that* $p_i \not\leq^x p_j$ *and* $p_k \not\leq^x p_j$*;*

2. *there are three indices* $i < j < k$ *of the same parity such that* $p_j \not\leq^x p_i$ *and* $p_j \not\leq^x p_k$*;*

3. *there are two indices* $i < j$ *of the same parity, and two indices* $i' < j'$ *of the other parity such that* $p_i \not\leq^x p_j$ *and* $p_{i'} \not\leq^x p_{j'}$.

Theorem 1 *[19] Suppose* G *is a triangle free graph such that for every vertex* x *of* G, $G - N[x]$ *is a bipartite graph with no induced* C_n *for* $n \geq 6$, *and any induced path of* $G - N[x]$ *is well-linked with respect to* x. *Then* G *is circular perfect.*

The proof of Theorem 1 is quite complicated. In Section 5, we shall use Theorem 1 to prove the completeness of the graph operations introduced in this paper.

4 Homomorphism of edge coloured graphs

Definition 4 *A 2-edge-coloured graph is a triple* $H = (X, E_1, E_2)$, *where* X *is a set of vertices, and* E_1 *and* E_2 *are two sets of edges (i.e., unordered pairs of* V*) such that* $E_1 \cap E_2 = \emptyset$.

A 2-edge-coloured graph can be obtained from a graph by colouring its edges by 2 colours. We shall say those edges in E_1 are *red edges* and those edges in E_2 are *blue edges*.

Definition 5 *Suppose* $H = (X, E_1, E_2)$ *and* $G = (V, E_1', E_2')$ *are 2-edge-coloured graphs. A mapping* $f : X \to V$ *is called a* homomorphism *from* H *to* G *if for every* $xy \in E_1$, $f(x)f(y) \in E_1'$, *and for every* $xy \in E_2$, $f(x)f(y) \in E_2'$.

Thus a homomorphism from a 2-edge-coloured graph to another 2-edge-coloured graph is a vertex to vertex mapping, which preserves the edge relation of each colour, i.e., each red edge of H is mapped to a red edge of G and each blue edge of H is mapped to a blue edge of G.

Now for $i = 1, 4, 5, 6, 7$ (readers are advised to refer to the depiction of Operation i), we replace each $G_j - e_j$ by a blue edge, and colour the other edges by red. Thus we obtain a 2-edge-coloured graph. We say such a 2-edge-coloured graph is *associated to* Operation i. Of course, for each i, there are 2-edge-coloured graphs of different sizes associated to Operation i. With an abuse of notation, we shall denote by H_i any 2-edge-coloured graph associated to Operation i.

To be precise, we can define the 2-edge-coloured graphs H_i as follows:

- The vertices of H_1 are $x, c_0, c_1, \cdots, c_{2k}$ for some integer $k \geq 1$. For $j = 0, 1, \cdots, 2k$, $c_j c_{j+1} \in E_1$ and $xc_j \in E_2$.

- The vertices of H_4 are $x, c_0, c_1, \cdots, c_{2k-1}$ for some $k \geq 3$. For $j = 0, 1, \cdots, 2k-1$, $c_j c_{j+1} \in E_1$ and $xc_j \in E_2$ and $x_j x_{j+3} \in E_2$.

- The vertices of H_5 are $x, u, v, p_0, p_1, \cdots, p_{2k+1}$ for some $k \geq 1$. For $j = 0, 1, \cdots, 2k$, $p_j p_{j+1} \in E_1$. For $j = 0, 1, \cdots, 2k+1$, $xp_j \in E_2$. For $j = 0, 1, \cdots, 2k-2$, $p_j p_{j+3} \in E_2$. Moreover, $xu, xv, p_{2k+1}v, p_2u \in E_1$ and $vp_{2k-1}, up_0 \in E_2$.

- The vertices of H_6 are $x, u, v, p_0, p_1, \cdots, p_{2k}$ for some $k \geq 2$. For $j = 0, 1, \cdots, 2k-1$, $p_j p_{j+1} \in E_1$. For $j = 0, 1, \cdots, 2k$, $xp_j \in E_2$. For $j = 0, 1, \cdots, 2k-3$, $p_j p_{j+3} \in E_2$. Moreover, $xu, xv, p_{2k}v, p_0u \in E_1$ and $vp_2, up_2 \in E_2$.

- The vertices of H_7 are $x, u, v, p_0, p_1, \cdots, p_{2k}$ for some $k \geq 2$. For $j = 0, 1, \cdots, 2k-1$, $p_j p_{j+1} \in E_1$. For $j = 0, 1, \cdots, 2k$, $xp_j \in E_2$. For $j = 0, 1, \cdots, 2k-3$, $p_j p_{j+3} \in E_2$. Moreover, $xu, xv, p_2v, p_2u \in E_1$ and $vp_{2k}, up_0 \in E_2$.

