

# Construction of graphs with given circular flow numbers

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## Abstract

Suppose  $r \geq 2$  is a real number. A proper  $r$ -flow of a directed multi-graph  $\vec{G} = (V, E)$  is a mapping  $f : E \rightarrow R$  such that (i) for every edge  $e \in E$ ,  $1 \leq |f(e)| \leq r - 1$ ; (ii) for every vertex  $v \in V$ ,  $\sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e) = 0$ . The circular flow number of a graph  $G$  is the least  $r$  for which an orientation of  $G$  admits a proper  $r$ -flow. The well-known 5-flow conjecture is equivalent to the statement that every bridgeless graph has circular flow number at most 5. In this paper, we prove that for any rational number  $r$  between 2 and 5, there exists a graph  $G$  with circular flow number  $r$ .

## 1 Introduction

Graphs considered in this paper may contain parallel edges but no loops. Suppose  $G = (V, E)$  is a graph. By assigning a direction to each edge of  $G$ ,

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we obtain a directed graph  $\vec{G} = (V, E)$ , which is called an *orientation* of  $G$ .

Suppose  $\vec{G} = (V, E)$  is a directed graph. A *weight function* on  $\vec{G}$  is a mapping  $f : E \rightarrow \mathbb{R}$  which assigns to each edge a real number. For a vertex  $v$  of a directed graph  $\vec{G} = (V, E)$ , let  $E^+(v)$  (respectively  $E^-(v)$ ) be the set of edges with their tails (respectively heads) at  $v$ , and let

$$f^+(v) = \sum_{e \in E^+(v)} f(e), \quad f^-(v) = \sum_{e \in E^-(v)} f(e).$$

A *flow* on  $\vec{G}$  is a weight function  $f$  such that each vertex  $v$  satisfies the *conservation law*:

$$f^+(v) - f^-(v) = 0.$$

A *proper  $r$ -flow* is a flow  $f$  such that for each edge  $e$ ,  $1 \leq |f(e)| \leq r - 1$ .

It is easy to see that if an orientation of a graph  $G$  admits a proper  $r$ -flow, then every orientation of  $G$  admits a proper  $r$ -flow. So the existence of a proper  $r$ -flow reflects the structure of the graph rather than the orientation. We say a graph  $G$  admits a proper  $r$ -flow if an orientation (and hence every orientation) of  $G$  admits a proper  $r$ -flow. It is easy to see that if  $G$  has a bridge, then  $G$  does not admit a proper  $r$ -flow for any real number  $r$ . So discussions about proper  $r$ -flow are restricted to bridgeless graphs.

**Definition 1.1** *Suppose  $G$  is a bridgeless graph. The circular flow number  $\Phi_c(G)$  of  $G$  is defined to be the infimum of those  $r$  for which  $G$  admits a proper  $r$ -flow, i.e.,*

$$\Phi_c(G) = \inf\{r : G \text{ admits a proper } r\text{-flow}\}.$$

The concept of circular flow number was introduced in [3] (where it is called the fractional flow number) as a dual concept of circular chromatic number (see definition below). Some problems concerning this parameter are discussed in [10, 15, 17]. It is known [3] that the infimum in the definition of circular flow number can be replaced by the minimum, and that the circular flow number of a finite bridgeless graph  $G$  is always a rational number.

If  $r = k$  is an integer and  $f$  is a proper  $k$ -flow with  $f(e)$  being an integer for each edge  $e$ , then we call  $f$  a *nowhere-zero  $k$ -flow*. The *flow number*  $\Phi(G)$  of  $G$  is the least integer  $k$  such that  $G$  has a nowhere-zero  $k$ -flow. Although a proper  $k$ -flow  $f$  may not be a nowhere zero  $k$ -flow (as  $f(e)$  may take non-integer value), the following result shows that the existence of a proper  $k$ -flow and the existence of a nowhere zero  $k$ -flow are equivalent.

**Lemma 1.1** [3] *If  $r$  is an integer and  $G$  admits a proper  $r$ -flow, then  $G$  admits a nowhere-zero  $r$ -flow.*

By definition, if a graph  $G$  admits a proper  $r$ -flow, then  $G$  admits a proper  $r'$ -flow for any  $r' \geq r$ . Therefore a graph  $G$  admits a nowhere-zero  $k$ -flow (for some integer  $k$ ) if and only if  $\Phi_c(G) \leq k$ . Hence  $\Phi(G) = \lceil \Phi_c(G) \rceil$ . The parameter  $\Phi_c(G)$  may be viewed as a refinement of  $\Phi(G)$ , and it contains more information about the structure of  $G$ . There are many results show that the circular flow number is a very natural refinement of flow number. For example, it is known (cf. [3]) that  $\Phi_c(G) = \min\{\max\{|C|/|C^-|, |C|/|C^+|\}\}$  and  $\Phi(G) = \min\{\max\{\lceil |C|/|C^-| \rceil, \lceil |C|/|C^+| \rceil\}\}$ . Here the minimum is taken over all acyclic orientations, and the maximum is taken over all circuits  $C$  of  $G$ .  $C^+$  and  $C^-$  denote the set of forward edges and the set of backward edges of  $C$ , respectively.

The flow number of graphs has been studied extensively in the literature. A fundamental problem in this area is “which integers are the possible values for the flow numbers of graphs ?” It is known that there are graphs with flow numbers 2, 3, 4 and 5. A well-known conjecture due to Tutte [11] asserts that these are the only possible values for the flow numbers of graphs.

**Conjecture 1.1** *Every bridgeless graph  $G$  admits a nowhere-zero 5-flow.*

The most significant progress concerning this conjecture was obtained by P. Seymour [9]:

**Theorem 1.1** *Every bridgeless graph  $G$  admits a nowhere-zero 6-flow.*

The question remains open whether or not there exists a graph  $G$  with  $\Phi(G) = 6$ .

Since the circular flow number of any graph is at least 2, Tutte’s 5-flow conjecture is equivalent to the statement that the circular flow number of a bridgeless graph is always between 2 and 5. Then a natural question concerning the circular flow numbers of graphs is “which rationals are the possible values for the circular flow numbers of graphs ?” The following particular question was asked in [17]:

**Question 1.1** *Is it true that for any rational number  $r \in [2, 5]$ , there exists a graph  $G$  with  $\Phi_c(G) = r$  ?*

This paper answers Question 1.1 in the affirmative. The result implies that if the 5-flow conjecture is true, then the possible values for the circular flow numbers of graphs are exactly all the rational numbers in the interval  $[2, 5]$ .

The *circular chromatic number*  $\chi_c(G)$  of a graph  $G$  is the dual concept of the circular flow number, which can be defined as follows: A *tension* on  $\vec{G}$  is a weight function  $f$  such that for each cycle  $C$  of  $G$ ,

$$f(C^+) - f(C^-) = 0.$$

An *r-tension* is a tension  $f$  such that for each edge  $e$ ,  $1 \leq |f(e)| \leq r - 1$ . Similarly as proper  $r$ -flow, the existence of an  $r$ -tension does not depend the orientation, but rather reflects the structure of the graph. The circular chromatic number  $\chi_c(G)$  of  $G$  is defined as

$$\chi_c(G) = \inf\{r : G \text{ admits an } r\text{-tension}\}.$$

There are many equivalent definitions for the circular chromatic number. Readers are referred to [19] for alternative definitions and for the basic properties of  $\chi_c(G)$ . We remark that for any graph  $G$ ,  $\chi(G) = \lceil \chi_c(G) \rceil$ , and hence the circular chromatic number of a graph is a refinement of its chromatic number.

If  $G$  is a planar graph and  $G^*$  is the geometrical dual of  $G$ , then an  $r$ -tension of  $G$  corresponds to a proper  $r$ -flow of  $G^*$  and vice versa. Therefore  $\chi_c(G) = \Phi_c(G^*)$ . In [17], it is proved that for any rational number  $r \in [2, 4]$  there exists a planar graph  $G$  with  $\Phi_c(G) = r$ . (See [5] for a simpler construction of such graphs). Therefore, for each rational  $r \in [2, 4]$  there is a planar graph  $G$  with  $\Phi_c(G) = r$ . To answer Question 1.1, it suffices to consider those rational numbers  $r \in (4, 5]$ . In this paper, we shall construct a graph  $G$  with  $\Phi_c(G) = r$  for every rational number  $r \in (4, 5]$ .

Question 1.1 was also studied by Steffen [10]. It was shown in [10] that for any  $\epsilon > 0$  there exists a graph  $G$  with  $4 < \Phi_c(G) < 4 + \epsilon$ .

A nontrivial cubic graph  $G$  with  $\Phi(G) > 4$  is called a *snark* [2]. Here “nontrivial” is ambiguous. One interpretation is that a snark is cyclically 4-edge connected with girth at least 5 [13]. Search for snarks and study of properties of snarks have attracted considerable continuous attention. Indeed, the above mentioned result of Steffen [10] was proved by considering the circular flow numbers of a well-known family of snarks: the *flower snarks*. For any integer  $k \geq 1$ , the flower snark  $J_{2k+1}$  is the cubic graph with vertex set  $\{a_i, b_i, c_i, d_i : i = 0, 1, 2, \dots, 2k\}$ , and with edge set  $\{a_i a_{i+1}, a_i b_i, b_i c_i, b_i d_i, d_i c_{i+1}, c_i d_{i+1} : i = 0, 1, \dots, 2k\}$  (where the summation is modulo  $2k + 1$ ). It was shown in [10] that  $\Phi_c(J_{2k+1}) \leq 4 + 1/k$  for each  $k \geq 1$ .

The result in this paper provides a method for constructing graphs  $G$  with  $\Phi_c(G) = r$  for any  $r \in (4, 5]$ . By appropriately splitting vertices  $x$  of degree  $d(x) \geq 4$ , we obtain snarks  $G'$  with  $\Phi_c(G') \geq r$ . This implies that for each  $r < 5$ , there exist infinitely many snarks  $G$  with  $\Phi_c(G) \geq r$ .

## 2 Two-terminal graphs and rooted $r$ -flow

Graphs considered in the remaining of this paper are assumed to be connected, but may have bridges.