Suppose $G = (V, E)$ is a graph. Let G' be the 2-edge-coloured graph obtained from G by colouring the edges of G red, and for each nonadjacent pair of vertices of G, join them in G' with a blue edge, i.e., $G' = (V, E'_1, E'_2)$ where $E'_1 = E$ and $E'_2 = \binom{V}{2} - E$. We call G' the 2-edge-coloured graph associated to G.

If H is a 2-edge-coloured graph, G' is a 2-edge-coloured graph associated to G, then a homomophism from H to G' can also be viewed as a mapping from $V(H)$ to $V(G)$ such that every red edge of G is sent to an edge of G, and every blue edge is sent to a pair of distinct nonadjacent vertices.

Lemma 7 *Suppose $G = (V, E)$ is a graph, and $i \in \{1, 4, 5, 6, 7\}$. Let G' be the 2-edge-coloured graph associated to G. Assume that there is a homomorphism f from a 2-edge coloured graph H_i to G'. Assume that the blue edge set of H_i is $E_2(H_i) = \{e_1, e_2, \cdots, e_m\}$. For each blue edge $e_j = xy$ of H_i, let $G_j = G + f(x)f(y)$. Then G can be obtained from $G_1, G_2, \cdots, G_m$ by Operation i.*

Proof. We consider the case that $i = 1$. Assume the vertices of G are $v_1, v_2, \cdots, v_t$. Let h be a homomorphism from H_1 to G'. The blue edges of H_1 are xc_j for $j = 0, 1, \cdots, 2k$. Let $G_j = G + h(x)h(c_j)$. Let $e_j = h(x)h(c_j)$. Then $G_j - e_j$ is simply a copy of G. We denote the vertices of $G_j - e_j$ by $v_1^j, v_2^j, \cdots, v_t^j$, where v_l^j is the copy of the vertex v_l in $G_j - e_j$. Let $f : S_1 \to Z$ be the mapping defined as $f(v_l^j) = l$. For those vertices x of S_1 not in any copy of G (S_1 has only one such vertex, namely the vertex x in the center of Fig. 2), let $f(x) = l$ if $h(x) = v_l$. Then for each $l \in Z$, $f^{-1}(l)$ is an independent set, and $|f^{-1}(l) \cap V(G_j)| \leq 1$ for each j. Since h is a homomorphism from H_1 to a graph G, it is easy to verify that by identifying $f^{-1}(l)$ into a single vertex, we obtain the graph G. For $i = 4, 5, 6, 7$, the proof is the same. ∎

5 Completeness of Operations 1, 4, 5, 6, 7

Lemma 8 *Suppose $i \in \{1, 4, 5, 6, 7\}$, and that $\mathcal{H}$ is a class of graphs which is closed under operation i. If G is a graph such that for any two nonadjacent* *vertices x and y of G, the graph $G+xy \in \mathcal{H}$ and H_i admits a homomorphism to the 2-edge-coloured graph G' associated to G, then $G \in \mathcal{H}$.*

Proof. Assume f is a homomorphism from H_i to G', where G' is the 2-edge-coloured graph associated to G. Assume that the blue edge set of H_i is $E_2(H_i) = \{e_1, e_2, \cdots, e_m\}$. For each blue edge $e_j = xy$ of H_i, let $G_j = G + f(x)f(y)$. Since f is a homomorphism, so $f(x)$ and $f(y)$ are nonadjacent vertices of G. So by our hypothesis, $G_j \in \mathcal{H}$. By Lemma 7, G can be obtained from $G_1, G_2, \cdots, G_m$ by Operation i. As $\mathcal{H}$ is closed under Operation i, we conclude that $G \in \mathcal{H}$. ∎

Theorem 2 *Suppose $k/d < 3$. Let $\mathcal{G}(k/d)$ be the set of graphs that contains $K_{k/d}$, all graphs obtained from $K_{k/d}$ by identifying nonadjacent vertices, and is closed under the following operations:*

- *Adding vertices and edges;*
- *Applying Operations 1, 4, 5, 6, 7.*

Then $\mathcal{G}(k/d)$ consists of all graphs G with $\chi_c(G) \geq k/d$.

Proof. By Lemmas 2, 3, 4, 5, we know that all the graphs in $\mathcal{G}(k/d)$ has circular chromatic number at least k/d. So it suffices to prove that any graph G with $\chi_c(G) \geq k/d$ is contained in $\mathcal{G}(k/d)$.