**Definition 2.1** *A two-terminal graph is a triple  $(G; x, y)$  such that  $G$  is a graph,  $x, y$  are two distinct vertices of  $G$ . The two specified vertices  $x$  and  $y$  are called the terminals of  $G$ .*

**Definition 2.2** *Suppose  $(G; x, y)$  is a two-terminal graph with an arbitrary fixed orientation and  $r \geq 2$  is a real number. A rooted  $r$ -flow on  $(G; x, y)$  is a weight function  $f$  on  $G$  such that*

- for every edge  $e$ ,  $1 \leq |f(e)| \leq r - 1$ ;
- for every vertex  $v \in V(G) - \{x, y\}$ ,  $f^+(v) - f^-(v) = 0$ .

Intuitively, a rooted  $r$ -flow is almost a proper  $r$ -flow, except that the two terminals may violate the conservation law. It can also be viewed as a network flow with the two terminals as the source and sink.

Suppose  $r > 0$  and  $x$  is a real number. We denote by “ $x \bmod r$ ” the remainder of  $x$  divided by  $r$ , i.e., “ $x \bmod r$ ” is the unique real number  $0 \leq x' < r$  such that  $x - x'$  is a multiple of  $r$ .

**Definition 2.3** *Given a rooted  $r$ -flow  $f$  of a two-terminal graph  $(G; x, y)$ , the value of  $f$ , denoted by  $\sigma(f)$ , is defined as*

$$\sigma(f) = (f^+(x) - f^-(x)) \bmod r.$$

For a two-terminal graph  $(G; x, y)$  and a real number  $r \geq 2$ , let

$$L_r(G; x, y) = \{\sigma(f) : f \text{ is a rooted } r\text{-flow on } (G; x, y)\}.$$

$L_r(G; x, y)$  (denoted by  $L_r(G)$  when the two terminals are clear from the context) is called the  $r$ -label set of  $(G; x, y)$ .

The following facts follow easily from the definitions:

1. For a rooted  $r$ -flow of  $(G; x, y)$ ,  $\sigma(f) = (f^-(y) - f^+(y)) \bmod r$ .

2.  $G$  admits a proper  $r$ -flow if and only if  $0 \in L_r(G; x, y)$  (for some two vertices  $x, y$  of  $G$ ).
3. Let  $G|_{xy}$  be the graph obtained from  $G$  by identifying  $x$  and  $y$  into a single vertex. Then a rooted  $r$ -flow on  $(G; x, y)$  induces a proper  $r$ -flow on  $G|_{xy}$ . Therefore  $G|_{xy}$  admits a proper  $r$ -flow if and only if  $L_r(G; x, y) \neq \emptyset$ .

So to determine whether or not a graph  $G$  admits a proper  $r$ -flow, one may calculate the  $r$ -label set  $L_r(G; x, y)$  for proper terminals  $x, y$ . In general, it is very difficult to calculate  $L_r(G; x, y)$  for a two-terminal graph  $(G; x, y)$ . However, if  $G$  is built on some simple structured graphs by using simple graph operations, then it may be possible to determine its label set. This is indeed what we shall do in our proofs. The graph operations we shall use in this paper are the parallel join and series join operations defined as follows:

- (Parallel join): Let  $(G; x, y)$  and  $(G'; x', y')$  be two disjoint two-terminal graphs. Let  $G''$  be the graph obtained from the union of  $G$  and  $G'$  by identifying  $x$  and  $x'$  into a single vertex  $x''$ , and identifying  $y$  and  $y'$  into a single vertex  $y''$ . The two-terminal graph  $(G''; x'', y'')$  is called the *parallel join* of  $(G; x, y)$  and  $(G'; x', y')$ .
- (Series join): Let again  $(G; x, y)$  and  $(G'; x', y')$  be two disjoint two-terminal graphs. Let  $G''$  be the graph obtained from the union of  $G$  and  $G'$  by identifying  $y$  and  $x'$  into a single vertex. The two-terminal graph  $(G''; x, y')$  is called obtained the *series join* of  $(G; x, y)$  and  $(G'; x', y')$ .

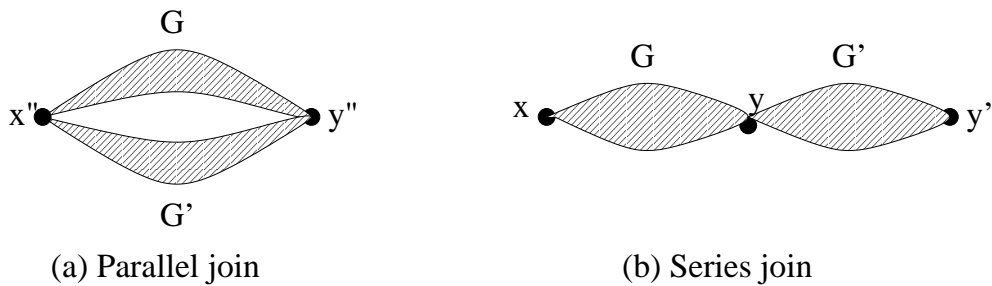


Figure 1: Illustration of the parallel join and series join

If the  $r$ -label sets of  $(G; x, y)$  and  $(G'; x', y')$  are known, then there is an easy way of determining the  $r$ -label set of their parallel join or series join.

First we introduce the concept of modular  $r$ -flows. A *proper modular  $r$ -flow* is a weight function  $f$  such that for each edge  $e$ ,  $1 \leq f(e) \bmod r \leq r - 1$ ,

and for each vertex  $v$ ,  $\sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e) \equiv 0 \pmod{r}$ . A *rooted modular  $r$ -flow* of a two-terminal graph is defined similarly. It is easy to show (see [14] for the discussion of integer flows) that a graph admits a proper  $r$ -flow if and only if  $G$  admits a proper modular  $r$ -flow. The  $r$ -label set  $L_r(G)$  of a two-terminal graph can be equivalently defined as

$$L_r(G; x, y) = \{\sigma(f) : f \text{ is a rooted modular } r\text{-flow of } (G; x, y)\}.$$

Using these facts, it is easy to prove the following lemma:

**Lemma 2.1**    1. If  $G''$  is the series join of  $G$  and  $G'$  then

$$L_r(G'') = L_r(G) \cap L_r(G').$$

2. If  $G''$  is the parallel join of  $G$  and  $G'$  then

$$L_r(G'') = L_r(G) + L_r(G').$$

Here the addition  $A + B$  of two subsets  $A, B$  of  $[0, r)$  is defined as

$$A + B = \{(a + b) \bmod r : a \in A, b \in B\}.$$

Note that the addition is commutative and associative. So the addition of more than two subsets, say  $A_1 + A_2 + \cdots + A_t$ , is well defined.

By Lemma 2.1, to calculate the label set  $L_r(G)$  of a graph  $G$ , one frequently needs to add two subsets of  $C^r$ . The following notion simplifies the calculation of the addition of two subsets.

Suppose  $r \geq 2$  is a real number. We shall view the real numbers in the set  $[0, r)$  forming a circle (by identifying 0 and  $r$ ), and denote this circle by  $C^r$ . For two distinct real numbers  $a, b \in [0, r)$ , we denote by  $[a, b]_r$  the interval of  $C^r$  goes from  $a$  to  $b$  along the clockwise (i.e., the increasing) direction. To be precise, if  $a < b$  then the interval  $[a, b]_r$  contains all the real numbers  $t$  such that  $a \leq t \leq b$ ; if  $a > b$  then the interval  $[a, b]_r$  contains all the real numbers  $t$  such that either  $a \leq t < r$  or  $0 \leq t \leq b$ . When the real number  $r$  is clear from the context, we shall write  $[a, b]$  for  $[a, b]_r$ .

For  $r = 3.5$ , the intervals  $[0.5, 2]$  and  $[2.5, 0.2]$  are depicted in Fig. 2. The length  $\ell([a, b]_r)$  of an interval  $[a, b]_r$  of  $C^r$  is just the geometric length of that interval on  $C^r$ , i.e., if  $a < b$  then  $\ell([a, b]_r) = b - a$ ; if  $a > b$  then  $\ell([a, b]_r) = b + r - a$ .

Lemma 2.2 below (quoted from [7]) says that the sum of intervals of  $C^r$  can be easily determined.

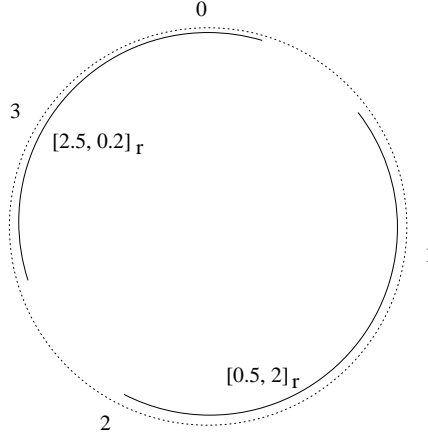


Figure 2: Illustration of the circle  $C^r$  for  $r = 3.5$  and intervals of  $C^r$

**Lemma 2.2** *Suppose  $A_i = [a_i, b_i]$  for  $i = 1, 2, \dots, t$ . If  $\sum_{i=1}^t \ell([a_i, b_i]) \geq r$  then  $A_1 + A_2 + \dots + A_t = C^r$ ; if  $\sum_{i=1}^t \ell([a_i, b_i]) < r$  then  $A_1 + A_2 + \dots + A_t = [a_1 + a_2 + \dots + a_t, b_1 + b_2 + \dots + b_t]$ , where the summations are carried out modulo  $r$ .*

Note that if  $A = \emptyset$  then for any set  $B$ ,  $A + B = \emptyset$ .

The following lemma shows that the  $r$ -label set of a two-terminal graph is “symmetric”.

**Lemma 2.3** *For any two-terminal graph  $(G; x, y)$ ,  $t \in L_r(G)$  if and only if  $(r - t) \bmod r \in L_r(G)$ .*

**Proof.** If  $f$  is a rooted  $r$ -flow with  $\sigma(f) = t$ , then  $g(v) = -f(v)$  is a rooted  $r$ -flow with  $\sigma(g) = r - t$ . ■

### 3 Graphs $G$ with $4 < \Phi_c(G) < 5$

This section proves the main result of this paper:

**Theorem 3.1** *For each rational  $4 < p/q \leq 5$ , there is a graph  $G$  with  $\Phi_c(G) = p/q$ .*

The case  $p/q = 5$  was known (a proof of the following folklore result first appeared in [10]):

**Theorem 3.2** *Let  $G$  be the Petersen graph. Then  $\Phi_c(G) = 5$ .*

Let  $G$  be the Petersen graph and let  $e = xy$  be an edge of  $G$ . In the remaining, we use  $(H; x, y)$  to denote the two-terminal graph, where  $H = G - e$ . The following lemma is also easy:

**Lemma 3.1** *For the graph  $H$  defined above,  $\Phi_c(H) = 4$ .*

The rest of this section is devoted to the proof of Theorem 3.1. By Theorem 3.2, it suffices to consider fractions  $p/q \in (4, 5)$ . First we consider an ordering of such fractions, by defining the parents of each fraction  $p/q \in (4, 5)$ .