The case that $k/d = 2$ is trivial. Thus in the following we assume that $2 < k/d < 3$. First we observe that the graph K_3 can be obtained from $K_{k/d}$ by identify some nonadjacent vertices. Therefore $K_3 \in \mathcal{G}(k/d)$ and hence any graph containing a triangle belongs to $\mathcal{G}(k/d)$.

Thus it suffices to prove that any triangle free graph G with $\chi_c(G) \geq k/d$ belongs to $\mathcal{G}(k/d)$.

Assume to the contrary that there is a triangle free graph G with $\chi_c(G) \geq k/d$ and $G \notin \mathcal{G}(k/d)$. By choosing G as a counterexample with maximum number of edges (on the same vertex set), we may assume that for any two nonadjacent vertices x and y of G, the graph $G + xy \in \mathcal{G}(k/d)$. We shall derive a contradiction by proving that $G \in \mathcal{G}(k/d)$.

Assume that $\chi_c(G) = k'/d' \geq k/d$. First we assume that $\omega_c(G) = k'/d'$. Then G contains a copy of $K_{k'/d'}$. It is well-known (cf. [2, 9, 17]) that $K_{k/d}$ admits a homomorphism to $K_{k'/d'}$. Hence $K_{k/d}$ admits a homomorphism to G. Let f be a homomorphism from $K_{k/d}$ to G. By identifying those vertices of $K_{k/d}$ which are mapped to the same vertex of G, and by adding some vertices and edges (if necessary), we obtain a copy of G. Thus $G \in \mathcal{G}(k/d)$.

Thus we may assume that G is not circular perfect. By Theorem 1, there is a vertex x of G such that $G - N[x]$ contains either an odd cycle, or an induced cycle C_n for some $n \geq 6$, or an induced path P which is not well-linked with respect to x.

If $G - N[x]$ contains an odd cycle, then H_1 admits a homomorphism to G', where G' is the 2-edge-coloured graph associated to G. By Lemma 8, $G \in \mathcal{G}(k/d)$.

Assume that $G - N[x]$ contains no odd cycles. If $G - N[x]$ contains an induced cycle $C_n = (c_0, c_1, \cdots, c_{n-1})$ for some $n \geq 6$, then H_4 admits a homomorphism to G', where G' is the 2-edge-coloured graph associated to G. By Lemma 8, $G \in \mathcal{G}(k/d)$.

Assume now that $G - N[x]$ has an induced path $P = (p_0, p_1, \cdots, p_n)$ which is not well-linked with respect to x. By Lemma 6, one of the following hold:

1. there are three indices $i < j < k$ of the same parity such that $p_i \not\leq^x p_j$ and $p_k \not\leq^x p_j$;

2. there are three indices $i < j < k$ of the same parity such that $p_j \not\leq^x p_i$ and $p_j \not\leq^x p_k$;

3. there are two indices $i < j$ of the same parity, and two indices $i' < j'$ of the other parity such that $p_i \not\leq^x p_j$ and $p_{i'} \not\leq^x p_{j'}$.

In the first case, without loss of generality, we may assume that P is a minimal path which is not well-linked with respect to x. Then we must have $i = 0$ and $k = n$. Thus $n \geq 4$ is even. If $p_0 \leq^x p_2$ or $p_n \leq^x p_2$, then $P[p_2, p_n]$ is also not well-linked with respect to x, contrary to the minimality of P. Thus we have $p_0 \not\leq^x p_2$ and $p_n \not\leq^x p_2$. Therefore H_7 admits a homomorphism to G', where G' is the 2-edge-coloured graph associated to G. By Lemma 8, $G \in \mathcal{G}(k/d)$.

In the second case, similarly as above, we may assume that n is even $p_2 \not\leq^x p_0$ and $p_2 \not\leq^x p_n$. Then H_6 admits a homomorphism to G', where G' is the 2-edge-coloured graph associated to G. By Lemma 8, $G \in \mathcal{G}(k/d)$.

In the third case, without loss of generality, we may assume that $j = i+2$ and $j' = i' + 2$. Indeed, in case $j \geq i + 4$, then if $p_i \not\leq^x p_{i+2}$ we can simply replace j by $i + 2$, if $p_i \leq^x p_{i+2}$ we can replace i by $i + 2$.

Now by assuming that P is a minimal path which is not well-linked with respect to x, it is easy to see that n is odd, $p_0 \not\leq^x p_2$ and $p_n \not\leq^x p_{n-2}$. Then H_5 admits a homomorphism to G', where G' is the 2-edge-coloured graph associated to G. By Lemma 8, $G \in \mathcal{G}(k/d)$. This completes the proof of Theorem 2. ∎

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