It is well-known that for each fraction  $p/q$  with  $p, q > 1$  (and  $(p, q) = 1$ ), there are unique integers  $0 < a < p$  and  $0 < b < q$  such that  $pb - aq = 1$ . We call the fraction  $a/b$  the *lower parent* of  $p/q$ , denote it by  $p_l(p/q) = a/b$ . Let  $a' = p - a$  and  $b' = q - b$ . The fraction  $a'/b'$  is called the *upper parent* of  $p/q$ , and denoted by  $p_u(p/q) = a'/b'$ . Observe that

$$p_l(p/q) < p/q < p_u(p/q).$$

Moreover, since  $a'b - ab' = (p - a)b - a(q - b) = pb - qa = 1$ , so if  $a < a'$  then  $p_l(a'/b') = a/b$ . It is also easy to see that in case  $a > a'$  then  $p_l(a'/b') < a/b$ . So in any case,  $p_l(a'/b') \leq a/b$ . Similarly,  $p_u(a/b) \geq a'/b'$ .

Now we are ready to construct graphs  $G$  with  $4 < \Phi_c(G) < 5$ . The two-terminal graph  $(H; x, y)$  is the basic building block for our construction. First we shall calculate the label set of  $(H; x, y)$ . The following lemma is easy and its proof is omitted (see [14] for proofs of similar results).

**Lemma 3.2** *If a two-terminal graph  $(G; x, y)$  has a rooted  $r$ -flow  $f$ , then there is an orientation  $\vec{G}$  of  $G$  which has a rooted  $r$ -flow  $f'$  with  $f'(e) > 0$  for all edges  $e$  of  $\vec{G}$ , and with  $\sigma(f) = \sigma(f')$ . Moreover, if  $\sigma(f) \geq 0$ , then there is a directed path from  $x$  to  $y$  in  $\vec{G}$ .*

**Lemma 3.3** *For  $4 < r < 5$ ,  $L_r(H; x, y) = [4, r - 4]$ . For  $r < 4$ ,  $L_r(H; x, y) = \emptyset$ .*

**Proof.** Suppose  $4 < r < 5$ . Since  $\Phi_c(H) = 4$ , it follows that  $0 \in L_r(H; x, y)$ . By Lemma 3.2, there is an orientation  $\vec{H}$  of  $H$  such that there is a proper 4-flow  $f$  on  $\vec{H}$  such that for each edge  $1 \leq f(e) \leq 3$ . Moreover, there is a directed path, say  $P = (e_1, e_2, \dots, e_k)$ , from  $x$  to  $y$  in  $\vec{H}$ . For any

$0 < \epsilon \leq r - 4$ , let  $h$  be the weight function on  $\vec{H}$  defined as  $h(e_i) = \epsilon$  for  $i = 1, 2, \dots, k$ , and  $h(e) = 0$  for other edges  $e$  of  $\vec{H}$ . Then it is straightforward to verify that the weight function  $g(e) = f(e) + h(e)$  is a rooted  $r$ -flow with  $\sigma(g) = \epsilon$ . Therefore  $[0, r - 4] \subseteq L_r(H)$ . By Lemma 2.3,  $[4, r - 4] \subseteq L_r(H)$ . (The weight function  $g$  defined above is said to be obtained from  $f$  by adding weight  $\epsilon$  to the directed path  $P$ ).

Assume to the contrary of this lemma that there exists a real number  $\beta \in L_r(H) - [4, r - 4]$ . By Lemma 2.3, we may assume that  $r/2 \leq \beta < 4$ . Let  $f$  be a rooted  $r$ -flow on  $\vec{H}$  with  $\sigma(f) = \beta$ . Let  $\vec{G}$  be the orientation of the Petersen graph obtained from  $\vec{H}$  by adding a directed edge  $e$  from  $y$  to  $x$ . Let  $g$  be the weight function on  $\vec{G}$  defined as  $g(e') = f(e')$  if  $e' \neq e$ , and  $g(e) = \beta$ . Then  $g$  is a modular  $r'$ -flow, where  $r' = \max\{r, \beta + 1\} < 5$ . This is in contrary to Theorem 3.2.  $\blacksquare$

**Lemma 3.4** *For each integer  $k \geq 2$ , there is a two-terminal graph  $(H_k; x, y)$  such that for  $4 \leq r < ((k - 1)4 + 1)/(k - 1)$ ,*

$$L_r(H_k) = [4k - (k - 1)r, kr - 4k].$$

Moreover, for  $r < 4$ ,

$$L_r(H_k) = \emptyset.$$

**Proof.** Let  $H_k$  be the parallel join of  $k$  copies of  $H$ . Since  $L_r(H) = [4, r - 4]$  which is an interval of length  $2(r - 4)$ , by Lemma 2.1 and Lemma 2.2,

$$L_r(H_k) = [4k - (k - 1)r, kr - 4k].$$

(Because  $r < ((k - 1)4 + 1)/(k - 1)$ , the sum of the lengths of  $k$  copies of the interval  $[4, r - 4]$  is equal to  $2k(r - 4) < r$ . Also note that  $(k - 1)r < 4k < kr$ . So  $4k \bmod r = 4k - (k - 1)r$ ).  $\blacksquare$

**Lemma 3.5** *Suppose  $4 < p/q < 5$  and that  $a/b = p_l(p/q)$ ,  $a'/b' = p_u(p/q)$ . There exists a two-terminal graph  $(G; x, y)$  such that for  $a/b \leq r < a'/b'$ ,*

$$L_r(G) = [p - 1 - (q - 1)r, qr - p + 1].$$

Moreover, for  $r < a/b$ ,

$$L_r(G) = \emptyset.$$

**Proof.** We shall prove this lemma by induction on the denominator  $q$ . If  $p/q = (4k + 1)/k$  for some integer  $k$ , then  $p_l(p/q) = 4/1$  and  $p_u(p/q) =$

$(4(k-1)+1)/(k-1)$ . We let  $G$  be the graph  $H_k$  defined in the proof of Lemma 3.4. The conclusion follows from Lemma 3.4.

Assume now that  $p/q \neq (4k+1)/k$  for any integer  $k$ . Then  $q \geq 3$  and  $p_l(p/q) = a/b > 4$  (hence  $b \geq 2$ ). By our induction hypothesis, there is a two terminal graph  $(G'; x', y')$  such that for  $p_l(a/b) \leq r < p_u(a/b)$ ,

$$L_r(G') = [a-1-(b-1)r, br-a+1].$$

Let  $Q$  be the series join of  $G'$  and  $K_2$ . As  $L_r(K_2) = [1, r-1]$ , by Lemma 2.1, for  $r < a/b$ ,  $L_r(Q) = \emptyset$ , and for  $a/b \leq r < a'/b' \leq p_u(a/b)$ ,

$$\begin{aligned} L_r(Q) &= L_r(G') \cap L_r(K_2) \\ &= [a-1-(b-1)r, br-a+1] \cap [1, r-1] \\ &= [1, br-a+1] \cup [a-1-(b-1)r, r-1]. \end{aligned}$$

Fig. 3 illustrates the intersection of  $L_r(G')$  and  $L_r(K_2)$ .

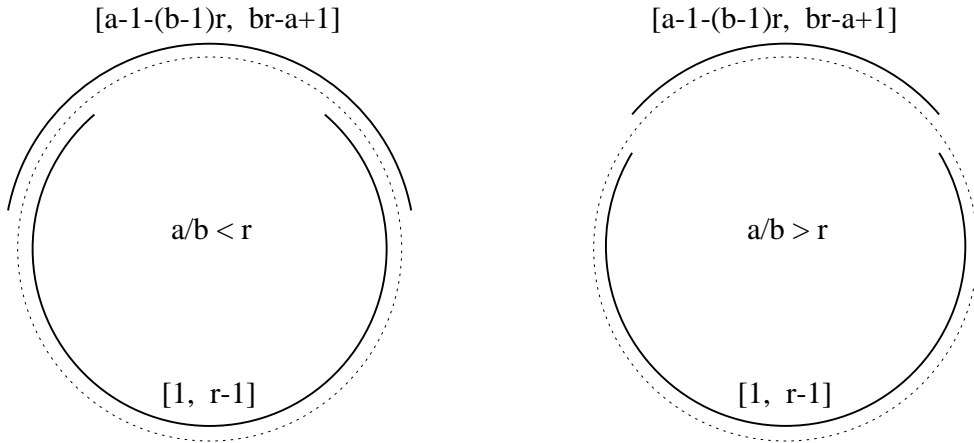


Figure 3: Illustration of the intersection of  $L_r(G') \cap L_r(K_2)$

Let  $R$  be the parallel join of two copies of  $Q$ . By Lemma 2.1,

$$\begin{aligned} L_r(R) &= ([1, br-a+1] \cup [a-1-(b-1)r, r-1]) \\ &\quad + ([1, br-a+1] \cup [a-1-(b-1)r, r-1]) \\ &= [2, 2br-2a+2] \cup [2a-2-(2b-1)r, r-2] \cup [a-(b-1)r, br-a]. \end{aligned}$$

Fig. 4 is an illustration of the label set  $L_r(R)$ . (Note that there are two possibilities for the relative positions of the intervals).

Let  $G^*$  be the series join of  $R$  and  $H$  (where  $H = G - e$  is as defined in Lemma 3.3). Then

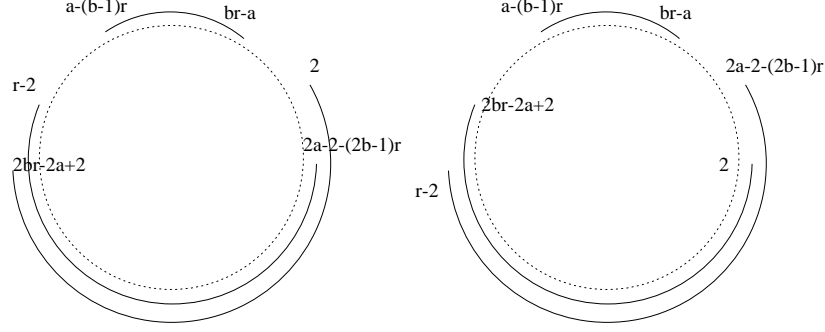


Figure 4: Illustration of  $L_r(R)$

$$\begin{aligned}
L_r(G^*) &= L_r(R) \cap L_r(H) \\
&= ([2, 2br - 2a + 2] \cup [2a - 2 - (2b - 1)r, r - 2] \\
&\quad \cup [a - (b - 1)r, br - a]) \cap [4, r - 4]
\end{aligned}$$

We shall show that the intersection is equal to  $[a - (b - 1)r, br - a]$ , i.e., that the positions of the intervals are more or less as indicated in Fig. 5. (Note that compared to Fig. 4, we do not know which of the two numbers 2 and  $2a - 2 - (2b - 1)r$  is smaller. Fig. 5 only show one of the two possibilities).

By assumption,  $r - 4 < 2$  and  $r - 2 < 4$ . Moreover, since  $r < a'/b' \leq (a + 1)/b$ , we have  $r - 4 < 2a - 2 - (2b - 1)r$  and  $2br - 2a + 2 < 4$ . Since  $r < a'/b' \leq (a - 4)/(b - 1)$ , we have  $r - 4 > br - a$  and  $4 < a - (b - 1)r$ . Therefore

$$L_r(G^*) = [a - (b - 1)r, br - a].$$

(see Fig. 5 for an illustration).

Now we consider two cases. If  $q = b + 1$  then  $p = a + 4 - 1$  and  $p_u(p/q) = 5/1$ . In this case, we let  $G$  be the parallel join of  $G^*$  and  $H$ . By Lemma 2.1, for  $a/b \leq r < (p - 4)/(q - 1) \leq (a - 4)/(b - 1)$ ,

$$\begin{aligned}
L_r(G) &= L_r(G^*) + L_r(H) \\
&= [a - (b - 1)r, br - a] + [4, r - 4] \\
&= [(a + 5) - 1 - ((b + 1) - 1)r, (b + 1)r - (a + 5) + 1] \\
&= [p - 1 - (q - 1)r, qr - p + 1].
\end{aligned}$$

If  $q \geq b + 2$  then  $a'/b' = (p - a)/(q - b) = p_u(p/q) < 5$ . By the induction hypothesis, there is a two-terminal  $G''$  such that for  $p_l(a'/b' \leq r < (a' - 4)/(b' - 1)$ ,

$$L_r(G'') = [a' - 1 - (b' - 1)r, b'r - a' + 1].$$

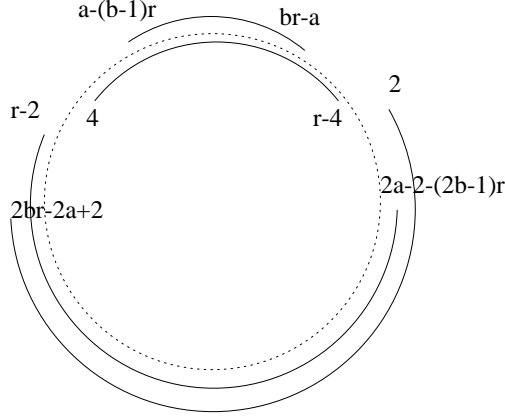


Figure 5: Illustration of the involved intervals

Let  $G$  be the parallel join of  $G^*$  and  $G''$ . By Lemma 2.1, for  $a/b \leq r < a'/b' \leq p_u(a/b)$ ,

$$\begin{aligned}
L_r(G) &= L_r(G^*) + L_r(G'') \\
&= [a - (b - 1)r, br - a] + [a' - 1 - (b' - 1)r, b'r - a' + 1] \\
&= [(a + a') - 1 - (b + b' - 1)r, (b + b')r - (a + a') + 1] \\
&= [p - 1 - (q - 1)r, qr - p + 1].
\end{aligned}$$

This completes the proof of Lemma 3.5. ■

**Proof of Theorem 3.1:** Suppose  $4 < p/q < 5$ . By Lemma 3.5, there is a two-terminal graph  $(G; x, y)$  with

$$L_r(G) = [p - 1 - (q - 1)r, qr - p + 1].$$

Let  $(G'; x', y')$  be the series join of  $G$  and  $K_2$ . Since  $L_r(K_2) = [1, r - 1]$ , by using Lemma 2.1, it is easy to verify that  $L_r(G') \neq \emptyset$  if and only if  $r \geq p/q$ . Let  $G'' = G'|_{x'y'}$ . Then  $\Phi_c(G'') = p/q$ . This completes the proof of Theorem 3.1. ■

**Remark:** The graphs  $G$  with  $\Phi_c(G) = r$  constructed here are not cubic. It is known [14] that if  $G$  is a bridgeless graph and  $x$  is a vertex of degree  $d_G(x) \geq 4$ , then there is a splitting at  $x$  so that the resulting graph is still bridgeless. Here a splitting at  $x$  means to add a new vertex  $x'$  and move two or more edges incident to  $x$  so that they be incident to  $x'$ , and then “smooth” any degree 2 vertices. By keep splitting at vertices of degree  $\geq 4$  we obtain a bridgeless cubic graph  $G'$  with  $\Phi_c(G') \geq \Phi_c(G) = r > 4$ . It is possible that  $\Phi_c(G') > \Phi_c(G)$ . The graph shown in Fig. 6 is a graph constructed as in the proof of Theorem 3.1 with  $\Phi_c(G) = 4.5$ . The two cubic graphs  $G'$  and  $G''$  are obtained from  $G$  by splitting at vertices  $u$  and  $v$ . Therefore

$\Phi_c(G'), \Phi_c(G'') \geq \Phi_c(G) = 4.5$ . The orientations and the assignment of weights to the edges in the figure show that each of  $G'$  and  $G''$  admits a proper 4.5-flow. Therefore  $\Phi_c(G') = \Phi_c(G'') = 4.5$ .

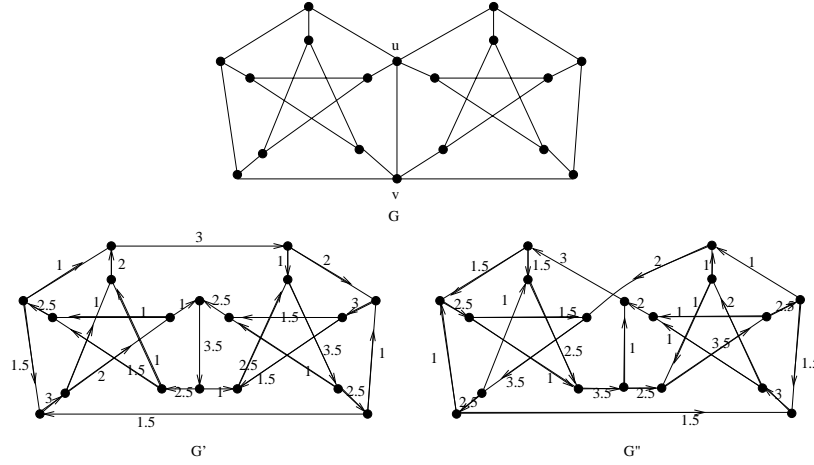


Figure 6: A graph  $G$  with  $\Phi_c(G) = 4.5$  and two cubic graphs  $G', G''$  obtained from  $G$  with  $\Phi_c(G') = \Phi_c(G'') = 4.5$

In Fig. 7,  $H$  is a graph constructed as in the proof of Theorem 3.1 with  $\Phi_c(H) = 14/3$ . The graph  $H'$  is a cubic graph obtained from  $H$  by replacing each vertex  $x$  of degree  $d(x) \geq 4$  by a circuit of length  $d(x)$ , then let each edge incident to  $x$  be incident to a distinct vertex of the circuit. Therefore  $H'$  is a cubic graph with  $\Phi_c(H') \geq \Phi_c(H) = 14/3$ . It can be shown that indeed  $\Phi_c(H') = 14/3$ .

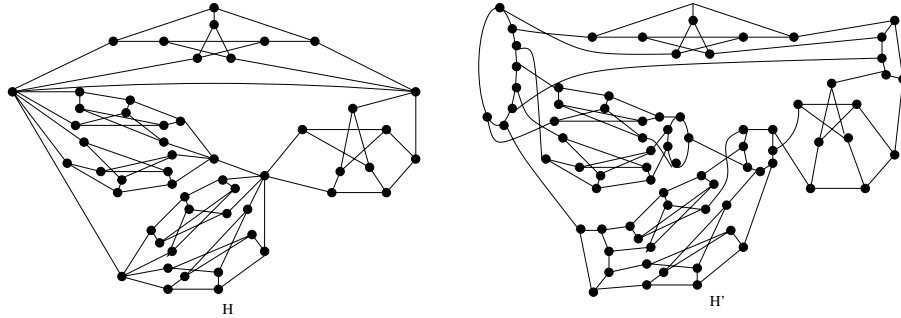


Figure 7: A graph  $H$  with  $\Phi_c(H) = 14/3$  and a snark  $H'$  obtained from  $H$  with  $\Phi_c(H') = 14/3$

Note that the method for constructing a graph  $G$  with  $\Phi_c(G) = r$  given in the proof of Theorem 3.1 is not deterministic. There are steps where one can make a choice. For example, when we take the parallel join of two two-terminal graphs  $(G; x, y)$  and  $(G'; x', y')$ , we can either identify  $x$  with  $x'$ ,

and  $y$  and  $y'$ , or identify  $x$  with  $y'$ , and  $y$  and  $x'$ . Different choice results in different graphs. The procedure of obtaining a cubic graph  $G'$  from  $G$  by either splitting at vertices of degree  $\geq 4$ , or by replacing a vertex by a circuit, is also non-deterministic. By appropriately making these choices (as shown in the examples above), the cubic graphs  $G'$  obtained will be cyclically 4-edge connected with girth at least 5. In other words, the resulting graphs are snarks. Snarks have been studied extensively in the literature. However, not much is known about their circular flow numbers, except that it follows from the definition that  $\Phi_c(G) > 4$  for each snark  $G$ . For example, no snark other than the Petersen graph was known to have circular flow number  $> 4.5$ . Our result provides a method for constructing infinitely many snarks  $G$  with  $\Phi_c(G) \geq r$  for any  $r < 5$ . However, the following question remains open:

**Question 3.1** *Is it true that for any  $4 < r < 5$  there exists a snark  $G$  with  $\Phi_c(G) = r$ ?*

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